Hiroshima Math. J.
31 (2001), 201–212

A note on the multilinear oscillatory singular integral operators

Zefu Chu, Guoen Hu and Zhibo Lu

(Received December 17, 1999)
(Revised November 13, 2000)

Abstract. In this paper, we consider the $L^p(\mathbf{R}^n)$ boundedness for a class of multilinear oscillatory singular integral operators with polynomial phases. We show that if the polynomial phases are non-trivial and the homogeneous kernels satisfy a certain minimum size condition, then the $L^p(\mathbf{R}^n)$ boundedness for the multilinear oscillatory singular integral operators can be deduced from the $L^p(\mathbf{R}^n)$ boundedness for the corresponding local multilinear singular integral operators.

1. Introduction

We will work on $\mathbf{R}^n (n \geq 2)$. Let $P(x, y)$ be a real-valued polynomial on $\mathbf{R}^n \times \mathbf{R}^n$, $\Omega(x)$ be homogeneous of degree zero which has a mean value zero on the unit sphere S^{n-1}. Define the oscillatory singular integral operator

$$Tf(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy. \tag{1}$$

It is well-known that the operators of this type have arisen in the study of Hilbert transforms along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms, etc. A celebrated result of Ricci and Stein [9] says that if $\Omega \in \mathrm{Lip}_1(S^{n-1})$, then T is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$, with a bound depending only on n, p and $\deg P$ (the total degree of P), not on the coefficients of the polynomial. Chanillo and Christ [2] showed that $\Omega \in \mathrm{Lip}_1(S^{n-1})$ is also sufficient for T to be a bounded mapping from L^1 to weak L^1, and the bound depends only on n and $\deg P$. Lu and Zhang [7] improved the result of Ricci and Stein, and proved that if $\Omega \in \bigcup_{q>1} L^q(S^{n-1})$, then T is bounded on $L^p(\mathbf{R}^n)$ with a bound $C(n, p, \deg P)$ for $1 < p < \infty$.

In this paper, we will study the multilinear operators defined by

$$T_{A_1,\dots,A_k} f(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{j=1}^{k} R_{m_j+1}(A_j, x, y) f(y)dy, \tag{2}$$

2000 *Mathematics Subject Classification*. 42B20

Key words and phrases. multilinear operator, oscillatory singular integral, BMO.

The research was supported by the NSF of China (19701039) and the NSF of Henan Province.

where k and m_j $(j=1,\dots,k)$ are positive integers, $m=\sum_{j=1}^{k} m_j$, A_j $(j=1,\dots,k)$ has derivatives of order m_j in $\mathrm{BMO}(\mathbf{R}^n)$, $R_{m_j+1}(A_j;x,y)$ denotes the (m_j+1)-th Taylor series remainder of A_j at x about y, that is,

$$R_{m+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Operators of this type have been studied in [3], [4], [6] and many other works. It is easy to see that the operator $T_{A_1,\dots,A_k}$ is closely related to the oscillatory singular integral operator defined by (1) and the multilinear singular integral operator defined by

$$\tilde{T}_{A_1,\dots,A_k} f(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{j=1}^{k} R_{m_j+1}(A_j;,x,y) f(y)dy. \tag{3}$$

Using good-λ-inequality techniques, Cohen and Gosselin [5] showed that if Ω satisfies a certain vanishing moment and $\Omega \in \mathrm{Lip}_1(S^{n-1})$, then for $1<p<\infty$,

$$\|\tilde{T}_{A_1,A_2} f\|_p \le \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{\mathrm{BMO}(\mathbf{R}^n)} \right) \|f\|_p.$$

In [3], Chen, Hu and Lu considered the $L^p(\mathbf{R}^n)$ boundedness for the operator T_{A_1,A_2} and proved that if $\Omega \in \bigcup_{q>1} L^q(S^{n-1})$, and the polynomial $P(x,y)$ is non-trivial, then the $L^p(\mathbf{R}^n)$ boundedness for T_{A_1,A_2} can be obtained from the $L^p(\mathbf{R}^n)$ boundedness for the local multilinear singular integral operator

$$S_{A_1,A_2} f(x) = \int_{|x-y|\le 1} \frac{\Omega(x-y)}{|x-y|^{n+m_1+m_2}} \prod_{j=1}^{2} R_{m_j+1}(A_j;,x,y) f(y)dy,$$

