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Estimation in a random effects model with parallel polynomial growth curves

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Abstract. In this paper we consider an extended growth curve model with two hierarchical within-individuals design matrices, which is useful in analyzing mean profiles of several groups with parallel polynomial growth curves. The covariance structure based on a random effects model is assumed. The maximum likelihood estimators (MLE's) are obtained under the random effects covariance structure. The efficiency of the MLE is discussed. A numerical example is also given.

1. Introduction

Suppose that a response variable x has been measured at p different occasions on each of N individuals, and each individual belongs to one of k groups. Let $\boldsymbol{x}_j^{(g)} = [x_{1j}^{(g)}, \dots, x_{pj}^{(g)}]'$ be a p-vector of measurements on the j-th individual in the g-th group, and assume that $\boldsymbol{x}_j^{(g)}$'s are independently distributed as $\boldsymbol{N}_p(\boldsymbol{\mu}^{(g)}, \Sigma)$, where Σ is an unknown $p \times p$ positive definite matrix, $j = 1, \dots, N_g$, $g = 1, \dots, k$. Further, we assume that mean profiles of k groups are parallel polynomial growth curves, i.e.,

$$\boldsymbol{\mu}^{(g)} = \xi^{(g)} \mathbf{1}_p + B' \boldsymbol{\xi}_2, \qquad g = 1, \dots, k, \tag{1.1}$$

where $\mathbf{1}_p$ is a p-vector of ones,

$$B = \begin{bmatrix} \mathbf{1}_p' \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & \cdots\cdots & 1 \\ t_1 & \cdots\cdots & t_p \\ \vdots & & \vdots \\ t_1^{q-1} & \cdots\cdots & t_p^{q-1} \end{bmatrix} \tag{1.2}$$

is a $q \times p$ within-individuals design matrix of rank q $(\le p)$. Yokoyama and Fujikoshi [10] considered a parallel profile model with

$$\boldsymbol{\mu}^{(g)} = \xi^{(g)} \mathbf{1}_p + \boldsymbol{\mu}, \qquad g = 1, \dots, k.$$

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Therefore, the model (1.1) means that $\boldsymbol{\mu}$ has a linear structure. It may be noted that the mean structure (1.1) includes two hierarchical within-individuals design matrices. Without loss of generality, we may assume that $\xi^{(k)} = 0$. In the following we shall do this. Let

$$X = [\boldsymbol{x}_1^{(1)}, \dots, \boldsymbol{x}_{N_1}^{(1)}, \dots\dots, \boldsymbol{x}_1^{(k)}, \dots, \boldsymbol{x}_{N_k}^{(k)}]', \qquad N = N_1 + \cdots + N_k.$$

Then the model of X can be written as

$$X \sim \boldsymbol{N}_{N\times p}(A_1\boldsymbol{\xi}_1\boldsymbol{1}_p' + \boldsymbol{1}_N\boldsymbol{\xi}_2'B, \Sigma \otimes I_N), \tag{1.3}$$

where

$$A_1 = \begin{bmatrix} \boldsymbol{1}_{N_1} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{1}_{N_{k-1}} \\ & \cdots\cdot & \\ & 0 & \end{bmatrix}$$

is an $N \times (k-1)$ between-individuals design matrix of rank $k-1$ $(\le N - p - 1)$, $\boldsymbol{\xi}_1 = [\xi^{(1)}, \dots, \xi^{(k-1)}]'$ and $\boldsymbol{\xi}_2$ are vectors of unknown parameters. The model (1.3) may be called a parallel growth curve model. This is a nested model based on the growth curve model with two different within-individuals design matrices. For a generalized nested model based on the growth curve model with several different within-individuals design matrices, see, e.g., von Rosen [9]. The model (1.3) with $B = I_p$ is a special case of mixed MANOVA-GMANOVA models considered by Chinchilli and Elswick [2], Kshirsagar and Smith [4, p. 85], etc. The mean structure of (1.3) can be written as

$$E(X) = [A_1 \quad \boldsymbol{1}_N]\begin{bmatrix} \boldsymbol{\xi}_{11} & 0 \\ \xi_{21} & \boldsymbol{\xi}_{22}' \end{bmatrix}B, \tag{1.4}$$

where $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_{11}$ and $\boldsymbol{\xi}_2' = [\xi_{21} \quad \boldsymbol{\xi}_{22}']$. We note that the model (1.3) is the ordinary growth curve model (Potthoff and Roy [5]) with a linear restriction on mean parameters.

