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# A generalization of Bôcher's theorem for polyharmonic functions

*Dedicated to Professor Maretsugu Yamasaki on the occasion of his sixtieth birthday*

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**Abstract.** In this paper we generalize Bôcher's theorem for polyharmonic functions $u$. In fact, if $u$ is polyharmonic outside the origin and satisfies a certain integral condition, then it is shown that $u$ is written as the sum of partial derivatives of the fundamental solution and a polyharmonic function near the origin.

## 1. Introduction

Let $\mathbf{R}^n$ be the $n$-dimensional Euclidean space with points $x = (x_1, x_2, \dots, x_n)$. For a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

We denote by $B(x, r)$ the open ball centered at $x$ with radius $r > 0$, whose boundary is written as $S(x, r) = \partial B(x, r)$. We also denote by $\mathbf{B}$ the unit ball $B(0, 1)$ and by $\mathbf{B}_0$ the punctured unit ball $\mathbf{B} - \{0\}$.

A real valued function $u$ on an open set $G \subset R^n$ is called polyharmonic of order $m$ on $G$ if $u \in C^{2m}(G)$ and $\Delta^m u = 0$ on $G$, where $m$ is a positive integer, $\Delta$ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$. We denote by $H^m(G)$ the space of polyharmonic functions of order $m$ on $G$. In particular, $u$ is harmonic on $G$ if $u \in H^1(G)$. The fundamental solution of $\Delta^m$ is written as $K_m$, that is,

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$$K_m(x) = \alpha_m \begin{cases} |x|^{2m-n} \log(1/|x|) & \text{if } 2m-n \text{ is an even nonnegative integer,} \\ |x|^{2m-n} & \text{otherwise,} \end{cases}$$

where the constant $\alpha_m$ is chosen such that $\Delta^m K_m$ is the Dirac measure $\delta$ at the origin.

Our aim of this paper is to prove the following theorem.

THEOREM. *If $u \in H^m(\mathbf{B}_0)$ satisfies*

$$\int_{\mathbf{B}_0} u(x)^+ |x|^s dx < \infty \tag{1}$$

*for some integer $s \geq 0$, then $u$ is of the form*

$$u = \sum_{|\mu| \leq s+2m-1} c(\mu) D^\mu K_m + h$$

*for some $h \in H^m(\mathbf{B})$, where $c(\mu)$ are constants and $u^+(x) = \max\{u(x), 0\}$.*

We shall show that $u$ in the theorem satisfies

$$\int_{\mathbf{B}_0} |u(x)||x|^s dx < \infty. \tag{2}$$

Hence in case $s \leq 0$, $u$ is integrable on $\mathbf{B}$. In case $s > 0$, we shall show that $u$ defines a distribution $T_u$ such that

$$\langle T_u, v \rangle = \lim_{r \to 0} \int_{\mathbf{B} - B(0,r)} u(x) v(x) dx \qquad \text{for } v \in C_0^\infty(\mathbf{B}).$$

Armitage [1] treated the case where $u \in H^m(\mathbf{B}_0)$ satisfies

$$\frac{1}{\omega_n r^{n-1}} \int_{S(0,r)} |u(x)| dS(x) = o(r^{-s'}) \qquad \text{as } r \to 0, \tag{3}$$

where $\omega_n$ denotes the surface area of $S(0,1)$. If $s' < s+n$, then (3) clearly implies (1).

As an easy consequence of the theorem we have the following result due to Ishikawa-Nakai-Tada [5]:

COROLLARY 1. *If $u$ is a harmonic function on $\mathbf{B}_0$ such that*

$$\limsup_{x \to 0} u(x)|x|^{n-1} \leq 0,$$

*then $u$ is of the form*

$$u = cK_1 + h$$

*for some $h \in H^1(\mathbf{B})$ and a constant $c$.*

As another application of our theorem, we obtain the following result.