(see [2, Theorem 2]). The purpose of this paper is to show that if $\Omega \in L(\log L)^{k+1}(S^{n-1})$, and P is non-trivial, then the $L^p(\mathbf{R}^n)$ boundedness for $T_{A_1,\dots,A_k}$ can be obtained from the $L^p(\mathbf{R}^n)$ boundedness for the local version of the operator $\tilde{T}_{A_1,\dots,A_k}$. Our main result in this paper can be stated as follows.

Theorem 1. *Let $1<p<\infty$, k and m_j $(j=1,2,\dots,k)$ be positive integers, $m=\sum_{j=1}^{k} m_j$, A_j $(j=1,2,\dots,k)$ be functions on $\mathbf{R}^n$ whose derivatives of order m_j are in* $\mathrm{BMO}(\mathbf{R}^n)$. *Suppose that Ω is homogeneous of degree zero and belongs to the space $L(\log L)^{k+1}(S^{n-1})$, that is,*

$$\int_{S^{n-1}} |\Omega(x')| \log^{k+1}(2+|\Omega(x')|)dx' < \infty,$$

and the operator

$$S_{A_1,\dots,A_k}f(x) = \int_{|x-y|\le 1} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{j=1}^{k} R_{m_j+1}(A_j; x, y) f(y)dy \tag{4}$$

is bounded on $L^p(\mathbf{R}^n)$. *Then for any real-valued non-trivial polynomial* $P(x, y)$, *the operator* T_A *defined by* (2) *is also bounded on* $L^p(\mathbf{R}^n)$, *with a bound depending on* n, p, m_j $(j = 1, \dots, k)$, $\prod_{j=1}^{k}(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)})$ *and* $\deg P$, *not on the coefficients of* P.

2. Proof of Theorem 1

We begin with some preliminary lemmas.

Lemma 1 (see [5]). *Let* $b(x)$ *be a function on* $\mathbf{R}^n$ *with derivatives of order* m *in* $L^q(\mathbf{R}^n)$ *for some* $n < q \le \infty$. *Then*

$$|R_m(b; x, y)| \le C_{m,n}|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{I}(x, y)|}\int_{\tilde{I}(x,y)} |D^\alpha b(z)|^q dz\right)^{1/q},$$

where $\tilde{I}(x, y)$ *is the cube centered at* x *with diameter* $5\sqrt{n}|x-y|$.

Lemma 2. *Let* $1 < p < \infty$, k *and* m_j $(j = 1, 2, \dots, k)$ *be positive integers,* $m=\sum_{j=1}^{k} m_j$, A_j $(j=1, 2, \dots, k)$ *be functions on* $\mathbf{R}^n$ *whose derivatives of order* m_j *are in* $\mathrm{BMO}(\mathbf{R}^n)$. *Suppose that* $\tilde{\Omega}$ *is homogeneous of degree zero and belongs to the space* $L^\infty(S^{n-1})$. *Set*

$$\lambda_{\tilde{\Omega},k} = \inf\left\{\lambda > 0 : \frac{\|\tilde{\Omega}\|_1}{\lambda} \log^k\left(2 + \frac{\|\tilde{\Omega}\|_\infty}{\lambda}\right) \le 1\right\}.$$

Then for any $r > 0$, *the operator*

$$U_{A_1,\dots,A_k;r}f(x) = r^{-n-m} \int_{r/2<|x-y|\le r} |\tilde{\Omega}(x-y)| \prod_{j=1}^{k} |R_{m_j+1}(A_j; x, y)||f(y)|dy \tag{5}$$

is bounded on $L^p(\mathbf{R}^n)$ *with a bound* $C(n, m, p)\lambda_{\tilde{\Omega},k} \prod_{j=1}^{k}(\sum_{|\alpha_j|=m} \|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)})$.

