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Metacyclic groups of automorphisms of compact Riemann surfaces

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Abstract. Let $G_s^0(n) = \langle a, b : a^n = 1 = b^2, b^{-1}ab = a^s \rangle$ and for n even $G_s^1(n) = \langle a, b : a^n = 1, a^{n/2} = b^2, b^{-1}ab = a^s \rangle$. In this paper we compute the minimum genus $g^* \geq 2$ of a compact Riemann surface that admits a metacyclic group $G_s^0(n)$ or $G_s^1(n)$ of biholomorphic homeomorphisms.

1. Introduction

It is known that for any compact Riemann surface of genus ≥ 2 the group of biholomorphic homeomorphisms, which we call automorphisms, is finite [9, p. 66] and that every finite group can be so realized [2 and 3]. Therefore, the following problem arises: Given a finite group G what is the minimum genus $g^* \geq 2$ of a compact Riemann surface that admits G as a group of automorphisms?

We solve this problem for G metacyclic group that belongs to two special classes of extensions of a cyclic group by an involution namely the classes

$$G_s^0(n) = \langle a, b : a^n = 1 = b^2, b^{-1}ab = a^s \rangle \qquad \text{and}$$

$$G_s^1(n) = \langle a, b : a^n = 1, a^{n/2} = b^2, b^{-1}ab = a^s \rangle \qquad \text{with } n \text{ even.}$$

The solution of this problem for these two classes is given in Theorems 3.3 and 4.2 respectively.

The same problem has been solved for cyclic and abelian groups in [4] and [8] respectively. This paper treats the non-abelian case for the first time.

2. The Fuchsian group approach

We shall approach the problem using Fuchsian groups. All the details and the proofs of the following well-known facts can be found in [7] and [6], see also [5].

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A Fuchsian group Γ is a discrete subgroup of the group of linear fractional transformation $LF(2,\mathbf{R})$

$$z \rightarrow \frac{az+b}{cz+d}$$

$a,b,c,d \in \mathbf{R}$ and $ad-bc=1$. Such a transformation is an automorphism of the complex upper half plane D. Since we will be only interested in the case where D/Γ is compact, we shall use the term Fuchsian group to mean a discrete subgroup of $LF(2,\mathbf{R})$ with a compact orbit space so that a Fuchsian group has a presentation

$$\Gamma = \left\langle a_1, b_1, \dots, a_\gamma, b_\gamma, x_1, \dots, x_r : x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{\gamma}[a_i, b_i] \cdot \prod_{i=1}^{r} x_i = 1 \right\rangle.$$

The positive integers, $m_1, \dots, m_r$, are called the periods of the group and γ is called the orbit genus. This group is written as $\Gamma = (\gamma; m_1, \dots, m_r)$. If $r=0$, there are no periods and the group is called a Fuchsian surface group.

Every Fuchsian group $\Gamma = (\gamma; m_1, \dots, m_r)$ has a fundamental region F_Γ in the complex upper half plane with a strictly positive non-Euclidean measure, $\mu(F_\Gamma)$, given by $\mu(F_\Gamma) = 2\pi\left(2(\gamma-1) + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)$.

The following theorem is our starting point for the computation of g^*.

THEOREM 2.1. *A finite group G is a group of biholomorphic automorphisms of a compact Riemann surface of genus $g \geq 2$ if and only if G is isomorphic to a factor group Γ/K where Γ is a Fuchsian group (with a compact orbit space by our convention) and K is a Fuchsian surface group with orbit genus g.*

By the above theorem, we have

$$|G| = \frac{\mu(F_K)}{\mu(F_\Gamma)} \qquad \text{and} \qquad \frac{2(g-1)}{|G|} = 2(\gamma-1) + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)$$

where F_K and F_Γ are the fundamental regions for K and Γ respectively, and $\Gamma = (\gamma; m_1, \dots, m_r)$.

For any finite group G and any natural number $r \geq 0$, we define

$$V_{\gamma; m_1, \dots, m_r}(G) = \left| \left\{ (A_1, B_1, \dots, A_\gamma, B_\gamma, E_1, \dots, E_r) \in G^{2\gamma+1} : |E_i| = m_i,\ 1 \leq i \leq r, \right.\right.$$
$$\left.\left. \prod_{i=1}^{\gamma}[A_i, B_i] \cdot \prod_{i=1}^{r} E_i = 1 \text{ and } \langle A_1, B_1, \dots, A_\gamma, B_\gamma, E_1, \dots, E_r \rangle = G \right\} \right|$$

where $|E_i|$ denotes the order of E_i in G.

Now G is isomorphic to Γ/K where Γ and K are as before if and only if $V_{\gamma;m_1,\dots,m_r}(G) > 0$ and $2(\gamma-1) + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) > 0$.

For any natural number $n \geq 1$, let $d(n) = \{1 \leq k \leq n : k \mid n\}$ and define for all $r \in Z_+$

$$A_r = \left\{(\gamma; m_1, \dots, m_r) \in Z_+ \times (d(|G|))^r : V_{\gamma;m_1,\dots,m_r}(G) > 0 \text{ and } 2(\gamma-1) + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) > 0\right\}.$$

Thus the minimum genus $g^* \geq 2$ of a compact Riemann surface that admits a group of biholomorphic automorphisms isomorphic to G is given by

$$\frac{2(g^*-1)}{|G|} = \min\left(2(\gamma-1) + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right) \tag{2.1}$$

where the minimum is taken over all ordered $r+1$ tuples $(\gamma; m_1, \dots, m_r) \in A_r$ for all $r \geq 0$.

The following proposition says that the number s is an invariant of the isomorphism type of $G_s^0(n)$ and $G_s^1(n)$ respectively.

