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A Liouville theorem for polyharmonic functions

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ABSTRACT. We give a short, elementary proof of a theorem which shows that if u is a polyharmonic function on $\mathbf{R}^d$ and the growth of u^+ is suitably restricted, then u must be a polynomial.

1. Introduction

A typical point of $\mathbf{R}^d$, where $d \geq 2$, is denoted by $x = (x_1, \dots, x_d)$. We write Δ for the Laplace operator $\sum_{j=1}^{d} \partial^2/\partial x_j^2$ and define $\Delta^1 = \Delta$ and $\Delta^{p+1} = \Delta\Delta^p$ when $p \in \mathbf{N}$. A function $u : \mathbf{R}^d \to \mathbf{R}$ is called *polyharmonic* of order p if $u \in C^{2p}(\mathbf{R}^d)$ and $\Delta^p u \equiv 0$ on $\mathbf{R}^d$. We denote the vector space of all such functions by $\mathscr{H}^p$. Thus, in particular, $\mathscr{H}^1$ is the space of all harmonic functions on $\mathbf{R}^d$. The positive part of a function $u : \mathbf{R}^d \to \mathbf{R}$ is denoted by u^+; that is, $u^+(x) = \max\{u(x), 0\}$ for each x in $\mathbf{R}^d$.

A classical Liouville theorem for harmonic functions may be stated as follows: if $u \in \mathscr{H}^1$ and u^+ is bounded on $\mathbf{R}^d$, then u is constant. A generalization, due to Kuran [4, Theorem 2], shows that if $u \in \mathscr{H}^p$ and u^+ is bounded on $\mathbf{R}^d$, then u is a polynomial of degree at most $2p - 2$. Several authors have given further generalizations. To describe their results, we introduce some more notation. The open ball and the sphere of radius r centred at the origin 0 of $\mathbf{R}^d$ are denoted by $B(r)$ and $S(r)$. We denote d-dimensional Lebesgue measure by λ and $(d-1)$-dimensional surface measure by σ. Some known results are summarized in the following theorem.

THEOREM A. *Let $u \in \mathscr{H}^p$, where $p \in \mathbf{N}$, and let s be a number such that $s > 2p - 2$. The following statements are equivalent:*

(1) *u is a polynomial of degree less than s;*

(2) $\lim_{r \to +\infty} r^{-s-d+1} \int_{S(r)} u^+ \, d\sigma = 0$;

(3) $\liminf_{r \to +\infty} r^{-s-d} \int_{B(r)} u^+ \, d\lambda = 0$;

(4) $\liminf_{r \to +\infty} (r^{-s} \max\{u^+(x) : x \in S(r)\}) = 0$.

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Each of the conditions (2), (3), (4) is clearly necessary for (1). The sufficiency of (2), (3), (4) was established by Armitage [1, Theorem 1], Mizuta [5, Theorem 4], and Nakai and Tada [6, Theorem 1] respectively. (I have reformulated the statements in [1], [5] and [6] in order to facilitate comparisons.) We note that for a continuous (not necessarily polyharmonic function) u on $\mathbf{R}^d$, condition (2) implies (3), but there are no other implications between (2), (3) and (4). In this note we prove the following result.

THEOREM 1. *Let $u \in \mathscr{H}^p$, where $p \in \mathbf{N}$. If $s > 2p-2$ and*

(5) $\liminf_{r\to+\infty} r^{-s-d+1} \int_{S(r)} u^+ \, d\sigma = 0,$

then u is a polynomial of degree less than s.

For a continuous function $u : \mathbf{R}^d \to \mathbf{R}$ each of the conditions (2), (3), (4) implies (5). Thus Theorem 1 includes Theorem A. However, the main interest of Theorem 1 perhaps lies in the simplicity of its proof, which uses only the Almansi representation for polyharmonic functions (see, e.g., [3, p. 4]) and a few elementary facts about harmonic functions, for which we refer to [2].