Fujikoshi and Satoh [3] obtained the MLE's in the growth curve model with two different within-individuals design matrices when Σ has no structures, i.e., is any unknown positive definite. When there is no theoretical or empirical basis for assuming special covariance structures, we need to assume that Σ is any unknown positive definite. However, for analysis of repeated measures or growth curves, it has been imposed to consider certain parsimonious covariance structures. As one of such structures, we are interested in a random effects covariance structure (see, e.g., Rao [6]). In our model, the structure can be expressed as

$$\Sigma = \delta^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p, \tag{1.5}$$

where $\delta^2 \geq 0$ and $\sigma^2 > 0$. The covariance structure (1.5) can be introduced by assuming the following random effects model:

$$\boldsymbol{x}_j^{(g)} = (\xi^{(g)} + \eta_j^{(g)}) \mathbf{1}_p + B' \boldsymbol{\xi}_2 + \boldsymbol{\varepsilon}_j^{(g)}, \tag{1.6}$$

where $\eta_j^{(g)}$'s and $\boldsymbol{\varepsilon}_j^{(g)}$'s are independently distributed as $N(0, \delta^2)$ and $\boldsymbol{N}_p(\mathbf{0}, \sigma^2 I_p)$, respectively. Therefore, the covariance matrix of $\boldsymbol{x}_j^{(g)}$ is given by (1.5). This implies that

$$X \sim \boldsymbol{N}_{N \times p}(A_1 \boldsymbol{\xi}_1 \mathbf{1}_p' + \mathbf{1}_N \boldsymbol{\xi}_2' B, (\delta^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 I_p) \otimes I_N). \tag{1.7}$$

In this paper we consider the problems of estimating the unknown parameters $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, δ^2 and σ^2 when Σ has the structure (1.5). In § 2 we obtain a canonical form of (1.7). In § 3 we obtain the MLE's in the model (1.7), using a canonical form. In § 4 it is shown how much gains can be obtained for the maximum likelihood estimation of $\boldsymbol{\xi}_1$ by assuming a random effects covariance structure. In § 5 we give a numerical example of the results of § 4.

2. Transformation of the model

In order to transform (1.7) to a model which is easier to analyze, we use a canonical reduction. Let $H = [H_1 \quad N^{-1/2}\mathbf{1}_N \quad H_3]$ be an orthogonal matrix of order N such that

$$\begin{aligned}[A_1 \quad \mathbf{1}_N] &= [H_1 \quad N^{-1/2}\mathbf{1}_N] \begin{bmatrix} L_{11} & \mathbf{0} \\ \boldsymbol{l}_{21}' & N^{1/2} \end{bmatrix} \\ &= H_{(2)} L,\end{aligned}$$

where $H_1 : N \times (k-1)$, and $L_{11} : (k-1) \times (k-1)$ is a lower triangular matrix. Similarly, let $Q = [p^{-1/2}\mathbf{1}_p \quad Q_2' \quad Q_3']'$ be an orthogonal matrix of order p such that

$$\begin{aligned}\begin{bmatrix} \mathbf{1}_p' \\ B_2 \end{bmatrix} &= \begin{bmatrix} p^{1/2} & \mathbf{0}' \\ \boldsymbol{g}_{21} & G_{22} \end{bmatrix} \begin{bmatrix} p^{-1/2}\mathbf{1}_p' \\ Q_2 \end{bmatrix} \\ &= G Q_{(2)},\end{aligned}$$

where $Q_2 : (q-1) \times p$, and $G_{22} : (q-1) \times (q-1)$ is a lower triangular matrix. Then the mean structure of (1.7) can be written as

$$A_1 \boldsymbol{\xi}_1 \mathbf{1}_p' + \mathbf{1}_N \boldsymbol{\xi}_2' B = p^{-1/2} H_1 \boldsymbol{\theta}_1 \mathbf{1}_p' + N^{-1/2} \mathbf{1}_N \boldsymbol{\theta}_2' Q_{(2)}, \tag{2.1}$$

where

$$\boldsymbol{\theta}_1 = p^{1/2} L_{11} \xi_1, \qquad \boldsymbol{\theta}_2' = N^{1/2} \xi_2' G + \boldsymbol{l}_{21}' [\xi_1 \quad 0] G.$$