Proposition 2.2 [1, p. 176].

$$G_s^0(n) \cong G_{s'}^0(n) \Leftrightarrow s = s' \qquad \textit{and} \qquad G_s^1(n) \cong G_{s'}^1(n) \Leftrightarrow s = s'.$$

3. Determination of g^* for the Class $G_s^0(n)$

We consider $G_s^0(n) = \langle a, b : a^n = 1 = b^2, b^{-1}ab = a^s\rangle$ subject to $n \geq 3$, $1 < s < n$, $(s,n) = 1$, $s^2 \equiv 1 \pmod{n}$ and it has order $2n$.

Our goal is to solve equation (2.1) for g^* where $G = G_s^0(n)$ or, in other words, to compute the right-hand side of (2.1).

Let $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$, where $2 \leq p_1 < p_2 < \cdots < p_m$ and $\alpha_i > 0$ for $1 \leq i \leq m$, be the prime factorization of n.

First we consider the case $r = 0$. We observe that $2(\gamma-1) > 0$ forces γ to be ≥ 2 and that $[a,b][b,a] = 1$ and $G_s^0(n) = \langle a, b\rangle$ so that $V_2(G_s^0(n)) > 0$. Therefore

$$\min_{\gamma \in A_0} 2(\gamma-1) = 2 \tag{i}$$

Second we consider the case $\gamma = 0$ for all $r > 0$ with the elements $E_1, \dots, E_r \in G_s^0(n)$ such that $E_1E_2\dots E_r = 1$ and $\langle E_1, E_2, \dots, E_r\rangle = G_s^0(n)$. We compute the strictly positive minimum of the expression

$$-2+\sum_{i=1}^{r}\left(1-\frac{1}{|E_i|}\right) \tag{3.1}$$

with respect to all such choices. Clearly we may assume that $E_i \neq 1$ for $1 \le i \le r$ and that $r \ge 3$. Note also that the number of E_i's that lie outside $\langle a\rangle \subseteq G_s^0(n)$ is always ≥ 2 in each choice of $E_1, \dots, E_r \in G_s^0(n)$ as above.

Case $r = 3$

Here $s \neq n-1$ because $G_{n-1}^0(n) = D_{2n}$ and the expression (3.1) is necessarily negative in this case. In fact, we must have at least two involutions among the E_i's, $1 \le i \le 3$.

Therefore for $r = 3$ we must have the following situation:

$$1 \neq E_1 \in \langle a\rangle; \qquad E_2, E_3 \in G_s^0(n)\backslash\langle a\rangle; \qquad E_1E_2E_3 = 1; \qquad \langle E_1, E_2\rangle = G_s^0(n)$$

and $s \neq n-1$.

Observe that $\langle E_1, E_2^2\rangle = \langle a\rangle = Z_n$ because

$$2n = |\langle E_1, E_2\rangle| = |\langle E_2, \langle E_1, E_2^2\rangle\rangle| = 2|\langle E_1, E_2^2\rangle|;$$

and also $|E_3| = |E_1E_2| = 2|(E_1E_2)^2| = 2|E_1^{s+1}E_2^2|$. Therefore by denoting $x = E_1$ and $y = E_2^2$ the problem now is to compute

$$M = \max\left(\frac{1}{|x|}+\frac{1}{2|y|}+\frac{1}{2|x^{s+1}y|}\right)$$

where the maximum is taken over all $x \in Z_n\backslash\{1\}$ and $y \in Z_{n/(n,s+1)}$ such that the least common multiple $[|x|, |y|] = n$. Taking $d = [|x|, |y|]$ we have

$$\begin{aligned} M &= \max\left\{\frac{1}{|x|}+\frac{1}{2|y|}+\frac{1}{2|x^{s+1}y|} : |y| \;\Big|\; \frac{n}{(n,s+1)}\right\} \\ &= \max\left\{\frac{|y|}{nd}+\frac{1}{2|y|}+\frac{1}{(d,|x^{s+1}y|)}\cdot\frac{\left(\frac{n}{|y|}, s+1\right)d^2}{2|x|\,|y|} : |y| \;\Big|\; \frac{n}{(n,s+1)}\right\} \end{aligned}$$

since

$$\begin{aligned} |x^{s+1}y| &= (d, |x^{s+1}y|)|x^{d(s+1)}y^d| = (d, |x^{s+1}y|)\frac{|x|}{(d(s+1), |x|)}\cdot\frac{|y|}{d} \\ &= (d, |x^{s+1}y|)\frac{|x|\,|y|}{\left((s+1), \frac{n}{|y|}\right)d^2}. \end{aligned}$$

Hence

$$M \leq \max\left\{\frac{t}{nd} + \frac{1}{2t} + d\frac{(n,s+1)}{2n} : d|t|\frac{n}{(n,s+1)}\right\}. \tag{3.2}$$

Define $g(t) = \frac{t}{nd} + \frac{1}{2t}$, $t > 0$. Then $g'(t) = \frac{1}{nd} - \frac{1}{2t^2} = 0$ for $t = \sqrt{\frac{nd}{2}}$ which gives a unique local minimum for g. So, (3.2) becomes

$$M \leq \max\left\{\frac{1}{n} + \frac{1}{2d} + d\frac{(n,s+1)}{2n}, \frac{1}{d(n,s+1)} + (d+1)\frac{(n,s+1)}{2n} : d \,\middle|\, \frac{n}{(n,s+1)}\right\}. \tag{3.3}$$

