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An analogue of Hardy's theorem for the Harish-Chandra transform

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Abstract. A theorem of Hardy asserts that a function and its Fourier transform cannot both be very small. We prove analogues of Hardy's theorem for the Harish-Chandra transform for spherical functions on a non-compact semisimple Lie group and the Helgason transform on a Riemannian symmetric space of the non-compact type.

Introduction

Hardy's theorem for the Fourier transform [10] asserts that f and its Fourier transform $\hat{f}$ cannot both be very small. More precisely, if f is a measurable function on the real line such that $f(x) = O(e^{-(1/2)x^2})$ and $\hat{f}(x) = O(e^{-(1/2)x^2})$ as $|x| \to \infty$, then $f(x)$ is a constant multiple of $e^{-(1/2)x^2}$.

It follows easily from Hardy's result that if α and β are positive numbers, $\alpha\beta > 1/4$, $f(x) = O(e^{-\alpha x^2})$, and $\hat{f}(x) = O(e^{-\beta x^2})$ as $|x| \to \infty$, then $f = 0$ almost everywhere. Sitaram and Sundari [16] generalize the result for semisimple Lie groups under some restrictions on groups or functions. Subsequently, similar results for general cases were proved independently by Cowling, Sitaram, and Sundari [5], Ebata, Eguchi, Koizumi, and Kumahara [7], and Sengupta [14].

In this paper, we give an analogue of Hardy's original result for functions on a Riemannian symmetric space of the noncompact type. It is crucial in Hardy's theorem that the Fourier transform of the heat kernel

$$\frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$$

on the real line is e^{-tx^2}. Similar result is no longer true for the heat kernel on a Riemannian symmetric space of the noncompact type. Our idea is to use the heat kernel and its transform for estimating functions. Using known estimates for the heat kernel, some connections between results of us and those of Sitaram and Sundari will be discussed.

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The result for $SL(2,\mathbf{R})/SO(2)$ was given in our previous paper [15].

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1. The Harish-Chandra transform

In this section, we review on the elementary spherical function and the Harish-Chandra transform on a Riemannian symmetric space of the noncompact type. We refer the reader to Helgason [11] for details.

Let G be a noncompact connected semisimple Lie group with finite center and K be a maximal compact subgroup. Let $G = NAK$ be corresponding Iwasawa decomposition and $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ be corresponding decomposition of its Lie algebra. For $g \in G$, let $A(g) \in \mathfrak{a}$ denote the unique element such that $g \in Ne^{A(g)}K$. Let Σ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and Σ^+ denote the set of positive roots. Then $\mathfrak{n}$ is the direct sum of the root spaces for all positive roots. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} m_\alpha \alpha$, where m_α denotes the multiplicity of α. Let $\mathfrak{a}^*$ denote the dual of $\mathfrak{a}$ and $\mathfrak{a}_{\mathbf{C}}^*$ its complexification. Let W denote the Weyl group for Σ.

A function f on G is said to be spherical if $f(kgk') = f(g)$ for all k, $k' \in K$ and $g \in G$. As usual, we identify functions on G/K with right K-invariant functions on G and those on $K\backslash G/K$ with bi-K-invariant functions on G.

For $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, the function defined by

$$\phi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))}\, dk, \qquad g \in G \tag{1.1}$$

is called the elementary spherical function. Here dk denotes the Haar measure on K with total measure 1. ϕ_λ is a spherical function on G and satisfies

$$|\phi_\lambda(a)| \le \phi_{i\,\mathrm{Im}\,\lambda}(a) \le e^{\max_{w \in W}(-w\,\mathrm{Im}\,\lambda(\log a))}\phi_0(a), \qquad a \in A. \tag{1.2}$$

Let $\mathscr{C}(K\backslash G/K)$ denote the space of spherical functions f on G such that

$$\sup_{g \in G} |(1+|g|)^q \phi_0(g)^{-1}(Df)(g)| < \infty \tag{1.3}$$

for each integer $q \ge 0$ and each invariant differential operator D on G. Here $|g| = |\log a|$ if $g \in KaK$.