Corollary 2. *If* $u \in H^m(\mathbf{B}_0)$ *satisfies* (2) *for some integer s, then u is of the form*

$$u = \sum_{|\mu| \le s+2m-1} c(\mu) D^{\mu} K_m + h$$

*for some* $h \in H^m(\mathbf{B})$*, where* $c(\mu)$ *are constants.*

As to the behavior at infinity, we refer the reader to the recent papers Kishi-Futamura-Mizuta [4] and Nakai-Tada [7].

## 2. Lemmas

In this section we prepare some lemmas, which will be used in the proof of the theorem.

Lemma 1. *If* $u \in H^m(\mathbf{B}_0)$*, then*

$$\frac{1}{\omega_n r^{n-1}} \int_{S(0,r)} u(x)dS = \sum_{k=1}^{m} \{a_k r^{2(1-k)} K_m(r) + b_k r^{2(m+1-k)-n} + c_k r^{2(m-k)}\} \tag{4}$$

*for all* $r \in (0,1)$*, where* $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ *are constants and* $a_k = 0$ *when* $k > m + 1 - \frac{n}{2}$.

Proof. We prove this lemma by induction on $m$. In the case that $m = 1$, this is known; see e.g. [3, Lemma 3.10]. So we suppose that (4) holds for $m = l$, and take $u \in H^{l+1}(\mathbf{B}_0)$. Then $\Delta u \in H^l(\mathbf{B}_0)$ because $\Delta^l(\Delta u) = \Delta^{l+1} u = 0$, so that

$$\int_{S(0,t)} \Delta u \, dS = \omega_n \sum_{k=1}^{l} \{a_k t^{2(1-k)+n-1} K_l(t) + b_k t^{2(l+1-k)-1} + c_k t^{2(l-k)+n-1}\}$$

for $0 < t < 1$, where $a_k = 0$ when $k > l + 1 - \frac{n}{2}$. We integrate this equality with respect to $t$ from $r_1$ to $r_2$, where $0 < r_1 < r_2 < 1$, and obtain

$$\begin{aligned}
\int_{\{x: r_1 < |x| < r_2\}} \Delta u \, dx &= \int_{r_1}^{r_2} \left( \int_{S(0,t)} \Delta u \, dS \right) dt \\
&= \omega_n \sum_{k=1}^{l} \int_{r_1}^{r_2} \{a_k t^{2(1-k)+n-1} K_l(t) + b_k t^{2(l+1-k)-1} + c_k t^{2(l-k)+n-1}\} dt \\
&= \sum_{k=1}^{l} \{a'_k r_2^{2(1-k)+n} K_l(r_2) + b'_k r_2^{2(l+1-k)} + c'_k r_2^{2(l-k)+n}\} \\
&\quad - \sum_{k=1}^{l} \{a'_k r_1^{2(1-k)+n} K_l(r_1) + b'_k r_1^{2(l+1-k)} + c'_k r_1^{2(l-k)+n}\}
\end{aligned} \tag{5}$$

where $a'_k = 0$ when $k > l + 1 - \frac{n}{2}$. On the other hands, we have by Green's formula

$$\int_{\{x:r<|x|<r_2\}} \Delta u\, dx = c(r_2) - \int_{S(0,r)} \frac{x}{r} \cdot \nabla u(x) dS(x)$$
$$= c(r_2) - r^{n-1} \int_{S(0,1)} \zeta \cdot \nabla u(r\zeta) dS(\zeta)$$
$$= c(r_2) - r^{n-1} \frac{d}{dr}\left(\int_{S(0,1)} u(r\zeta) dS(\zeta)\right)$$
$$= c(r_2) - r^{n-1} \frac{d}{dr}\left(\frac{1}{r^{n-1}} \int_{S(0,r)} u\, dS\right),$$

where $0 < r < r_2$ and $c(r_2) = \int_{S(0,r_2)} \frac{x}{r_2} \cdot \nabla u(x) dS(x)$. Hence (5) leads to

$$\sum_{k=1}^{l} \{a'_k r^{2(1-k)+1} K_l(r) + b'_k r^{2(l+1-k)+1-n} + c'_k r^{2(l-k)+1}\} + d' r^{1-n}$$
$$= \frac{d}{dr}\left(\frac{1}{r^{n-1}} \int_{S(0,r)} u\, dS\right)$$