Define $F(d) = \frac{1}{2d} + d\frac{(n,s+1)}{2n}$, $d > 0$. Then $F'(t) = \frac{-1}{2d^2} + \frac{(n,s+1)}{2n} = 0$ for $d = \sqrt{\frac{n}{(n,s+1)}}$ which gives a unique local minimum for F. Therefore

$$\max\left\{\frac{1}{n} + \frac{1}{2d} + d\frac{(n,s+1)}{2n} : d \,\middle|\, \frac{n}{(n,s+1)}\right\} = \frac{1}{2} + \frac{1}{n} + \frac{(n,s+1)}{2n}.$$

Also define $G(d) = \frac{1}{d(n,s+1)} + d\frac{(n,s+1)}{2n}$, $d > 0$. Then $G'(d) = \frac{-1}{d^2(n,s+1)} + \frac{(n,s+1)}{2n} = 0$ for $d = \frac{\sqrt{2n}}{(n,s+1)}$ which gives a unique local minimum for G. Therefore

$$\begin{aligned}
&\max\left\{\frac{1}{d(n,s+1)} + (d+1)\frac{(n,s+1)}{2n} : d \,\middle|\, \frac{n}{(n,s+1)}\right\} \\
&\quad = \max\left(\frac{1}{(n,s+1)} + \frac{(n,s+1)}{n}, \frac{1}{2} + \frac{1}{n} + \frac{(n,s+1)}{2n}\right) \\
&\quad = \frac{1}{2} + \frac{1}{n} + \frac{(n,s+1)}{2n}
\end{aligned}$$

because

$$\begin{aligned}
&\frac{(n,s+1)}{2n} + \frac{1}{(n,s+1)} - \frac{1}{2} - \frac{1}{n} \\
&\quad = \frac{1}{2n(n,s+1)}\left((n,s+1)^2 - (n+2)(n,s+1) + 2n\right) \\
&\quad = \frac{1}{2n(n,s+1)}((n,s+1) - n)((n,s+1) - 2) \leq 0.
\end{aligned}$$

Hence equation (3.3) becomes $M \leq \frac{1}{2}+\frac{1}{n}+\frac{(n,s+1)}{2n}$.

Observe that for $x=a$ and $y=b^2=1$ this upper bound is attained in equation (3.2). So $M=\frac{1}{2}+\frac{1}{n}+\frac{(n,s+1)}{2n}$ and the solution of (3.1) for $r=3$ is

$$\min\left(-2+\sum_{i=1}^{3}\left(1-\frac{1}{|E_i|}\right)\right)=\frac{1}{2}-\frac{1}{n}-\frac{(n,s+1)}{2n}. \tag{a}$$

Case $r=4$

The only possible choices for $E_1,\dots,E_4 \in G_s^0(n)$ subject to the conditions mentioned earlier are of exactly two kinds:

(1) Two elements are contained in $\langle a\rangle$ and the other two elements are in $G_s^0(n)\backslash\langle a\rangle$.

(i) If no elements outside $\langle a\rangle$ are involutions, then $s\neq n-1$ and in this case the minimum of (3.1) is

$$\geq -2+2\left(1-\frac{1}{p_1}\right)+2\left(1-\frac{1}{2p_1}\right)=2-\frac{3}{p_1}\geq\frac{1}{2}.$$

So, this case can be ignored in view of (a) above.

(ii) If at least one of the elements outside $\langle a\rangle$ is an involution, then the two elements in $\langle a\rangle$ generate it and in this case the minimum of (3.1) is

$$\geq -1-\max\left\{\frac{1}{t}+\frac{1}{m}:[t,m]=n;t,m>1\right\}.$$

To compute this last quantity we need the following two Lemmas (see [4]).

Lemma 3.1.

$$\mathrm{Max}\left\{\frac{1}{r}+\frac{1}{s}:[r,s]=n\right\}=1+\frac{1}{n}.$$

Lemma 3.2. *Let* $n=p_1^{\alpha_1}\dots p_m^{\alpha_m}$, *where* $2\leq p_1<p_2<\dots<p_m$ *and* $\alpha_i>0$ *for* $1\leq i\leq m$, *be the prime factorization of* n. *Then*

$$\max\left\{\frac{1}{r}+\frac{1}{s}:[r,s]=n;r,s>1\right\}=\begin{cases}\frac{1}{p_1}+\frac{p_1}{n} & \text{if } \alpha_1=1 \text{ and } n \text{ not prime}\\ \frac{1}{p_1}+\frac{1}{n} & \text{if } \alpha_1>1 \text{ or } n \text{ prime.}\end{cases}$$

Now if $s=n-1$, then clearly the minimum of the expression (3.1) in this case is

$$\min\left(-2+\sum_{i=1}^{4}\left(1-\frac{1}{|E_i|}\right)\right)=\begin{cases}1-\left(\dfrac{1}{p_1}+\dfrac{p_1}{n}\right) & \text{if } \alpha_1=1 \text{ and } n \text{ not prime}\\ 1-\left(\dfrac{1}{p_1}+\dfrac{1}{n}\right) & \text{if } \alpha_1>1 \text{ or } n \text{ prime;}\end{cases} \tag{b}$$

and if $s \neq n-1$, then the minimum of the expression (3.1) is

$$\geq 1-\left(\frac{1}{p_1}+\frac{p_1}{n}\right)\geq \frac{1}{2}-\frac{1}{n}-\frac{p_1}{2n}\geq \frac{1}{2}-\frac{1}{n}-\frac{(n,s+1)}{2n}.$$

So, this case can be ignored in view of the minimum attained earlier in (a).

(2) The four elements are in $G_s^0(n)\backslash\langle a\rangle$.

Clearly in this case we need only to consider the case $s \neq n-1$ because the expression (3.1) is zero for $s=n-1$.