For $f \in \mathscr{C}(K\backslash G/K)$, we define the Harish-Chandra transform $\tilde{f}(\lambda)$ by

$$\tilde{f}(\lambda) = \int_G f(g)\phi_{-\lambda}(g)dg, \qquad \lambda \in \mathfrak{a}_{\mathbf{C}}^*. \tag{1.4}$$

Here dg denotes the (suitably normalized) Haar measure on G.

By the restriction mapping $G \to A$, $\mathscr{C}(K\backslash G/K)$ is isomorphic to the space $\mathscr{S}_W(A)$ of W-invariant rapidly decreasing functions on A.

The following theorem is due to Harish-Chandra.

Theorem 1.1. *For* $f \in \mathscr{C}(K\backslash G/K)$,

$$f(g) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \tilde{f}(\lambda)\phi_\lambda(g)|\boldsymbol{c}(\lambda)|^2 d\lambda, \tag{1.5}$$

where $\mathbf{c}(\lambda)$ *is the Harish-Chandra* $\mathbf{c}$*-function.*

The Harish-Chandra transform extends to an isometry of $L^2(K\backslash G/K)$ *onto* $L^2(\mathfrak{a}^*/W, |\boldsymbol{c}(\lambda)|^{-2}d\lambda)$.

Explicit formula for $\boldsymbol{c}(\lambda)$ is given by Gindikin and Karpelevič. For details, we refer to Helgason [11, Ch. IV], Gangolli and Varadarajan [9, Chapter 6], and references therein.

2. The heat kernel

Our main tool we shall use is the following h_t, which is an analogue of the heat kernel on the real line.

For $t > 0$, define the function $h_t(g)$ on G by

$$h_t(g) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \exp(-t(|\lambda|^2 + |\rho|^2))\phi_\lambda(g)|\boldsymbol{c}(\lambda)|^{-2} d\lambda. \tag{2.1}$$

We state some properties of h_t, which is due to Gangolli [8, Proposition 3.1].

Proposition 2.1. *The function* h_t *has the following properties:*

$$h_t \in \mathscr{C}(K\backslash G/K), \tag{2.2}$$

$$\tilde{h}_t(\lambda) = \exp(-t(|\lambda|^2 + |\rho|^2)), \tag{2.3}$$

$$Lh_t = \frac{\partial h_t}{\partial t}, \tag{2.4}$$

$$h_t * h_s = h_{s+t} \qquad t, s > 0. \tag{2.5}$$

Here L *denotes the Laplace-Beltrami operator on* G/K *and* $*$ *denotes the convolution product on* G/K.

Moreover, there is an estimate of the heat kernel obtained by Anker [1, 2]. For any $t_0 > 0$ there exists $C > 0$ such that

$$h_t(\exp H) \le Ct^{-n/2}(1 + |H|^2)^{(n-r)/2} \exp\left(-|\rho|^2 t - \langle \rho, H\rangle - \frac{|H|^2}{4t}\right) \tag{2.6}$$

for all $0 < t < t_0$ and $H \in \overline{\mathfrak{a}^+}$, where $n = \dim G/K$ and $r = \dim \mathfrak{a}$, and $\overline{\mathfrak{a}^+}$ denotes the closure of the positive Weyl chamber in $\mathfrak{a}$.

3. An analogue of Hardy's theorem

We now state and prove an analogue of Hardy's theorem for the Harish-Chandra transform.

THEOREM 3.1. *Let t be a fixed positive constant. If f is a K-invariant measurable function on G/K satisfying*

$$|f(a)| \leq Ch_t(a) \qquad \text{for all } a \in A \tag{3.1}$$

and

$$|\tilde{f}(\lambda)| \leq C \exp(-t|\lambda|^2) \qquad \text{for all } \lambda \in \mathfrak{a}^*, \tag{3.2}$$

where C is a positive constant, then f is a constant multiple of h_t.

PROOF. The proof goes along the line of the first proof of Hardy [10] in the Euclidean case that is based on the Phragmén-Lindelöf theorem.