2. Proof of Theorem 1

We note first that u has an Almansi representation

$$u(x) = \sum_{m=0}^{p-1} \|x\|^{2m} h_m(x) \qquad (x \in \mathbf{R}^d),$$

where $\|x\| = (x_1^2 + \cdots + x_d^2)^{1/2}$ and $h_m \in \mathscr{H}^1$ for each m. Hence, by the mean value property of harmonic functions ([2, Theorem 1.2.2]),

(6) $\int_{S(r)} u \, d\sigma = \sigma(S(r)) \sum_{m=0}^{p-1} r^{2m} h_m(0) = o(r^{s+d-1}) \qquad (r \to +\infty).$

Since $|u| = 2u^+ - u$, it follows from (5) and (6) that

(7) $\liminf_{r\to+\infty} r^{-s-d+1} \int_{S(r)} |u| d\sigma = 0.$

We write $\mathscr{H}\mathscr{P}_n$, where $n \in \{0\} \cup \mathbf{N}$, for the vector space of all homogeneous harmonic polynomials of degree n on $\mathbf{R}^d$. For future reference, we recall the orthogonality property

(8) $\int_{S(r)} HK \, d\sigma = 0 \ \ (H \in \mathscr{H}\mathscr{P}_\nu; K \in \mathscr{H}\mathscr{P}_n; \nu \neq n; r > 0);$

see [2, Lemma 2.2.1]. Each of the harmonic functions h_m is given on $\mathbf{R}^d$ by a series $\sum_{n=0}^{\infty} H_{m,n}$, where $H_{m,n} \in \mathscr{H}\mathscr{P}_n$, and the convergence of the series is uniform on every sphere $S(r)$; see [2; Theorem 2.2.4].

The proof of Theorem 1 will be complete if we show that $H_{\mu,\nu} \equiv 0$ for all values of μ and ν such that $2\mu + \nu \geq s$. We fix such integers μ and ν and, arguing towards a contradiction, we suppose that $H_{\mu,\nu} \not\equiv 0$. We define

$$F(r) = \int_{S(r)} H_{\mu,\nu} u \, d\sigma \qquad (r > 0)$$

and note first that by (7)

(9) $\liminf_{r \to +\infty} r^{-s-\nu-d+1} |F(r)| = 0.$

We also have

$$F(r) = \sum_{m=0}^{p-1} r^{2m} \int_{S(r)} H_{\mu,\nu} h_m \, d\sigma$$

$$= \sum_{m=0}^{p-1} r^{2m} \sum_{n=0}^{\infty} \int_{S(r)} H_{\mu,\nu} H_{m,n} \, d\sigma,$$

the change of order of integration and summation being justified by uniform convergence. Hence, by the orthogonality property (8),

$$F(r) = \sum_{m=0}^{p-1} r^{2m} \int_{S(r)} H_{\mu,\nu} H_{m,\nu} \, d\sigma$$

$$= \sum_{m=0}^{p-1} a_m r^{2m+2\nu+d-1},$$

where each a_m is a real number. Our assumption that $H_{\mu,\nu} \not\equiv 0$ implies that $a_\mu \neq 0$. Hence F is a polynomial of degree at least $2\mu + 2\nu + d - 1$. Since $2\mu + 2\nu + d - 1 \geq s + \nu + d - 1$, this contradicts (9), so the proof is complete.

3. Generalization

Nakai and Tada [6, Section 4] indicated a generalization of their result (the implication (4) $\Rightarrow$ (1) in Theorem A) to a wider class of functions, namely the class of functions $u : \mathbf{R}^d \to \mathbf{R}$ that are expressible in the form

(10) $u(x) = \sum_{m=0}^{q} \|x\|^m h_m(x),$

where $h_m \in \mathscr{H}^1$ for each m. They showed that if a function u has the form (10), then the harmonic functions h_m are uniquely determined by u, and if further u satisfies (4) for some $s > q$, then each h_m is a polynomial of degree less than $s - m$ (see [6, Theorem 3]). The proof of Theorem 1 can be easily adapted to show that if u is a function of the form (10) satisfying (5) for some $s > m$, then again each h_m is a polynomial of degree less than $s - m$.

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