(i) If the number of involutions in $E_1,\dots,E_4$ is ≤ 2, then the minimum of the expression (3.1) is $\geq -2+2\left(1-\dfrac{1}{2p_1}\right)+1=1-\dfrac{1}{p_1}\geq\dfrac{1}{2}$. So, this case can be ignored in comparison with (a).

(ii) If there are 3 involutions among $E_1,\dots,E_4$, this forces the fourth element to be an involution and the expression (3.1) is zero in this case.

Case $r \geq 5$

In this case if $s \neq n-1$, then the minimum of the expression (3.1) is $\geq -2+5/2=1/2$; and this can be ignored compared to the value obtained in equation (a).

Also if $s=n-1$ and n is even, then the minimum of the expression (3.1) is $\geq -2+5/2=1/2$. So, this case can be ignored compared to the value obtained in equation (b).

If $s=n-1$, n odd and $r\geq 6$, the minimum of the expression (3.1) is $\geq -2+6/2=1$. So, this case can be ignored in view of equation (b) as before.

There remains the case $s=n-1$, n odd and $r=5$. In this case we have either exactly two or exactly four elements of $E_1,\dots,E_5$ outside $\langle a\rangle$. In the first case the minimum of (3.1) is $\geq -2+2\cdot\dfrac{1}{2}+3\left(1-\dfrac{1}{p_1}\right)=2-\dfrac{3}{p_1}\geq 1$ and in the second case the minimum of (3.1) is $\geq -2+4\cdot\dfrac{1}{2}+\left(1-\dfrac{1}{p_1}\right)=1-\dfrac{1}{p_1}$. So, both cases can be ignored compared to the minimum attained in equation (b).

Summarizing the above analysis for the case $\gamma=0$ and $r>0$, we see that the minimum of the expression (3.1) is given by

$$\min\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{|E_i|}\right)\right)=\begin{cases}\dfrac{1}{2}-\dfrac{1}{n}-\dfrac{(n,s+1)}{2n} & \text{if } s\neq n-1\\ 1-\left(\dfrac{1}{p_1}+\dfrac{p_1}{n}\right) & \text{if } s=n-1,\ \alpha_1=1 \text{ and } n \text{ not prime}\\ 1-\left(\dfrac{1}{p_1}+\dfrac{1}{n}\right) & \text{if } s=n-1, \text{ and } \alpha_1>1 \text{ or } n \text{ prime.}\end{cases} \tag{ii}$$

Finally, we consider the case $\gamma \geq 1$ and $r \geq 1$. Here, we have to minimize the expression

$$2(\gamma-1)+\sum_{i=1}^{r}\left(1-\frac{1}{|E_i|}\right) \tag{3.4}$$

for all $\gamma \geq 1$ and all $r \geq 1$ and all choices of elements $a_1, b_1, \dots, a_\gamma, b_\gamma$; $E_1, \dots, E_r \in G_s^0(n)$ such that $\langle a_1, b_1, \dots, a_\gamma, b_\gamma; E_1, \dots, E_r\rangle = G_s^0(n)$ and $\prod_{i=1}^{\gamma}[a_i, b_i] \cdot E_1 \dots E_r = 1$.

By equations (i) and (ii) we may restrict ourselves only to the case $\gamma = 1$ and $r \geq 1$.

If $s \neq n-1$, then the minimum of the expression (3.4) is $\geq 1/2$ which can be ignored compared to the value of equation (ii). Also if $s = n-1$ and n is even, then the minimum of (3.4) is $\geq 1/2$ and this can be also ignored compared to (ii).

If $s = n-1$, n odd and $r \geq 2$ then the minimum of the expression (3.4) is ≥ 1 which can be ignored compared to (ii).

For the case $s = n-1$, n odd and $r = 1$ we have $E_1 \in \langle a\rangle$ and the minimum of (3.4) is $\geq 1 - \dfrac{1}{p_1}$. So, this case also can be ignored compared to the result of (ii).

From the equations (i) and (ii) and the above equation (2.1) now reads

$$\frac{2(g^*-1)}{2n}=\begin{cases}\dfrac{1}{2}-\dfrac{1}{n}-\dfrac{(n,s+1)}{2n} & \text{if } s\neq n-1\\ 1-\left(\dfrac{1}{p_1}+\dfrac{p_1}{n}\right) & \text{if } s=n-1,\ \alpha_1=1 \text{ and } n \text{ not prime}\\ 1-\left(\dfrac{1}{p_1}+\dfrac{1}{n}\right) & \text{if } s=n-1, \text{ and } \alpha_1>1 \text{ or } n \text{ prime.}\end{cases}$$

Therefore we have the following theorem.

THEOREM 3.3. *The minimum genus $g^* \geq 2$ of a compact Riemann surface that admits a group of biholomorphic automorphisms to type $G_s^0(n)$, $n \geq 3$, is given by:*

$$g^* = \begin{cases} \dfrac{n-(n,s+1)}{2} & \textit{if } s \neq n-1 \\ n+1-\dfrac{n}{p_1}-p_1 & \textit{if } s = n-1, \ \alpha_1 = 1 \textit{ and } n \textit{ not prime} \\ n-\dfrac{n}{p_1} & \textit{if } s = n-1, \textit{ and } \alpha_1 > 1 \textit{ or } n \textit{ prime} \end{cases}$$

where p_1 *is the minimum prime divisor of* n *and* $n = p_1^{\alpha_1} m$ *with* $(p_1, m) = 1$.

Corollary 3.4. *Hypotheses are the same as Theorem* 3.3.

i) *For* $G = D_{2n}$, $n \geq 3$, *we have*

$$g^* = \begin{cases} n+1-\dfrac{n}{p_1}-p_1 & \textit{if } \alpha_1 = 1 \textit{ and } n \textit{ not prime} \\ n-\dfrac{n}{p_1} & \textit{if } \alpha_1 > 1 \textit{ or } n \textit{ prime.} \end{cases}$$

ii) *For* $G = SD_{2^{n+1}}$, $n \geq 3$, *we have* $g^* = 2^{n-2}$.