By (1.2) and (2.6), $\tilde{f}$ is holomorphic on $\mathfrak{a}^*_{\mathbf{C}}$, if f satisfies (3.1). By (3.1) and (2.3), we have

$$\begin{aligned} |\tilde{f}(\lambda)| &\leq C \int_G h_t(g)\phi_{-i\,\mathrm{Im}\,\lambda}(g)dg \\ &= C \exp((|\mathrm{Im}\,\lambda|^2 - |\rho^2|)t) \\ &= C' \exp(|\mathrm{Im}\,\lambda|^2 t) \end{aligned} \tag{3.3}$$

for all $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, where C and C' are some constants.

Since $\tilde{f}$ satisfies estimates (3.2) and (3.3), it follows from [16, Lemma 2.1] that

$$\tilde{f}(\lambda) = C \exp(-t|\lambda|^2), \qquad \lambda \in \mathfrak{a} \tag{3.4}$$

for some constant C, hence f is a constant multiple of h_t by (2.3). □

In the Euclidean case, Hardy proved more general result: Let m be a non-negative integer. If f and $\hat{f}$ are both $O(x^m e^{-(1/2)x^2})$ as $|x| \to \infty$, then $f(x) = p(x)e^{-(1/2)x^2}$, where $p(x)$ is a polynomial of degree m.

We do not know whether an analogous result is true or not for the Harish-Chandra transform. Here we give a family of functions which satify conditions weaker than (3.1) and (3.2).

PROPOSITION 3.2. *Let $p(a)$ be a W-invariant polynomial function on A and* *define f by $f(kak') = p(a)h_t(a)$ for all $k, k' \in K$ and $a \in A$. Then for any fixed $t > 0$, f satisfies*

$$|f(a)| \leq Ce^{\gamma|\log a|}h_t(a) \qquad \text{for all } a \in A \tag{3.5}$$

and

$$|\tilde{f}(\lambda)| \leq C\exp(\delta|\lambda| - t|\lambda|^2) \qquad \text{for all } \lambda \in \mathfrak{a}^*, \tag{3.6}$$

where C, γ, and δ are positive constants.

Proof. (3.5) is obvious. By (2.3) and [4, Theorem 6.2 (6.8)],

$$\tilde{f}(\lambda) = \Lambda_p(\exp(-(|\lambda|^2 + |\rho|^2)t)),$$

where Λ_p is a difference operator which is a product of the Damazure-Lusztig operators. Thus $\tilde{f}$ is of the form

$$\tilde{f}(\lambda) = Q(\lambda)\exp(-|\lambda|^2 t),$$

where $Q(\lambda)$ is an analytic function of at most exponential growth. □

4. The case of the Helgason transform

In this section, we prove an analogue of Hardy's theorem for functions on G/K.

First, we review on the Helgason transform on G/K. For details, see Helgason [12, Chapter III].

Let M denote the centralizer of A in K and $A(x, b)$ denote the function on $G/K \times K/M$ defined by $A(gK, kM) = A(k^{-1}g)$.

Let $\mathscr{C}(G/K)$ denote the space of C^∞-functions on G/K satisfying (1.3) for each integer $q \geq 0$ and each invariant differential operator D on G. For $f \in \mathscr{C}(G/K)$, the Helgason transform $\tilde{f}(\lambda, b)$ is defined by

$$\tilde{f}(\lambda, b) = \int_{G/K} f(x)e^{(-i\lambda+\rho)(A(x,b))}\,dx, \qquad \lambda \in \mathfrak{a}^*_{\mathbf{C}},\ b \in K/M. \tag{4.1}$$

Here dx denotes the (suitably normalized) invariant measure on G/K. If $f \in \mathscr{C}(K\backslash G/K)$, then the Helgason transform $\tilde{f}(\lambda, b)$ does not depend on $b \in K/M$ and coincides with the Harish-Chandra transform $\tilde{f}(\lambda)$.