Proof. Note that for each $t > 0$,

$$\begin{aligned}\lambda_{t\tilde{\Omega},k} &= \inf\left\{\lambda > 0 : \frac{\|t\tilde{\Omega}\|_1}{\lambda} \log^k\left(2 + \frac{\|t\tilde{\Omega}\|_\infty}{\lambda}\right) \le 1\right\}\\ &= \inf\left\{t\tilde{\lambda} : \tilde{\lambda} > 0, \frac{\|t\tilde{\Omega}\|_1}{t\tilde{\lambda}} \log^k\left(2 + \frac{\|t\tilde{\Omega}\|_\infty}{t\tilde{\lambda}}\right) \le 1\right\}\\ &= t\lambda_{\tilde{\Omega},k}.\end{aligned}$$

Thus we may assume that $\lambda_{\tilde{\Omega},k} = 1/2$. Therefore,

$$\|\tilde{\Omega}\|_1 \log^k(2 + \|\tilde{\Omega}\|_\infty) \leq 1.$$

Define the operator E by

$$Eh(x) = \int_{|x-y|\leq 1} |\tilde{\Omega}(x-y)| h(y) dy.$$

Denote by E^* the adjoint operator of E, that is,

$$E^*h(x) = \int_{|x-y|\leq 1} |\tilde{\Omega}(y-x)| h(y) dy.$$

Let $b_1, b_2, \ldots, b_k \in \mathrm{BMO}(\mathbf{R}^n)$ and Q be a cube with side length 1. Denote by $m_Q(b_j)$ the mean value of b_j on Q. We claim that for $1 < p < \infty$, $\operatorname{supp} h \subset 10nQ$ and non-negative integer $l \leq k$,

$$\text{(6)} \qquad \int_Q |E^*h(x)|^p \prod_{j=1}^{l} |b_j(x) - m_Q(b_j)|^p dx$$
$$\leq C \log^{(-k+l)p}(2 + \|\tilde{\Omega}\|_\infty) \prod_{j=1}^{k} \|b_j\|^p_{\mathrm{BMO}(\mathbf{R}^n)} \|h\|_p^p,$$

with the interpretation that when $l = 0$, $\prod_{j=1}^{l} |b_j(x) - m_Q(b_j)| \equiv 1$. To prove (6), we can assume that $\|h\|_p = 1$. Choose $1 < r_j < \infty$ such that $\sum_{j=1}^{k} 1/r_j = 1$. By the well-known John-Nirenberg inequality, there is a positive constant $C_j = C(p, r_j, n)$ such that

$$\left(\int_Q |b_j(x) - m_Q(b_j)|^{2pr_j} dx \right)^{1/(2r_j)} \leq C_j \|b_j\|^p_{\mathrm{BMO}(\mathbf{R}^n)}.$$

We may also assume that $\|b_j\|^p_{\mathrm{BMO}(\mathbf{R}^n)} = 1/C_j$ for all $1 \leq j \leq k$. We shall carry out our argument by induction on l. If $l = 0$, the Young inequality gives that

$$\int_Q |E^*h(y)|^p dy \leq C \|\tilde{\Omega}\|_1^p \|h\|_p^p \leq C \log^{-kp}(2 + \|\tilde{\Omega}\|_\infty).$$

Now let $d \leq k - 1$ be a non-negative integer and assume that the estimate (6) holds for $l = d$. We will show that (6) holds for $l = d + 1$. Observe that $\Phi(t) = t \log^p(2 + t)$ is a Young function and its complementary Young function is $\Psi(t) \approx \exp t^{1/p}$. By the general Hölder inequality, it follows that

$$\int_Q |E^*h(x)|^p \prod_{j=1}^{d+1} |b_j(x) - m_Q(b_j)|^p dx$$

$$\leq C \inf\left\{\lambda > 0 : \int_Q \frac{|E^*h(x)|^p}{\lambda} \log^p\left(2+\frac{|E^*h(x)|^p}{\lambda}\right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq 1\right\}$$

$$\times \inf\left\{\lambda > 0 : \int_Q \exp\left(\frac{|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}}\right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq 2\right\},$$

(see [1] or [8]). Applying the Young inequality again, we have

$$\|E^*h\|_\infty \leq \|\tilde{\Omega}\|_\infty \|h\|_1 \leq C\|\tilde{\Omega}\|_\infty \|h\|_p \leq C\|\tilde{\Omega}\|_\infty.$$