Proof. i) Put $s = n-1$ in Theorem 3.3.

ii) Observe that the semi-dihedral group $SD_{2^{n+1}}$, $n \geq 3$, is defined by

$$SD_{2^{n+1}} = \langle a, b : a^{2^n} = 1 = b^2, b^{-1}ab = a^{2^{n-1}-1} \rangle.$$

So the result can be obtained by replacing n by 2^n and putting $s = 2^{n-1} - 1$ in Theorem 3.3.

4. Determination of g^* for the Class $G_s^1(n)$

We consider $G_s^1(n) = \langle a, b : a^n = 1, a^{n/2} = b^2, b^{-1}ab = a^s \rangle$ subject to $n \geq 4$, $2 \mid n$, $1 < s < n$, $(s,n) = 1$, $s^2 \equiv 1 \pmod{n}$ and it has order $2n$.

Our goal is to solve equation (2.1) for g^* where $G = G_s^1(n)$ or to compute the right-hand side of (2.1).

Let $n = 2^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, where $2 < p_2 < \cdots < p_m$ and $\alpha_i > 0$ for $1 \leq i \leq m$, be the prime factorization of n.

By virtue of the following proposition we may consider only the case $2 \nmid \dfrac{n}{(n,s+1)}$.

Proposition 4.1 [1, p. 176]. *Suppose* n *is even, then* $G_s^0(n) \cong G_{s'}^1(n) \Leftrightarrow s = s'$ *and* $2 \mid \dfrac{n}{(n,s+1)}$.

First we consider the case $r = 0$. Then clearly $2(\gamma - 1) > 0$ and $G_s^1(n) = \langle a, b \rangle$. Therefore

$$\min_{\gamma \in A_0} 2(\gamma - 1) = 2. \tag{i}$$

Second we consider the case $\gamma = 0$ for all $r > 0$ with the elements $E_1, \dots, E_r \in G_s^1(n)$ such that $E_1E_2\dots E_r = 1$ and $\langle E_1, \dots, E_r\rangle = G_s^1(n)$ and we compute the strictly positive minimum of the expression

$$-2 + \sum_{i=1}^{r}\left(1 - \frac{1}{|E_i|}\right) \tag{4.1}$$

with respect to all such choices. Clearly we may assume that $E_i \neq 1$ for $1 \leq i \leq r$ and that $r \geq 3$. Note also that the number of the E_i's that lie outside $\langle a\rangle \subseteq G_s^1(n)$ is always ≥ 2 in each choice of $E_1, \dots, E_r \in G_s^1(n)$ as above.

Case $r = 3$

Here we must have the following situation:

$$1 \neq E_1 \in \langle a\rangle; \quad E_2, E_3 \in G_s^1(n)\backslash\langle a\rangle; \quad E_1E_2E_3 = 1 \quad \text{and} \quad \langle E_1, E_2\rangle = G_s^1(n).$$

Observe that $\langle E_1, E_2^2\rangle = \langle a\rangle$ and that

$$\begin{aligned}|E_2| &= 2|E_2^2|\\ &= 4|E_2^2 \cdot a^{n/2}| \qquad \text{since } 2 \nmid \frac{n}{(n, s+1)}\\ &= 4|y| \qquad \text{where } y \in Z_{n/(n,s+1)} \text{ and also}\end{aligned}$$

$$|E_3| = |E_1E_2| = 2|(E_1E_2)^2| = 2|E_1^{s+1} \cdot E_2^2| = 4|E_1^{s+1} \cdot y|.$$

Now we distinguish two cases:

Case 1: $\alpha_1 \geq 2$

In this case we only have $\langle E_1, y\rangle = \langle a\rangle$. Therefore the problem now is to compute $N_n = \max\left(\frac{1}{|x|} + \frac{1}{4|y|} + \frac{1}{4|x^{s+1}y|}\right)$ where the maximum is taken over all $x \in Z_n\backslash\{1\}$ and $y \in Z_{n/(n,s+1)}$ such that $[|x|, |y|] = n$. Taking $d = (|x|, |y|)$ we have

$$\begin{aligned}N_n &= \max\left\{\frac{1}{|x|} + \frac{1}{4|y|} + \frac{1}{4|x^{s+1}y|} : |y| \;\Big|\; \frac{n}{(n, s+1)}\right\}\\ &\leq \max\left\{\frac{t}{nd} + \frac{1}{4t} + d\frac{(n, s+1)}{4n} : d|t|\frac{n}{(n, s+1)}\right\}.\end{aligned} \tag{4.2}$$

Analyzing equation (4.2) in exactly the same way as was done before for equation (3.2) we get

$$\mathrm{Max}\left\{\frac{t}{nd} + \frac{1}{4t} + d\frac{(n, s+1)}{4n} : d|t|\frac{n}{(n, s+1)}\right\} = \frac{1}{4} + \frac{1}{n} + \frac{(n, s+1)}{4n}.$$

Note that $(n, s+1) \geq 4$ (since $(s, n) = 1$ gives s odd and $2 \mid (n, s+1)$. If $(n, s+1) = 2$, then $\frac{n}{2}$ is odd since $2 \nmid \frac{n}{(n, s+1)}$ and $s^2 \equiv 1 \pmod{n}$ gives $\frac{n}{2} \mid s - 1$ and since $1 < s < n$ we must have $s = \frac{n}{2} + 1$ or s is even which is absurd).