for $0 < r < r_2$. We integrate both sides with respect to $r$ from $r_1$ to $r_2$ to obtain

$$\frac{1}{r_1^{n-1}} \int_{S(0,r_1)} u\, dS = \sum_{k=1}^{l} \{a''_k r_1^{2(1-k)+2} K_l(r) + b''_k r_1^{2(l+1-k)-n+2} + c''_k r_1^{2(l-k)+2}\}$$
$$+ b'' \log r_1 - d'' r_1^{2-n} + d''' \log r_1 + e$$
$$= \sum_{k=1}^{l+1} \{a''_k r_1^{2(1-k)} K_{l+1}(r_1) + b''_k r_1^{2(l+2-k)-n} + c''_k r_1^{2(l+1-k)}\},$$

where $0 < r_1 < r_2 < 1$, $a''_k$, $b''_k$, $c''_k$ are determined such that $a''_k = 0$ when $k > l + 2 - \frac{n}{2}$, $b'' = b'_k$ when $k = l + 2 - \frac{n}{2}$ and $d''' = d'$ when $n = 2$. But we see that the constants $a''_k$, $b''_k$, $c''_k$ are determined independently of $r_2$. Now the induction is completed.

With the aid of Lemma 1, we prove

Lemma 2. *If* $u \in H^m(\mathbf{B}_0)$ *and* $\lambda$ *is a multi-index, then*

$$\frac{1}{\omega_n r^{n-1}} \int_{S(0,r)} u x^{\lambda} \, dS$$

$$= \sum_{k=1}^{m+|\lambda|} \{A_k r^{2(1+|\lambda|-k)} K_m(r) + B_k r^{2(m+1+|\lambda|-k)-n} + C_k r^{2(m+|\lambda|-k)}\} \tag{6}$$

*for* $0 < r < 1$, *where* $A_k$, $B_k$, $C_k$ *are constants and* $A_k = 0$ *when* $k > m + 1 - \frac{n}{2}$.

Proof. We prove this by induction on the length $|\lambda|$. First we note from Lemma 1 that the conclusion is true for $|\lambda| = 0$.

Now let $|\lambda| = l + 1$ and write $\lambda = \lambda' + e_j$, where $|\lambda'| = l$ and $|e_j| = 1$. By the Gauss-Green formula we have

$$\int_{S(0,r_1)} u x^{\lambda'} (x_j/r_1) dS = -\int_{\{x: r_1 < |x| < r_2\}} \frac{\partial}{\partial x_j} \left(u(x) x^{\lambda'}\right) dx + \int_{S(0,r_2)} u x^{\lambda'} (x_j/r_2) dS. \tag{7}$$

Using the assumption on induction, we have

$$\int_{S(0,r)} \frac{\partial u}{\partial x_j}(x) x^{\lambda'} dS$$

$$= \sum_{k=1}^{m+|\lambda'|} \{A_k r^{2(1+|\lambda'|-k)+n-1} K_m(r) + B_k r^{2(m+1+|\lambda'|-k)-1} + C_k r^{2(m+|\lambda'|-k)+n-1}\}$$

and

$$\int_{S(0,r)} u(x) \frac{\partial}{\partial x_j} x^{\lambda'} dS$$

$$= \sum_{k=1}^{m+|\lambda'|-1} \{A'_k r^{2(|\lambda'|-k)+n-1} K_m(r) + B'_k r^{2(m+|\lambda'|-k)-1} + C'_k r^{2(m+|\lambda'|-1-k)+n-1}\},$$

where $A_k = A'_k = 0$ when $k > m + 1 - \frac{n}{2}$. As in the proof of Lemma 1, using polar coordinates in (7), we have the required conclusion for $\lambda$ with $|\lambda| = l + 1$.