Here we note that (ξ_1, ξ_2) is an invertible function of $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. In fact, ξ_1 and ξ_2 can be expressed in terms of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ as

$$\text{(2.2)} \quad \xi_1 = p^{-1/2} L_{11}^{-1} \boldsymbol{\theta}_1, \qquad \xi_2' = N^{-1/2} \boldsymbol{\theta}_2' G^{-1} - N^{-1/2} \boldsymbol{l}_{21}' [p^{-1/2} L_{11}^{-1} \boldsymbol{\theta}_1 \quad 0].$$

Using the above transformation, we can write a canonical form of (1.7) as

$$\text{(2.3)} \qquad Y = H' X Q' = \begin{bmatrix} \boldsymbol{y}_{11} & Y_{12} & Y_{13} \\ y_{21} & \boldsymbol{y}_{22}' & \boldsymbol{y}_{23}' \\ \boldsymbol{y}_{31} & Y_{32} & Y_{33} \end{bmatrix} \sim \boldsymbol{N}_{N \times p}(E(Y), \Psi \otimes I_N),$$

where means $E(Y)$ and covariance matrix Ψ are given by

$$\text{(2.4)} \qquad E(Y) = \begin{bmatrix} \boldsymbol{\theta}_{11} & 0 & 0 \\ \theta_{21} & \boldsymbol{\theta}_{22}' & \boldsymbol{0}' \\ \boldsymbol{0} & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{\theta}_{11} & 0 \\ \theta_{21} & \boldsymbol{\theta}_{22}' \end{bmatrix} = L \begin{bmatrix} \boldsymbol{\xi}_{11} & 0 \\ \xi_{21} & \boldsymbol{\xi}_{22}' \end{bmatrix} G,$$

$$\boldsymbol{\theta}_1 = \boldsymbol{\theta}_{11}, \qquad \boldsymbol{\theta}_2' = [\theta_{21} \quad \boldsymbol{\theta}_{22}']$$

and

$$\text{(2.5)} \qquad \Psi = Q \Sigma Q' = \begin{bmatrix} p\delta^2 + \sigma^2 & \boldsymbol{0}' \\ \boldsymbol{0} & \sigma^2 I_{p-1} \end{bmatrix}.$$

3. The MLE's

In this section we obtain the MLE's of ξ_1, ξ_2, δ^2 and σ^2 in the model (1.7), using (2.3). Let

$$U = [\boldsymbol{y}_{11} \quad Y_{12} \quad Y_{13}]' [\boldsymbol{y}_{11} \quad Y_{12} \quad Y_{13}] = \begin{bmatrix} u_{11} & \boldsymbol{u}_{12}' & \boldsymbol{u}_{13}' \\ \boldsymbol{u}_{21} & U_{22} & U_{23} \\ \boldsymbol{u}_{31} & U_{32} & U_{33} \end{bmatrix},$$

$$V = [y_{21} \quad \boldsymbol{y}_{22}' \quad \boldsymbol{y}_{23}']' [y_{21} \quad \boldsymbol{y}_{22}' \quad \boldsymbol{y}_{23}'] = \begin{bmatrix} v_{11} & \boldsymbol{v}_{12}' & \boldsymbol{v}_{13}' \\ \boldsymbol{v}_{21} & V_{22} & V_{23} \\ \boldsymbol{v}_{31} & V_{32} & V_{33} \end{bmatrix},$$

$$W = [\boldsymbol{y}_{31} \quad Y_{32} \quad Y_{33}]' [\boldsymbol{y}_{31} \quad Y_{32} \quad Y_{33}] = \begin{bmatrix} w_{11} & \boldsymbol{w}_{12}' & \boldsymbol{w}_{13}' \\ \boldsymbol{w}_{21} & W_{22} & W_{23} \\ \boldsymbol{w}_{31} & W_{32} & W_{33} \end{bmatrix}$$

and

$$T = U + W = \begin{bmatrix} t_{11} & \boldsymbol{t}'_{12} & \boldsymbol{t}'_{13} \\ \boldsymbol{t}_{21} & T_{22} & T_{23} \\ \boldsymbol{t}_{31} & T_{32} & T_{33} \end{bmatrix}.$$