Observe that for $x = a$ and $y = b^4 = 1$, this upper bound is attained in (4.2). Therefore $N_n = \frac{1}{4} + \frac{1}{n} + \frac{(n, s+1)}{4n}$ and the solution of (4.1) for $r = 3$ and $\alpha_1 \geq 2$ is

$$\min\left(-2 + \sum_{i=1}^{3}\left(1 - \frac{1}{|E_i|}\right)\right) = \frac{3}{4} - \frac{1}{n} - \frac{(n, s+1)}{4n}. \tag{c1}$$

Case 2: $\alpha_1 = 1$

In this case we only have $\langle E_1, y, a^{n/2} \rangle = \langle a \rangle$ and therefore the problem is to compute $k = \max\left(\frac{1}{|x|} + \frac{1}{4|y|} + \frac{1}{4|x^{s+1}y|}\right)$ where the maximum is taken over all $x \in Z_n \backslash \{1\}$ and $y \in Z_{n/(n,s+1)}$ such that $\langle x, y, a^{n/2} \rangle = \langle a \rangle$.

$$\begin{aligned}
k &= \max\left(\max_{|x| \text{ even}}\left(\frac{1}{|x|} + \frac{1}{4|y|} + \frac{1}{4|x^{s+1}y|}\right), \max_{|x| \text{ odd}}\left(\frac{1}{|x|} + \frac{1}{4|y|} + \frac{1}{4|x^{s+1}y|}\right)\right) \\
&= \max\left(\max\left\{\frac{1}{|x|} + \frac{1}{4|y|} + \frac{1}{4|x^{s+1}y|} : x \in Z_n \backslash \{1\}, y \in Z_{n/(n,s+1)}, [|x|, |y|] = n\right\},\right. \\
&\qquad \left.\max\left\{\frac{1}{|x|} + \frac{1}{4|y|} + \frac{1}{4|x^{s+1}y|} : x \in Z_{n/2} \backslash \{1\}, y \in Z_{(n/2)/(n/2,s+1)}, [|x|, |y|] = n/2\right\}\right) \\
&= \max(N_n, N_{n/2}) \\
&= \max\left(\frac{1}{4} + \frac{1}{n} + \frac{(n, s+1)}{4n}, \frac{1}{4} + \frac{2}{n} + \frac{(n, s+1)}{4n}\right) = \frac{1}{4} + \frac{2}{n} + \frac{(n, s+1)}{4n}.
\end{aligned}$$

Observe that for $x = a^2$ and $y = b^4 = 1$ the maximum is attained. Therefore the solution of (4.1) for $r = 3$ and $\alpha_1 = 1$ is

$$\min\left(-2 + \sum_{i=1}^{3}\left(1 - \frac{1}{|E_i|}\right)\right) = \frac{3}{4} - \frac{2}{n} - \frac{(n, s+1)}{4n}. \tag{c2}$$

Case $r = 4$

Here the only possible choices for $E_1, \dots, E_r \in G_s^1(n)$ subject to the conditions mentioned earlier are of exactly two kinds:

(1) Two elements are in $\langle a \rangle$ and the other two elements are in $G_s^1(n) \backslash \langle a \rangle$.

(i) There is no element of order 4 in the elements outside $\langle a \rangle$. In this case the minimum of (4.1) is $\geq -2+2(1-\frac{1}{8})+2(1-\frac{1}{2})=\frac{3}{4}$ since every element outside $\langle a\rangle$ has order divisible by 4 as shown in the previous case $r=3$. So, this case can be ignored in view of (c1) and (c2) above.

(ii) At least one of the elements outside $\langle a\rangle$ is of order 4.

If $\alpha_1 \geq 2$, then $\langle E_1, E_2\rangle = \langle a\rangle$ and in this case the maximum of (4.1) is $\geq -2+2\left(1-\frac{1}{4}\right)+2-\max\left\{\frac{1}{t}+\frac{1}{m}:[t,m]=n; t,m>1\right\} \geq \frac{3}{2}-\left(\frac{1}{2}+\frac{2}{n}\right) = 1-\frac{2}{n} \geq \frac{3}{4}-\frac{1}{n}$ which can be ignored in comparison with (c1).

If $\alpha_1 = 1$, then the case $\langle E_1, E_2\rangle = \langle a\rangle$ can be excluded as above and we may consider only the case $\langle E_1, E_2\rangle = \langle a^2\rangle$. In this case the minimum of (4.1) is $\geq -2+2\left(1-\frac{1}{4}\right)+2-\max\left\{\frac{1}{t}+\frac{1}{m}:[t,m]=\frac{n}{2}; t,m>1\right\}$

$$\begin{aligned} &\geq \frac{3}{2}-\left(\frac{1}{p_2}+\frac{2p_2}{n}\right) \text{ and } p_3\,|\,n && \text{or} \quad \frac{3}{2}-\left(\frac{1}{p_2}+\frac{2}{n}\right) \text{ by Lemma 3.2} \\ &\geq 1-\frac{1}{p_3} && \text{or} \quad 1-\frac{2}{n} \\ &\geq \frac{3}{4} && \text{or} \quad 1-\frac{2}{n}. \end{aligned}$$

So this minimum can also be ignored compared to the value of equation (c2).

(2) The four elements are in $G_s^1(n)\backslash\langle a\rangle$.

In this case the minimum of (4.1) is $\geq -2+4(1-\frac{1}{4})=1$. So this can be ignored compared to (c1) and (c2).

Case $r \geq 5$

In this case the minimum of (4.1) is $\geq -2+2(1-\frac{1}{4})+\frac{3}{2}=1$ and this case can be ignored as well.