Lemma 3. *If* $u \in H^m(\mathbf{B}_0)$ *satisfies* (2), *then the limit*

$$\lim_{r \to 0} \int_{\{x: r < |x| < 1\}} uv \, dx$$

*exists for every* $v \in C_0^{\infty}(\mathbf{B})$. *Further, the mapping*

$$T_u : v \mapsto \lim_{r \to 0} \int_{\{x: r < |x| < 1\}} uv \, dx$$

*defines a distribution on* $\mathbf{B}$. *In what follows we identify* $u$ *with* $T_u$.

PROOF. We write

$$\int_{\{x: r<|x|<1\}} uv \, dx = \int_{\mathbf{B}-B(0,r)} u \left( v - \sum_{|\mu| \le L} \frac{x^\mu}{\mu!} D^\mu v(0) \right) dx$$
$$+ \sum_{|\mu| \le L} \frac{D^\mu v(0)}{\mu!} \int_{\mathbf{B}-B(0,r)} u x^\mu \, dx = I(r) + J(r),$$

where $L$ is an integer such that $L \ge s - 1$. Since $v - \sum_{|\mu| \le L} \frac{x^\mu}{\mu!} D^\mu v(0) = O(|x|^{L+1})$, we have

$$|I(r)| \le \int_{\mathbf{B}-B(0,r)} |u| \left| v - \sum_{|\mu| \le L} \frac{x^\mu}{\mu!} D^\mu v(0) \right| dx \le C \int_{\mathbf{B}-B(0,r)} |u| |x|^{L+1} \, dx,$$

so that $\lim_{r \to 0} I(r)$ exists and is finite by (2).

In view of Lemma 2, we see that

$$\lim_{r \to 0} \int_{\{x: r<|x|<1\}} u(x) x^\mu \, dx = \lim_{r \to 0} \int_r^1 \left( \int_{S(0,t)} u(x) x^\mu \, dS(x) \right) dt$$

exists and is finite; this limit is denoted by $C(\mu)$. Hence $J(r)$ converges to

$$\sum_{|\mu| \le L} \frac{D^\mu v(0)}{\mu!} C(\mu)$$

as $r \to 0$. Therefore

$$\langle u, v \rangle \equiv \lim_{r \to 0} \int_{\{x: r<|x|<1\}} uv \, dx$$
$$= \int_{\mathbf{B}_0} u \left( v - \sum_{|\mu| \le L} \frac{x^\mu}{\mu!} D^\mu v(0) \right) dx + \sum_{|\mu| \le L} C(\mu) \frac{D^\mu v(0)}{\mu!}$$

is defined to be finite. Clearly, $u$ is a distribution on $\mathbf{B}$.

## 3. The proof of the main theorem

In this section we give a proof of the theorem.

(**I**) We first prove the theorem under the strong condition (2). We recall Green's formula for $\Delta^m$ (see e.g. [2], [8]):

$$\int_{\mathbf{B}-B(0,r)} (u\Delta^m v - v\Delta^m u)dx$$

$$= -\sum_{i=1}^{m} \int_{S(0,r)} \left[ (\Delta^{i-1}u) \frac{\partial(\Delta^{m-i}v)}{\partial n} - (\Delta^{m-i}v) \frac{\partial(\Delta^{i-1}u)}{\partial n} \right] dS \qquad (8)$$

for $u \in C^\infty(\mathbf{B}_0)$, $v \in C_0^\infty(\mathbf{B})$ and $0 < r < 1$, where $\partial/\partial n$ denotes the inner normal derivative. If $u \in H^m(\mathbf{B}_0)$ and $v \in C_0^\infty(\mathbf{B})$, then we obtain

$$\int_{\mathbf{B}-B(0,r)} u\Delta^m v\, dx = \sum_{j=1}^{n} \left\{ \sum_{|\lambda|+|\mu|=2m-1} C(\lambda, \mu, j) \int_{S(0,r)} D^\lambda u D^\mu v \frac{x_j}{r}\, dS \right\}$$

with constants $C(\lambda, \mu, j)$. For simplicity, we put $\psi(x) = D^\mu v(x)$, and use its Taylor expansion