Our induction assumption now gives that

$$\int_Q |E^*h(x)|^p \log^p\left(2+\frac{|E^*h(x)|^p}{\lambda}\right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx$$

$$\leq C \log^p\left(2+\frac{C\|\tilde{\Omega}\|_\infty^p}{\lambda}\right) \log^{(-k+d)p}(2+\|\tilde{\Omega}\|_\infty).$$

Set $\lambda_0 = \log^{(-k+d+1)p}(2+\|\tilde{\Omega}\|_\infty)$. An easy computation then leads to that

$$\int_Q |E^*h(x)|^p \log^p\left(2+\frac{|E^*h(x)|^p}{\lambda_0}\right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq C\lambda_0.$$

On the other hand, by the Hölder inequality,

$$\int_Q \exp\left(\frac{|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}}\right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx$$

$$\leq \left(\int_Q \exp\left(\frac{2|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}}\right) dx\right)^{1/2} \prod_{j=1}^{d} \left(\int_Q |b_j(x) - m_Q(b_j)|^{2pr_j} dx\right)^{1/(2r_j)}$$

$$\leq \left(\int_Q \exp\left(\frac{2|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}}\right) dx\right)^{1/2},$$

which together with the John-Nirenberg inequality implies that

$$\inf\left\{\lambda > 0 : \int_Q \exp\left(\frac{|b_{l+1}(x) - m_Q(b_{l+1})|}{\lambda^{1/p}}\right) \prod_{j=1}^{d} |b_j(x) - m_Q(b_j)|^p dx \leq 2\right\} \leq C,$$

Therefore,

$$\int_Q |E^*h(x)|^p \prod_{j=1}^{d+1} |b_j(x) - m_Q(b_j)|^p dx \le C \log^{(-k+d+1)p}(2 + \|\tilde{\Omega}\|_\infty).$$

We can now prove our Lemma 2. By dilation-invariance, it suffices to consider the case $r = 1$. Write $\mathbf{R}^n = \bigcup_j I_j$, where each I_j is a cube having side length 1 and the cubes have disjoint interiors. Let χ_j be the characteristic function of I_j. Set $f_j = f\chi_j$. Then

$$f(x) = \sum_j f_j(x), \qquad \text{a.e. } x \in \mathbf{R}^n.$$

Since the support of $U_{A_1,\dots,A_k;1} f_j$ is contained in a fixed multiple of I_j, the supports of various terms $\{U_{A_1,\dots,A_k;1} f_j\}$ have bounded overlaps, and so we have

$$\|U_{A_1,\dots,A_k;1} f\|_p^p \le C \sum_j \|U_{A_1,\dots,A_k;1} f_j\|_p^p.$$

Thus we may assume that supp $f \subset I$ for some cube I with side length 1. Set

$$\widetilde{A_j}(y) = A_j(y) - \sum_{|\alpha_j|=m_j} \frac{1}{\alpha_j!} m_I(D^{\alpha_j} A_j) y^{\alpha}.$$

A straightforward computation shows that for $x, y \in \mathbf{R}^n$,

$$R_{m_j+1}(A_j; x, y) = R_{m_j+1}(\widetilde{A_j}; x, y).$$

Choose $n < q < \infty$ Lemma 1 now tells us that

$$\begin{aligned}
&|R_{m_j}(\widetilde{A_j}; x, y)| \\
&\quad \le C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \left(\frac{1}{|\tilde{I}(x,y)|} \int_{\tilde{I}(x,y)} |D^{\alpha_j} A_j(z) - m_I(D^{\alpha_j} A_j)|^q dz \right)^{1/q} \\
&\quad \le C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \left(\frac{1}{|\tilde{I}(x,y)|} \int_{\tilde{I}(x,y)} |D^{\alpha_j} A_j(z) - m_{\tilde{I}(x,y)}(D^{\alpha_j} A_j)|^q dz \right)^{1/q} \\
&\qquad + C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} |m_I(D^{\alpha_j} A_j) - m_{\tilde{I}(x,y)}(D^{\alpha_j} A_j)| \\
&\quad \le C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} (\|D^{\alpha_j} A_j\|_{\mathrm{BMO}(\mathbf{R}^n)} + |m_I(D^{\alpha_j} A_j) - m_{\tilde{I}(x,y)}(D^{\alpha_j} A_j)|).
\end{aligned}$$