Summarizing the above analysis for the case $\gamma=0$ and $r>0$, we see that the minimum of the expression (4.1) is given by

$$\min\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{|E_i|}\right)\right) = \begin{cases} \frac{3}{4}-\frac{1}{n}-\frac{(n,s+1)}{4n} & \text{if } \alpha_1 \geq 2 \\ \frac{3}{4}-\frac{2}{n}-\frac{(n,s+1)}{4n} & \text{if } \alpha_1 = 1. \end{cases} \tag{ii}$$

Finally, we consider the case $\gamma \geq 1$ and $r \geq 1$. Here we consider minimizing the expression

$$2(\gamma-1)+\sum_{i=1}^{r}\left(1-\frac{1}{|E_i|}\right) \tag{4.3}$$

for all $\gamma \geq 1$ and all $r \geq 1$ and all choices of elements $a_1, b_1, \ldots, a_\gamma, b_\gamma$; $E_1, \ldots, E_r \in G_s^1(n)$ such that $\langle a_1, b_1, \ldots, a_\gamma, b_\gamma; E_1, \ldots, E_r \rangle = G_s^1(n)$ and $\prod_{i=1}^{\gamma} [a_i, b_i] \cdot E_1 \ldots E_r = 1$.

By equations (i) and (ii) we may restrict ourselves only to the case $\gamma = 1$ and $r = 1$. Therefore, we consider the following situation:

$$[x_1, x_2] \cdot E_1 = 1 \qquad \text{where } x_1, x_2, E_1 \in G_s^1(n) \backslash \{1\} \text{ and } x_1 \notin \langle a \rangle.$$

Case 1: $x_1 \notin \langle a \rangle$ and $x_2 \in \langle a \rangle$

In this case we have $|x_1| = 4|y|$ where $|y| \,\Big|\, \dfrac{n}{(n, s+1)}$ and $\langle y \,.\, a^{n/2}, x_2 \rangle = \langle a \rangle$. Also note that $[x_1, x_2]^{-1} = x_2^{s-1}$ so that our problem is to compute

$$L = \min\left\{ |x_2^{s-1}| : y, x_2 \in \langle a \rangle, |y| \,\Bigg|\, \frac{n}{(n, s+1)}, \langle y \,.\, a^{n/2}, x_2 \rangle = \langle a \rangle, x_2 \neq 1 \right\}$$

$$= \min\left\{ \frac{|x_2|}{(s-1, |x_2|)} : y, x_2 \in \langle a \rangle, |y| \,\Bigg|\, \frac{n}{(n, s+1)}, [2|y|, |x_2|] = n, x_2 \neq 1 \right\}$$

$$= \min\left\{ \frac{n}{\left(\dfrac{n(s-1)}{|x_2|}, n\right)} : 1 \neq |x_2| = \frac{n}{2|y|}(2|y|, |x_2|), |y| \,\Bigg|\, \frac{n}{(n, s+1)} \right\}.$$

If $\alpha_1 \geq 2$, we have $2 \,|\, |x_2|$ and $\dfrac{n}{|x_2|} \,|\, |y|$, then

$$L \geq \min\left\{ \frac{n}{(|y|(s-1), n)} : |y| \,\Bigg|\, \frac{n}{(n, s+1)} \right\} = \frac{(n, s+1)}{(n, s+1, s-1)}.$$

If $\alpha_1 = 1$ and $|x_2|$ even, then $L \geq \dfrac{(n, s+1)}{(n, s+1, s-1)}$ as before.

If $\alpha_1 = 1$ and $|x_2|$ is odd, we have $\dfrac{n}{|x_2|} \,|\, 2|y|$, and hence

$$L \geq \min\left\{ \frac{n}{(2|y|(s-1), n)} : |y| \,\Bigg|\, \frac{n}{(n, s+1)} \right\}$$

$$= \min\left\{ \frac{n}{(|y|(s-1), n)} : |y| \,\Bigg|\, \frac{n}{(n, s+1)} \right\} \qquad \text{since } s-1 \text{ is even}$$

$$= \frac{(n, s+1)}{(n, s+1, s-1)}.$$

This lower bound is attained for $x_1 = ab$ and $x_2 = a^{n/(n,s+1)}$ and we get

$$L = \frac{(n,s+1)}{(n,s+1,s-1)}.$$

Case 2: $x_1, x_2 \notin \langle a \rangle$

In this case putting $x_1 = a^i b$ and $x_2 = a^j b$ we have $[x_1, x_2]^{-1} = a^{(i-j)(s-1)}$ so that our problem is to compute

$$\begin{aligned} P &= \min\{|a^{(i-j)(s-1)}| : \langle a^i b, a^j b\rangle = G_s^1(n)\} \\ &= \min\{|a^{t(s-1)}| : \langle a^{j+t} b, a^t\rangle = G_s^1(n)\} = \min|x^{s-1}| \end{aligned}$$

where the minimum is taken over all $1 \neq x \in \langle a \rangle$ such that $\langle y \,.\, a^{n/2}, x\rangle = \langle a \rangle$ and $|y| \mid \frac{n}{(n,s+1)}$. So $P = L = \frac{(n,s+1)}{(n,s+1,s-1)}$.