It is easy to see that the MLE's of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are given by

$$\hat{\boldsymbol{\theta}}_1 = \boldsymbol{y}_{11}, \qquad \hat{\boldsymbol{\theta}}'_2 = \boldsymbol{y}'_{2(12)},$$

where $\boldsymbol{y}'_{2(12)} = [y_{21} \quad \boldsymbol{y}'_{22}]$. Hence the MLE's of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are given by

$$\text{(3.1)} \quad \hat{\boldsymbol{\xi}}_1 = p^{-1/2} L_{11}^{-1} \boldsymbol{y}_{11}, \qquad \hat{\boldsymbol{\xi}}'_2 = N^{-1/2} \boldsymbol{y}'_{2(12)} G^{-1} - N^{-1/2} \boldsymbol{l}'_{21} [p^{-1/2} L_{11}^{-1} \boldsymbol{y}_{11} \quad 0].$$

Using a technique similar to the one in estimating variance components in a one-way random effects model by maximum likelihood (see, e.g., Searle, Casella and McCulloch [7, p. 148]), we can obtain the MLE's of δ^2 and σ^2. Let $L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \sigma^2, \delta^2)$ be the likelihood function of Y. Then we have

$$\begin{aligned} g(\sigma^2, \delta^2) &= -2 \log L(\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2, \sigma^2, \delta^2) \\ &= Np \log(2\pi) + N \log(p\delta^2 + \sigma^2) + \frac{w_{11}}{p\delta^2 + \sigma^2} \\ &\quad + N(p-1) \log \sigma^2 + \frac{1}{\sigma^2} (\operatorname{tr} T_{(23)(23)} + \operatorname{tr} V_{33}), \end{aligned}$$

where

$$T_{(23)(23)} = \begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix}.$$

The minimum of $g(\sigma^2, \delta^2)$ with respect to $\delta^2 \geq 0$ and $\sigma^2 > 0$ is achieved at

$$\begin{aligned} \hat{\delta}^2 &= \max\left\{ \frac{1}{p} \left[\frac{1}{N} w_{11} - \frac{1}{N(p-1)} (\operatorname{tr} T_{(23)(23)} + \operatorname{tr} V_{33}) \right], 0 \right\}, \\ \text{(3.2)} \qquad \hat{\sigma}^2 &= \min\left\{ \frac{1}{N(p-1)} (\operatorname{tr} T_{(23)(23)} + \operatorname{tr} V_{33}), \right. \\ &\qquad \left. \frac{1}{N + N(p-1)} (w_{11} + \operatorname{tr} T_{(23)(23)} + \operatorname{tr} V_{33}) \right\} \end{aligned}$$

(see, e.g., Arnold [1, p. 251]). Therefore, the MLE's of δ^2 and σ^2 are given by (3.2).

Now we express the MLE's given in (3.1) and (3.2) in terms of the original observations. Let S_w and S_t be the matrices of the sums of squares and products due to the within variation and total variation, i.e.,

$$S_w = X'H_3H_3'X = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})',$$

$$S_t = X'(H_1H_1' + H_3H_3')X = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})',$$

where $\bar{\boldsymbol{x}}^{(g)}$ and $\bar{\boldsymbol{x}}$ are the sample mean vectors of observations of the g-th group and all the groups, respectively. Further, let

$$\tilde{A}_1 = \left(I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N'\right)A_1, \qquad \tilde{B}_2 = B_2\left(I_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}_p'\right). \tag{3.3}$$

Then, from the definitions of L and G it is easily seen that

$$H_1 = \tilde{A}_1L_{11}^{-1}, \qquad \boldsymbol{l}_{21}' = \frac{1}{\sqrt{N}}\mathbf{1}_N'A_1, \qquad Q_2 = G_{22}^{-1}\tilde{B}_2, \qquad \boldsymbol{g}_{21} = \frac{1}{\sqrt{p}}B_2\mathbf{1}_p.$$

Using these results, we have the following theorem.