Theorem 4.1. *For $f \in \mathscr{C}(G/K)$,*

$$f(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*}\int_{K/M} e^{(i\lambda+\rho)(A(x,b))}\tilde{f}(\lambda, b)|\boldsymbol{c}(\lambda)|^{-2}d\lambda db, \qquad x \in G/K. \tag{4.2}$$

We now state and prove an analogue of Hardy's theorem for the Helgason transform.

Theorem 4.2. *Let t be a fixed positive constant. If f is a measurable function on G/K satisfying*

$$|f(g)| \le Ch_t(g) \qquad \text{for all } g \in G \tag{4.3}$$

and

$$|\tilde{f}(\lambda, b)| \le C \exp(-t|\lambda|^2) \qquad \text{for all } \lambda \in \mathfrak{a}^*, \ b \in K/M, \tag{4.4}$$

where C is a positive constant, then $\tilde{f}(\lambda, b) = h(b) \exp(-t|\lambda|^2)$, where h is a bounded function on K/M.

Proof. The proof is similar to that of Theorem 3.1. We give an outline of the proof. By (4.3), $\tilde{f}(\lambda, b)$ is holomorphic in $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$ with

$$|\tilde{f}(\lambda, b)| \le C' \exp(|\mathrm{Im}\,\lambda|^2 t), \tag{4.5}$$

hence it follows from (4.4) and (4.5) that

$$\tilde{f}(\lambda, b) = h(b) e^{-t|\lambda|^2}, \qquad h \in L^\infty(K/M). \tag{4.6}$$

□

Dym and McKean [6] stated Hardy's result in the following form: Let α and β be positive constants and assume that f is a function on the real line satisfying

$$|f(x)| \le Ce^{-\alpha x^2}$$

and

$$|\hat{f}(y)| \le Ce^{-\beta y^2}$$

for some positive constant C. Then

(1) If $\alpha\beta > 1/4$, then $f = 0$.
(2) If $\alpha\beta = 1/4$, then f is a constant multiple of $e^{-\alpha x^2}$.
(3) If $\alpha\beta < 1/4$, then there are infinitely many f that are linearly independent.

We give an analogue of (1) for the Helgason transform.

Corollary 4.3. *Let α and β be positive constants and assume that f is a measurable function on G/K satisfying*

$$|f(g)| \le Ch_{1/(4\alpha)}(g) \qquad \text{for all } g \in G \tag{4.7}$$

and

$$|\tilde{f}(\lambda, b)| \le C \exp(-\beta|\lambda|^2) \qquad \text{for all } \lambda \in \mathfrak{a}^*, \ b \in K/M, \tag{4.8}$$

where C is a positive constant. If $\alpha\beta > 1/4$, then $f = 0$ almost everywhere.

Proof. Since $\beta > 1/(4\alpha)$, f satisfies the assumptions of Theorem 4.2 with $t = 1/(4\alpha)$, and hence $\tilde{f}(\lambda, b) = h(b)\exp(-|\lambda|^2/(4\alpha))$, which contradicts (4.8).

This corollary also follows from (2.6) and [16, Theorem 4.1]. □

An analogue of (3) of the above statement of Dym and McKean might be true for the Helgason transform. Here we give an affirmative answer for some symmetric spaces, where conjectural lower bounds for the heat kernel is true.

Corollary 4.4. *Assume that G is complex, G/K is of rank one, or $G = SL(3, \mathbf{R})$. Let α and β be positive constants. Suppose f is a measurable function on G/K satisfying* (4.7) *and* (4.8), *where C is a positive constant. If $\alpha\beta < 1/4$, then there are infinitely many such f that are linearly independent.*

Proof. A lower bound for the heat kernel is known for each of symmetric space cited above (see [3] and references therein). Choose α' such that $\alpha < \alpha' < 1/(4\beta)$. Let $p(a)$ be a W-invariant polynomial function on A and define f by $f(kak') = p(a)h_{1/(4\alpha')}(a)$ for all $k, k' \in K$ and $a \in A$. It follows from Proposition 3.2 and [3, (3)] that each f satisfies (4.7) and (4.8). Therefore the desired result follows. □