$$\psi(x) = \sum_{|\nu| \le l} \frac{x^\nu}{\nu!} D^\nu \psi(0) + R_{l+1}(x),$$

where $l = s + |\lambda|$. Then we have

$$\int_{S(0,r)} D^\lambda u D^\mu v \frac{x_j}{r}\, dS$$

$$= \int_{S(0,r)} D^\lambda u(x) R_{l+1}(x) \frac{x_j}{r}\, dS + \frac{1}{r} \sum_{|\nu| \le l} \frac{1}{\nu!} D^\nu \psi(0) \int_{S(0,r)} D^\lambda u(x) x^\nu x_j\, dS$$

$$= I(\psi) + J(\psi).$$

To evaluate $I(\psi)$, we need the following lemma, which is an easy consequence of [6, Lemma 8.4.5].

Lemma 4. *If $u \in H^m(\mathbf{B}_0)$ and $0 < r < 2/3$, then*

$$\int_{S(0,r)} |D^\lambda u| dS \le Cr^{-|\lambda|-1} \int_{\{x: r/2 < |x| < 3r/2\}} |u| dx$$

*with a positive constant $C$.*

From Lemma 4 it follows that

$$
\begin{aligned}
|I(\psi)| &= \left| \int_{S(0,r)} D^{\lambda} u(x) R_{l+1}(x) \frac{x_j}{r}\, dS \right| \\
&\le C r^{l+1} \int_{S(0,r)} |D^{\lambda} u| dS \\
&\le C' r^{l+1-(|\lambda|+1)-s} \int_{\{x:0<|x|<2r\}} |u(x)||x|^{s} dx,
\end{aligned}
$$

so that $I(\psi)$ tends to zero as $r \to 0$, since $l = s + |\lambda|$.

On the other hand, we see from Lemma 2 that

$$
\begin{aligned}
J(\psi) &= \frac{1}{r} \sum_{|\nu| \le l} \frac{1}{\nu!} D^{\nu}\psi(0) \int_{S(0,r)} D^{\lambda} u(x) x^{\nu} x_j \, dS \\
&= \frac{1}{r} \sum_{|\nu| \le l} \frac{1}{\nu!} D^{\nu}\psi(0) \left\{ \sum_{k=1}^{m+|\nu|+1} (A_k(\lambda,\nu,j) r^{2(2+|\nu|-k)+n-1} K_m(r) \right. \\
&\quad \left. + B_k(\lambda,\nu,j) r^{2(m+2+|\nu|-k)-1} + C_k(\lambda,\nu,j) r^{2(m+1+|\nu|-k)+n-1}) \right\} \\
&\to \sum_{|\nu| \le l} C'(\lambda,\nu,j) D^{\nu}\psi(0) \qquad \text{as } r \to 0,
\end{aligned}
$$

since $A_k(\lambda,\nu,j) = 0$ when $k > m + 1 - \frac{n}{2}$. Hence it follows that

$$
\begin{aligned}
\langle u, \Delta^m v \rangle &= \lim_{r\to 0} \int_{\{x:r<|x|<1\}} u \Delta^m v \, dx \\
&= \sum_{|\lambda+\mu|=2m-1, |\nu| \le s+|\lambda|} C''(\lambda,\nu) D^{\mu+\nu} v(0) \\
&= \sum_{|\lambda| \le s+2m-1} C'''(\lambda) D^{\lambda} v(0). \qquad (9)
\end{aligned}
$$

Finally let us find constants $c(\mu)$ for which $u - \sum_{|\mu| \le s+2m-1} c(\mu) D^{\mu} K_m$ is polyharmonic of order $m$. For this purpose, we have by (9)

$$
\begin{aligned}
&\left\langle \Delta^m \left( u - \sum_{|\mu| \le s+2m-1} c(\mu) D^{\mu} K_m \right), v \right\rangle \\
&= \left\langle u - \sum_{|\mu| \le s+2m-1} c(\mu) D^{\mu} K_m, \Delta^m v \right\rangle \\
&= \langle u, \Delta^m v \rangle - \sum_{|\mu| \le s+2m-1} c(\mu) \langle D^{\mu} K_m, \Delta^m v \rangle
\end{aligned}
$$