Note that if $y \in I$ and $|x-y| \le 1$, then $\tilde{I}(x,y) \subset 100nI$. This in turn implies that for $y \in I$ and $1/2 \le |x-y| \le 1$,

$$|m_I(D^{\alpha_j}A_j) - m_{\tilde{I}(x,y)}(D^{\alpha_j}A_j)| \le C\|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)}.$$

Thus in this case, we have

$$|R_{m_j}(\widetilde{A_j};x,y)| \le C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)} \le C\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)}.$$

Let

$$\phi(y) = \prod_{j=1}^{k}\left(\sum_{|\alpha_j|=m_j}(\|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)} + |D^{\alpha_j}A_j(y) - m_I(D^{\alpha_j}A_j)|)\right).$$

We can write

$$U_{A_1,\dots,A_k;1}f(x) \le E(|\phi f|)(x).$$

A standard duality arguement and the Hölder inequality then show that

$$\begin{aligned}\|U_{A_1,\dots,A_k;1}f\|_p &\le \sup_{\operatorname{supp} h\subset 10nI,\,\|h\|_{p'}\le 1}\left|\int E(|\phi f|)(x)h(x)dx\right| \\ &= \sup_{\operatorname{supp} h\subset 10nI,\,\|h\|_{p'}\le 1}\int |E^*h(y)\phi(y)f(y)|dy \\ &\le C\|f\|_p \sup_{\operatorname{supp} h\subset 10nI,\,\|h\|_{p'}\le 1}\|\phi E^*h\|_{p'},\end{aligned}$$

where p' is the dual exponent of p, i.e. $p' = p/(p-1)$. Invoking the estimate (6) for $0 \le l \le k$, we finally obtain

$$\|U_{A_1,\dots,A_k;1}f\|_p \le C\prod_{j=1}^{k}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)}\right)\|f\|_p.$$

This completes the proof of Lemma 2.

Proof of Theorem 1. Without loss of generality, we may assume that for $1 \le j \le k$,

$$\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}(\mathbf{R}^n)} = 1.$$

Let k_0 be a positive integer and $P(x,y)$ be a real-valued non-trivial polynomial having degree k_0 in x and degree l_0 in y. Write

$$P(x,y) = \sum_{|\mu|=k_0,\,|\nu|=l_0} a_{\mu,\nu}x^\mu y^\nu + R(x,y),$$

where $R(x, y)$ is a real-valued polynomial which has degree less k_0 in x. By dilation-invariance, we may assume that $\sum_{|\mu|=k_0, |\nu|=l_0} |a_{\mu\nu}| = 1$. Splite $T_{A_1,\dots,A_k}$ as

$$\begin{aligned} T_{A_1,\dots,A_k} f(x) &= \int_{|x-y|\le 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy \\ &\quad + \sum_{j=1}^{\infty} \int_{2^{j-1}<|x-y|\le 2^j} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy \\ &= T^0_{A_1,\dots,A_k} f(x) + \sum_{j=1}^{\infty} T^j_{A_1,\dots,A_k} f(x). \end{aligned}$$

We first consider the operator $T^j_{A_1, A_2,\dots,A_k}$ for $j \ge 1$. Let $E_0 = \{x' \in S^{n-1}, |\Omega(x')| \le 2\}$ and $E_l = \{x' \in S^{n-1}, 2^l < |\Omega(x')| \le 2^{l+1}\}$ for positive integer l. Let Ω_l be the restriction of Ω on E_l. Define the operator $T^j_{A_1,\dots,A_k;l}$ by

$$T^j_{A_1,\dots,A_k;l} f(x) = \int_{2^{j-1}<|x-y|\le 2^j} e^{iP(x,y)} \frac{\Omega_l(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy.$$

To estimate the $L^p(\mathbf{R}^n)$ boundedness for $T^j_{A_1,\dots,A_k;l}$, we will use the following lemma.