Corollary 4.5. *Assume that G is complex, G/K is of rank one, or $G = SL(3, \mathbf{R})$. Let α and β be positive constants. Suppose f is a measurable function on G/K satisfying*

$$|f(kak')| \le Ce^{-\alpha|\log a|^2} \quad \textit{for all } k, k' \in K, \ a \in A \tag{4.9}$$

and (4.8), *where C is a positive constant. If $\alpha\beta < 1/4$, then there are infinitely many such f that are linearly independent.*

Proof. Choose α' such that $\alpha < \alpha' < 1/(4\beta)$. By (2.6), there is a constant C' such that

$$h_{1/(4\alpha')}(kak') \le C'e^{-\alpha|\log a|^2}$$

for all $k, k' \in K$ and $a \in A$. Therefore, by Corollary 4.4, there are infinitely many independent f satisfying $f(kak') \le Ce^{-\alpha|\log a|^2}$ and (4.8). □

Remark 4.6. After we have finished our work, Mr. Mitsuhiko Ebata kindly sent a copy of preprint of Narayanan and Ray [13]. They also give an attention to the heat kernel and prove that Theorem 4.2 remains to be true if (4.3) is replaced by

$$|f(kak')| \le Ce^{-\alpha|\log a|^2}\phi_0(a)(1 + |\log a|)^r \qquad \text{for all } k, k' \in K, \ a \in A,$$

where r is a positive constant.

References

[1] J.-Ph. Anker, "Le noyau de la chaleur sur les espaces symmétriques $U(p,q)/U(p) \times U(q)$" in Harmonic Analysis, Proceedings, Luxembourg, 1987, ed. by Eymard and J.-P. Pier, Lecture Notes in Math. **1359**, Springer, Berline, 1988, pp. 60–82.

[2] J.-Ph. Anker, Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces, Duke Math. J. **65** (1992), pp. 257–297.

[3] J.-Ph. Anker and L. Ji, Comportement exact du noyau de la chaleur et de la fonction de Green sur les espaces symétriques non compacts, C. R. Acad. Sci. Paris, Série I **326** (1998), pp. 153–156.

[4] I. Cherednik, Inverse Harish-Chandra transform and difference operators, Internat. Math. Res. Notices **15** (1997), pp. 733–750.

[5] M. Cowling, A. Sitaram and M. Sundari, Hardy's uncertainty principle on semisimple groups, Pacific J. Math. **192** (2000), pp. 293–296.

[6] H. Dym and H. P. McKean, Fourier series and integrals, Academic Press, New York, 1972.

[7] M. Ebata, M. Eguchi, S. Koizumi, and K. Kumahara, A generalization of the Hardy theorem to semisimple Lie groups, Proc. Japan Acad. Ser. A Math. Sci. **75** (1999), pp. 113–114.

[8] R. Gangolli, Asymptotic behavior of spectra of compact quotient of certain symmetric spaces, Acta math. **121** (1968), pp. 151–192.

[9] R. Gangolli and V. S. Varadarajan, Harmonic Analysis of Spherical Functions on Real Reductive Groups, Springer-Verlag, Berlin, 1988.

[10] G. H. Hardy, A theorem concerning Fourier transforms, J. London Math. Soc. **8** (1933), pp. 227–231.

[11] S. Helgason, Groups and Geometric Analysis, Academic Press, New York, 1984.

[12] S. Helgason, Geometric Analysis on Symmetric Spaces, American Mathematical Society, 1994.

[13] E. K. Narayanan and S. K. Ray, Hardy's theorem on symmetric spaces of non-compact type, preprint.

[14] J. Sengupta, An analogue of Hardy's theorem for semi-simple Lie groups, Proc. Amer. Math. Soc. **128** (2000), pp. 2493–2499.

[15] N. Shimeno, An analogue of Hardy's theorem on the Poincaré disk, to appear in Bull. Okayama Univ. of Science.

[16] A. Sitaram and M. Sundari, An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple Lie groups, Pacific J. Math. **177** (1997), pp. 187–200.

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