$$= \sum_{|\lambda| \le s+2m-1} C'''(\lambda) D^\lambda v(0) - \sum_{|\mu| \le s+2m-1} c(\mu) \langle D^\mu (\Delta^m K_m), v \rangle$$

$$= \sum_{|\lambda| \le s+2m-1} C'''(\lambda) D^\lambda v(0) - \sum_{|\mu| \le s+2m-1} c(\mu) \langle D^\mu \delta, v \rangle$$

$$= \sum_{|\lambda| \le s+2m-1} C'''(\lambda) D^\lambda v(0) - \sum_{|\mu| \le s+2m-1} c(\mu) (-1)^{|\mu|} \langle \delta, D^\mu v \rangle$$

$$= \sum_{|\mu| \le s+2m-1} C'''(\mu) D^\mu v(0) - \sum_{|\mu| \le s+2m-1} c(\mu) (-1)^{|\mu|} D^\mu v(0).$$

Hence if we take $c(\mu) = (-1)^{|\mu|} C'''(\mu)$, then

$$\Delta^m \left( u - \sum_{|\mu| \le s+2m-1} c(\mu) D^\mu K_m \right) = 0$$

as required.

(**II**) Now we assume that (1) holds for $u \in H^m(\mathbf{B}_0)$. Using polar coordinates and Lemma 1, we have

$$\int_{\mathbf{B}-B(0,r)} u(x)|x|^s dx$$

$$= \int_r^1 t^s \left( \int_{S(0,t)} u(x) dS \right) dt$$

$$= \int_r^1 \omega_n t^{s+n-1} \sum_{k=1}^m \{ a_k t^{2(1-k)} K_m(t) + b_k t^{2(m+1-k)-n} + c_k t^{2(m-k)} \} dt$$

$$= \sum_{k=1}^m \{ a_k' r^{2(1-k)+s+n} K_m(r) + b_k' r^{2(m+1-k)+s} + c_k' r^{2(m-k)+s+n} \} + d,$$

where $a_k' = 0$ for $k > m + 1 - \dfrac{n}{2}$. Hence $\int_{\mathbf{B}-B(0,r)} u(x)|x|^s dx$ is bounded for $0 < r < 1$. Therefore (1) implies (2). In view of the first half of the proof, $u$ is of the form

$$u = \sum_{|\mu| \le s+2m-1} c(\mu) D^\mu K_m + h$$

for some $h \in H^m(\mathbf{B})$. The proof of the theorem is now completed.

## 4. Proofs of Corollaries 1 and 2

In this section we give proofs of Corollaries 1 and 2.

Proof of Corollary 1. Assume that $u$ is a harmonic function on $\mathbf{B}_0$ such that

$$u(x) \le o(|x|^{-n+1}) \qquad \text{as } x \to 0. \tag{10}$$

Then we have

$$\int_{\mathbf{B}_0} u(x)^+ dx < \infty,$$

so that (1) holds for $s = 0$. Hence our therem shows that $u$ is of the form

$$u = \sum_{|\mu| \le 1} c(\mu) D^\mu K_1 + h$$

for some $h \in H^1(\mathbf{B})$ and some constants $c(\mu)$. In view of (10), we see that $c(\mu) = 0$ when $|\mu| = 1$, which proves the corollary.

Proof of Corollary 2. Assume that $u \in H^m(\mathbf{B}_0)$ satisfies (2) for an integer $s$. Then

$$\int_{\mathbf{B}_0} |u(x)||x|^{s'} dx < \infty$$

for a nonnegative integer $s' \ge s$. Hence it follows from our theorem that $u$ is the form

$$u = \sum_{|\mu| \le s'+2m-1} c(\mu) D^\mu K_m + h$$

for some $h \in H^m(\mathbf{B})$. According to (2), $\sum_{s+2m-1 < |\mu| \le s'+2m-1} c(\mu) D^\mu K_m$ should disappear, which completes the proof of Corollary 2.

## 5. Remark

Remark 1. If $u \in H^m(\mathbf{B}_0)$, then $u$ is expressed as Laurent series expansion:

$$u(x) = \sum_{\mu} c(\mu) D^\mu K_m(x) + h(x) \qquad (h \in H^m(\mathbf{B}))$$

(cf. [3, Chapter 10]).