Lemma 3. *Let the polynomial $P(x, y)$, k, m_u and A_u $(u = 1, \dots, k)$ be the same as above, $\tilde{\Omega}$ be homogeneous of degree zero and belong to the space $L^\infty(S^{n-1})$. Define the operator*

$$V_j f(x) = \int_{1<|x-y|\le 2} e^{iP(2^j x, 2^j y)} \frac{\tilde{\Omega}(x-y)}{|x-y|^{n+m}} \prod_{u=1}^{k} R_{m_u+1}(A_u; x, y) f(y) dy.$$

Then for $1 < p < \infty$, there exists positive constants C and δ which are depending only on n, p and $\deg P$ *such that*

$$\|V_j f\|_p \le C \|\tilde{\Omega}\|_\infty 2^{-\delta j} \|f\|_p.$$

For the case of $k = 1$, this lemma was proved essentially in [3, page 43–46]. For general positive integer k, Lemma 3 can be proved by induction on k. We omit the details.

We now estimate $T^j_{A_1,\dots,A_k;l}$. Note that for $b \in \mathrm{BMO}(\mathbf{R}^n)$ and $t > 0$, $b_t(x) = b(tx)$ also belongs to the space $\mathrm{BMO}(\mathbf{R}^n)$ and $\|b_t\|_{\mathrm{BMO}(\mathbf{R}^n)} = \|b\|_{\mathrm{BMO}(\mathbf{R}^n)}$. Thus by dilation-invariance and Lemma 3,

$$\|T^j_{A_1,\dots,A_k;l} f\|_p \le C 2^{-\delta j} 2^l \|f\|_p. \tag{7}$$

On the other hand, Lemma 2 states that

$$\|T^j_{A_1,\dots,A_k;l}f\|_p \le C\lambda_{\Omega_l,k}\|f\|_p. \tag{8}$$

Set $\lambda_l^k = l^k\|\Omega_l\|_1 + 2^{-l}$. A trivial computation gives that

$$\frac{\|\Omega_l\|_1}{\lambda_l^k}\log^k\left(2+\frac{\|\Omega_l\|_\infty}{\lambda_l^k}\right) \le \frac{\|\Omega_l\|_1}{l^k\|\Omega_l\|_1}\log^k\left(2+\frac{\|\Omega_l\|_\infty}{2^{-l}}\right) \le C,$$

which in turn implies

$$\lambda_{\Omega_l,k} \le C(l^k\|\Omega_l\|_1 + 2^{-l}). \tag{9}$$

Our hypothesis on Ω now says that $\sum_{l>0} l^{k+1}\|\Omega_l\|_1 < \infty$. Let N be a positive integer such that $N > 2\delta^{-1}$. Combining the inequalities (7) and (8) yields that

$$\begin{aligned}\left\|\sum_{j\ge1}\sum_{l\ge0}T^j_{A_1,\dots,A_k;l}f\right\|_p &\le \sum_{j\ge1}\|T^j_{A_1,\dots,A_k;0}f\|_p + \sum_{l>0}\sum_{j>Nl}\|T^j_{A_1,\dots,A_k;l}f\|_p\\ &\quad+\sum_{l>0}\sum_{1\le j\le Nl}\|T^j_{A_1,\dots,A_k;l}f\|_p\\ &\le C\sum_{j\ge1}2^{-\delta j}\|f\|_p + C\sum_{l>0}2^l\sum_{j\ge Nl}2^{-\delta j}\|f\|_p\\ &\quad+C\sum_{l>0}l\lambda_{\Omega_l,k}\|f\|_p \le C\|f\|_p.\end{aligned}$$

Now we turn our attention to the operator $T^0_{A_1,\dots,A_k}$. The estimate for this term follows from the following lemma directly.

Lemma 4. *Let $1 < p < \infty$, and $S_{A_1,\dots,A_k}$ be defined by* (4) *with $\Omega \in L(\log L)^k(S^{n-1})$. Suppose that $S_{A_1,\dots,A_k}$ is bounded on $L^p(\mathbf{R}^n)$. Then for any real-valued polynomial $\tilde{P}(x,y)$, the operator*

$$W_{A_1,\dots,A_k}f(x) = \int_{|x-y|\le1} e^{i\tilde{P}(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}\prod_{u=1}^k R_{m_u+1}(A_u;x,y)f(y)dy,$$

is bounded on $L^p(\mathbf{R}^n)$ with a bound $C(n,m,p,\deg\tilde{P})$.