Cases (1) and (2) show that the minimum of the expression (4.3) is given by

$$\min\left(2(\gamma - 1) + \sum_{i=1}^{r}\left(1 - \frac{1}{|E_i|}\right)\right) = 1 - \frac{(n,s+1,s-1)}{(n,s+1)}. \tag{iii}$$

From (i), (ii), and (iii) we get

$$\frac{2(g^*-1)}{2n} = \begin{cases} \frac{1}{2} - \frac{1}{n} - \frac{(n,s+1)}{2n} & \text{if } 2 \mid \frac{n}{(n,s+1)} \\ \min\left\{\frac{3}{4} - \frac{2}{n} - \frac{(n,s+1)}{4n}, 1 - \frac{(n,s+1,s-1)}{(n,s+1)}\right\} & \text{if } 2 \nmid \frac{n}{(n,s+1)} \text{ and } \alpha_1 = 1 \\ \min\left\{\frac{3}{4} - \frac{1}{n} - \frac{(n,s+1)}{4n}, 1 - \frac{(n,s+1,s-1)}{(n,s+1)}\right\} & \text{if } 2 \nmid \frac{n}{(n,s+1)} \text{ and } \alpha_1 \geq 2. \end{cases}$$

Therefore we have the following theorem.

THEOREM 4.2. *The minimum genus $g^* \geq 2$ of a compact Riemann surface that admits a group of biholomorphic automorphisms of type $G_s^1(n)$ is given by:*

$$g^* = \begin{cases} \frac{n-(n,s+1)}{2} & \textit{if } 2 \mid \frac{n}{(n,s+1)} \\ \min\left\{\frac{3n-(n,s+1)}{4} - 1, n+1-n\frac{(n,s+1,s-1)}{(n,s+1)}\right\} & \textit{if } 2 \nmid \frac{n}{(n,s+1)} \textit{ and } \alpha_1 = 1 \\ \min\left\{\frac{3n-(n,s+1)}{4}, n+1-n\frac{(n,s+1,s-1)}{(n,s+1)}\right\} & \textit{if } 2 \nmid \frac{n}{(n,s+1)} \textit{ and } \alpha_1 \geq 2 \end{cases}$$

where $n = 2^{\alpha_1} m$ with m odd.

Corollary 4.3. *For* $G = DC_{2n}$, $n \geq 4$, *we have*

$$g^* = \begin{cases} \dfrac{n}{2} - 1 & \textit{if} \ \ \alpha_1 = 1 \\ \dfrac{n}{2} & \textit{if} \ \ \alpha_1 \geq 2 \end{cases}$$

In particular for $G = Q_{2^{n+1}}$, $n \geq 2$, we have $g^* = 2^{n-1}$.

5. List of g^* for the groups of order $n \leq 15$

The following table is a list of all groups G of order n, $2 \leq n \leq 15$, and the corresponding minimum genus g^* of a compact Riemann surface that admits G as a group of biholomorphic automorphisms:

n	G	g^*
2	Z_2	2 (Harvey)
3	Z_3	2 (Harvey)
4	Z_4	2 (Harvey)
	$Z_2 \times Z_2$	2 (Maclachlan)
5	Z_5	2 (Harvey)
6	Z_6	2 (Harvey)
	S_3	2
7	Z_7	3 (Harvey)
8	Z_8	2 (Harvey)
	$Z_2 \times Z_4$	3 (Maclachlan)
	$Z_2 \times Z_2 \times Z_2$	3 (Maclachlan)
	D_8	2
	Q_8	2
9	Z_9	3 (Harvey)
	$Z_3 \times Z_3$	4 (Maclachlan)
10	Z_{10}	2 (Harvey)
	D_{10}	4
11	Z_{11}	5 (Harvey)
12	Z_{12}	3 (Harvey)
	$Z_2 \times Z_6$	2 (Maclachlan)
	A_4	3
	D_{12}	2
	DC_{12}	2
13	Z_{13}	6 (Harvey)
14	Z_{14}	3 (Harvey)
	D_{14}	6
15	Z_{15}	4 (Harvey)

PROOF. All the values of g^* of this table follow directly from Harvey's and Maclachlan's results together with Corollaries 3.4 and 4.3 except $G = A_4$.

Suppose that A_4 acts as a group of biholomorphic automorphisms on a compact Riemann surface X of genus 2, then one can immediately see that X must be a ramified Galois covering of the Riemann sphere ramified over $\{a_1, \dots, a_r\}$, $r \geq 3$, and that at least two of these points have ramification index $= 3$ so that equation (2.1) now gives

$$\frac{1}{6} = \min\left(-2 + \left(2 - \frac{2}{3}\right) + \left(1 - \frac{1}{m_3}\right) + \cdots + \left(1 - \frac{1}{m_r}\right)\right)$$

which is impossible for $r \geq 4$ and also for $r = 3$ because A_4 has no element of order 6.

Now observe that there exists a ramified Galois covering of genus 3 of the Riemann sphere with Galois group isomorphic to A_4 where the ramification occurs over exactly four points with ramification indicies 3, 3, 2, 2 respectively. This shows that for $G = A_4$, $g^* = 3$.

References

[1] B. G. Basmaji, On the isomorphisms of two metacyclic groups, Proc. Amer. Math. Soc. **22** (1969), 175–182.

[2] W. Burnside, Theory of Groups of Finite Order, New York, 1955.

[3] L. Greenberg, Maximal Groups and Signatures, Ann. of Math. Studies No. **79** (1974), 207–226.

[4] W. J. Harvey, Cyclic groups of automorphisms of a compact Riemann surface, Quart. J. Math. (2) **17** (1966), 86–97.

[5] W. J. Harvey, Discrete Groups and Automorphic Functions, Academic Press, 1977.

[6] J. Lehner, Discontinuous Groups and Automorphic Functions, Amer. Math. Soc. Surveys No. VIII, New York, 1964.

[7] A. M. Macbeath, Fuchsian Groups, cyclostyled notes (Dundee 1961).

[8] C. Maclachlan, Abelian groups of automorphisms of compact Riemann surfaces, Proc. London Math. Soc. (3) **15** (1965), 699–712.

[9] P. Samuel, Lectures on Old and New Results on Algebraic Curves, Tata Institute of Fundamental Research, Bombay, 1966.

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