To show this, fix $x \in \mathbf{B}_0$ and find from (8) (Green's formula for $\Delta^m$)

$$
\begin{aligned}
0 &= \int_{B(0,r_2)-B(0,r_1)-B(x,r)} (u(y)\Delta^m K_m(x-y) - K_m(x-y)\Delta^m u(y))dy \\
&= \sum_{i=1}^{m} \int_{S(0,r_2)} \left[ (\Delta^{i-1}u(y)) \frac{\partial(\Delta^{m-i}K_m(x-y))}{\partial n_y} \right. \\
&\qquad \left. - (\Delta^{m-i}K_m(x-y)) \frac{\partial(\Delta^{i-1}u(y))}{\partial n_y} \right] dS(y) \\
&\quad - \sum_{i=1}^{m} \int_{S(0,r_1)} \left[ (\Delta^{i-1}u(y)) \frac{\partial(\Delta^{m-i}K_m(x-y))}{\partial n_y} \right. \\
&\qquad \left. - (\Delta^{m-i}K_m(x-y)) \frac{\partial(\Delta^{i-1}u(y))}{\partial n_y} \right] dS(y) \\
&\quad - \sum_{i=1}^{m} \int_{S(x,r)} \left[ (\Delta^{i-1}u(y)) \frac{\partial(\Delta^{m-i}K_m(x-y))}{\partial n_y} \right. \\
&\qquad \left. - (\Delta^{m-i}K_m(x-y)) \frac{\partial(\Delta^{i-1}u(y))}{\partial n_y} \right] dS(y) \\
&= \alpha(r_2,x) - \beta(r_1,x) - \gamma(r,x)
\end{aligned}
$$

for $0 < r_1 < |x| < r_2$ and $r > 0$ with $B(x,r) \subset B(0,r_2) - B(0,r_1)$. Note that $\lim_{r\to 0}\gamma(r,x) = cu(x)$ for some constant $c$ and $\lim_{r_2\to 1}\alpha(r_2,x) \in H^m(\mathbf{B})$. Further, using the Taylor expansion for $K_m$ and Lemma 2, we have

$$\lim_{r_1\to 0}\beta(r_1,x) = \sum_{\mu} c(\mu)D^{\mu}K_m(x),$$

where $c(\mu)$ are constans. Hence we see that

$$u(x) = \sum_{\mu} c'(\mu)D^{\mu}K_m(x) + h(x) \qquad (h \in H^m(\mathbf{B})).$$

Our aim in this paper has been to find conditions which assure that the series in the above expression contains only finite terms.

Open problem. Under the weaker condition that

$$\liminf_{r\to 0}\ r^{-s-n+1}\int_{S(0,r)} u(x)^+ dS < \infty$$

instead of (1), we do not know whether $u \in H^m(\mathbf{B}_0)$ is a polynomial or not.

## References

[1] D. H. Armitage, On polyharmonic functions in $R^n - \{0\}$, J. London Math. Soc. (2) **8** (1974), 561–569.
[2] N. Aronszajn, T. M. Creese and L. J. Lipkin, Polyharmonic functions, Clarendon Press, 1983.
[3] S. Axler, P. Bourdon and W. Ramey, Harmonic function theory, Springer-Verlag, 1992.
[4] T. Futamura, K. Kishi and Y. Mizuta, A generalization of the Liouville theorem to polyharmonic functions, to appear in J. Math. Soc. Japan **53** (2001).
[5] Y. Ishikawa, M. Nakai and T. Tada, A form of classical Picard principle, Proc. Japan Acad. Ser. A Math. Sci. **72** (1996), 6–7.
[6] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.
[7] M. Nakai and T. Tada, A form of classical Liouville theorem for polyharmonic functions, Hiroshima Math. J. **30** (2000), 205–213.
[8] M. Nicolesco, Recherches sur les fonctions polyharmoniques, Ann. Sci. École Norm Sup. **52** (1935), 183–220.

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