Proof. We follow along the same line as in the proof of Lemma 6 in [3]. We shall carry out the arguement by a double induction on the degree in x and y of the polynomial. Obviously, Lemma 4 holds if the polynomial $\tilde{P}(x,y)$ depends only on x or only on y. Let u and v be two positive integers and the polynomial $\tilde{P}(x, y)$ have degree u in x and v in y. We assume that Lemma 4 is known for all polynomials which are sums of monomials of degree less than u in x times monomials of any degree in y, together with monomials which are of degree u in x times monomials which are of degree less than v in y.

We can now write

$$\tilde{P}(x, y) = \sum_{|\mu|=u,\,|v|=v} b_{\mu v} x^{\mu} y^{v} + P_0(x, y),$$

where $P_0(x, y)$ satisfies the inductive assumption. We consider the following two cases.

Case I. $\sum_{|\mu|=u,\,|v|=v} |b_{\mu v}| \le 1$. As in the proof of Lemma 2, we may assume that supp $f \subset I$ for some cube I centered at x_0 and having side length 1. By translation-invariance (note that our result is independent of the coefficients of the polynomial), we may assume that supp $f \subset I_0$, the cube centered at the origin and having side length 1. Set

$$\bar{P}(x, y) = P_0(x, y) + \sum_{|\mu|=u,\,|v|=v} b_{\mu v} y^{\mu+v}.$$

Observe that if $|x - y| \le 1$ and $y \in I_0$, then

$$|e^{i\tilde{P}(x,y)} - e^{i\bar{P}(x,y)}| \le C|x - y|.$$

Thus,

$$\begin{aligned} |W_{A_1,\dots,A_k} f(x)| &\le \left| \int_{|x-y|\le 1} e^{i\bar{P}(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{j=1}^{k} R_{m_j+1}(A_j; x, y) f(y)dy \right| \\ &\quad + C \int_{|x-y|\le 1} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} \prod_{j=1}^{k} |R_{m_j+1}(A_j; x, y)||f(y)|dy \\ &\le \left| \int_{|x-y|\le 1} e^{i\bar{P}(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} \prod_{j=1}^{k} R_{m_j+1}(A_j; x, y) f(y)dy \right| \\ &\quad + C \sum_{j=0}^{\infty} 2^{-j} U_{A_1,\dots,A_k;2^{-j}} f(x), \end{aligned}$$

where $U_{A_1,\dots,A_k;2^{-j}}$ is defined by (5). Set

$$U^{l}_{A_1,\dots,A_k;2^{-j}} f(x) = 2^{-j(n+m)} \int_{2^{-j-1}<|x-y|\le 2^{-j}} |\Omega_l(x-y)| \prod_{u=1}^{k} |R_{m_u+1}(A_u; x, y)||f(y)|dy.$$

It follows from Lemma 2 and the inequality (9) that

$$\sum_{j=0}^{\infty} 2^{-j}\|U_{A_1,\dots,A_k;2^{-j}}f\|_p \le C\sum_{j=0}^{\infty} 2^{-j}\sum_{l\ge 0}\|U^l_{A_1,\dots,A_k;2^{-j}}f\|_p$$

$$\le C\|f\|_p + \sum_{j=0}^{\infty} 2^{-j}\sum_{l\ge 1}\lambda_{\Omega_l,k}\|f\|_p \le C\|f\|_p.$$

This via the induction hypothesis tells us that

$$\|W_{A_1,\dots,A_k}f\|_p \le C(n,m,p,\deg\tilde{P})\|f\|_p.$$

Case II. $\sum_{|\mu|=u,|v|=v}|b_{\mu v}| > 1$. Set $J = (\sum_{|\mu|=u,|v|=v}|b_{\mu v}|)^{1/(u+v)}$. Let

$$Q(x,y) = \sum_{|\mu|=u,|v|=v}\frac{b_{\mu v}}{J^{u+v}}x^\mu y^v + P_0(x/J, y/J).$$

Then $\tilde{P}(x,y) = Q(Jx, Jy)$. Define the operator

$$\tilde{W}_{A_1,\dots,A_k}f(x) = \int_{|x-y|\le J} e^{iQ(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}\prod_{u=1}^{k}R_{m_u+1}(A_u;x,y)f(y)dy.$$

By dilation-invariance, it suffices to prove that

$$\|\tilde{W}_{A_1,\dots,A_k}f\|_p \le C(n,m,p,\deg\tilde{P})\|f\|_p. \tag{10}$$

We splite the operator $\tilde{W}_{A_1,\dots,A_k}$ as

$$\begin{aligned}\tilde{W}_{A_1,\dots,A_k}f(x) &= \int_{|x-y|\le 1} e^{iQ(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}\prod_{u=1}^{k}R_{m_u+1}(A_u;x,y)f(y)dy\\ &\quad+\sum_{j=1}^{j_0}\int_{2^{j-1}<|x-y|\le 2^j} e^{iQ(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}\prod_{u=1}^{k}R_{m_u+1}(A_u;x,y)f(y)dy\\ &\quad+\int_{2^{j_0}<|x-y|\le J} e^{iQ(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}\prod_{u=1}^{k}R_{m_u+1}(A_u;x,y)f(y)dy\\ &= \tilde{W}^{\mathrm{I}}f(x) + \tilde{W}^{\mathrm{II}}f(x) + \tilde{W}^{\mathrm{III}}f(x),\end{aligned}$$

where j_0 is the positive integer such that $2^{j_0} < J \le 2^{j_0+1}$. The conclusion of Case I applies to $\tilde{W}^{\mathrm{I}}$, so

$$\|\tilde{W}^{\mathrm{I}}f\| \le C(n,m,p,\deg\tilde{P})\|f\|_p.$$

By the inequalities (7), (8) and (9) as in the estimate for $\sum_{j \geq 1} T^{j}_{A_1, A_2, \ldots, A_k}$, we can obtain that

$$\|\tilde{W}^{\mathrm{II}} f\|_p \leq C(n, m, p, \deg \tilde{P}) \|f\|_p.$$

On the other hand, it follows from Lemma 2 and the estimate (9) that

$$\|\tilde{W}^{\mathrm{III}} f\|_p \leq C(n, m, p, \deg \tilde{P}) \|f\|_p.$$

This leads to the estimate (10), and completes the proof of Lemma 4.

Acknowledgement

The authors would like to thank the referee for some valuable suggestions and corrections.

References

[1] R. A. Adams, Sobolev Space, Academic Press, New York, 1975.
[2] S. Chanillo and M. Christ, Weak $(1, 1)$ bounds for oscillatory singular integrals, Duke Math. J. **55** (1987), 141–155.
[3] W. Chen, G. Hu and S. Lu, Criterion of (L^p, L^r) boundedness for a class of multilinear oscillatory singular integrals, Nagoya Math. J. **149** (1998), 33–51.
[4] W. Chen, G. Hu and S. Lu, On a multilinear oscillatory singular integral (II), Chin Ann. of Math. **18:A** (1997), 73–82.
[5] J. Cohen and J. Gosselin, A BMO estimate for multilinear oscillatory singular integral, Illinois J. of Math. **30** (1986), 445–464.
[6] G. Hu, S. Lu and D. Yang, On a class of multilinear oscillatory singular integral operators, J. Math. Soc. Japan **48** (1998), 623–637.
[7] S. Lu and Y. Zhang, Criterion of L^p boundedness for a class of oscillatory singular integrals with rough kernels, Rev. Mat. Iberoamericana **8** (1992), 201–219.
[8] C. Pérez, Endpoint estimates for commutators of singular integral operators, J. Funct. Anal. **128** (1995), 163–185.
[9] F. Ricci and E. M. Stein, Harmonic analysis on nilpotent groups and singular integrals (I), oscillatory integrals, J. of Funct. Anal. **73** (1987), 179–194.

Zefu Chu, Guoen Hu and Zhibo Lu
Department of Applied Mathematics
University of Information Engineering
P.O. Box 1001-747, Zhengzhou 450002
People's Republic of China
Z. Chu: chuzf@163.net
G. Hu: huguoen@371.net
Z. Lu: luzhb21@sina.com