THEOREM 3.1. *The MLE's of $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, δ^2 and σ^2 in the extended growth curve model* (1.7) *are given as follows:*

$$\hat{\boldsymbol{\xi}}_1 = \frac{1}{p}(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'X\mathbf{1}_p,$$

$$\hat{\xi}_{21} = \frac{1}{p}\left[\bar{\boldsymbol{x}}'\{I_p - \tilde{B}_2'(\tilde{B}_2\tilde{B}_2')^{-1}B_2\} - \frac{1}{N}\mathbf{1}_N'A_1(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'X\right]\mathbf{1}_p,$$

$$\hat{\boldsymbol{\xi}}_{22}' = \bar{\boldsymbol{x}}'\tilde{B}_2'(\tilde{B}_2\tilde{B}_2')^{-1},$$

$$\hat{\delta}^2 = \max\left\{\frac{1}{p}\left[\frac{1}{N}s_1^2 - \frac{1}{N(p-1)}s_2^2\right], 0\right\},$$

$$\hat{\sigma}^2 = \min\left\{\frac{1}{N(p-1)}s_2^2, \frac{1}{Np}(s_1^2 + s_2^2)\right\},$$

where $\tilde{A}_1$ and $\tilde{B}_2$ are given by (3.3), *and s_1^2 and s_2^2 are defined by*

$$s_1^2 = \frac{1}{p}\mathbf{1}_p'S_w\mathbf{1}_p$$

and

$$s_2^2 = \operatorname{tr} S_t - \frac{1}{p}\mathbf{1}_p'S_t\mathbf{1}_p + N\bar{\boldsymbol{x}}'\left\{I_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}_p' - \tilde{B}_2'(\tilde{B}_2\tilde{B}_2')^{-1}\tilde{B}_2\right\}\bar{\boldsymbol{x}},$$

respectively.

We note that the MLE's $\hat{\delta}^2$ and $\hat{\sigma}^2$ are not unbiased. The usual unbiased estimators of δ^2 and σ^2 may be defined by

$$
(3.4)\qquad
\begin{aligned}
\tilde{\delta}^2 &= \frac{1}{p}\left\{\frac{1}{N-k}w_{11} - \frac{1}{N(p-1)-(q-1)}(\mathrm{tr}\ T_{(23)(23)} + \mathrm{tr}\ V_{33})\right\},\\
\tilde{\sigma}^2 &= \frac{1}{N(p-1)-(q-1)}(\mathrm{tr}\ T_{(23)(23)} + \mathrm{tr}\ V_{33}),
\end{aligned}
$$

respectively. There is the possibility that the use of $\tilde{\delta}^2$ can lead to a negative estimate of δ^2, while $\hat{\delta}^2$ is non-negative. As a modification of the MLE's, we propose the estimators obtained from the MLE's by replacing N and $N(p-1)$ by $N-k$ and $N(p-1)-(q-1)$, respectively, in (3.2). The modified MLE's, which are based on the joint distribution of w_{11} and $(\mathrm{tr}\ T_{(23)(23)} + \mathrm{tr}\ V_{33})$ only, may be called restricted maximum likelihood estimators (REMLE's). These estimators can be expressed in terms of the original observations, again using the notations in Theorem 3.1.

4. Efficiency of $\hat{\boldsymbol{\xi}}_1$

Next we consider the efficiency of the MLE for $\boldsymbol{\xi}_1$ in the case when the covariance structure (1.5) is assumed. When no special assumptions about Σ are made, the MLE of $\boldsymbol{\xi}_1$ is given by

$$
(4.1)\qquad \tilde{\boldsymbol{\xi}}_1 = (\mathbf{1}_p' S_w^{-1}\mathbf{1}_p)^{-1}(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1' X S_w^{-1}\mathbf{1}_p.
$$

The estimators $\hat{\boldsymbol{\xi}}_1$ and $\tilde{\boldsymbol{\xi}}_1$ have the following properties.

Theorem 4.1. *In the extended growth curve model* (1.7) *it holds that both the estimators* $\hat{\boldsymbol{\xi}}_1$ *and* $\tilde{\boldsymbol{\xi}}_1$ *are unbiased, and*

$$
\mathrm{Var}(\hat{\boldsymbol{\xi}}_1) = \frac{1}{p}(p\delta^2+\sigma^2)(\tilde{A}_1'\tilde{A}_1)^{-1},
$$

$$
\mathrm{Var}(\tilde{\boldsymbol{\xi}}_1) = \frac{1}{p}(p\delta^2+\sigma^2)\left(1+\frac{p-1}{N-k-p}\right)(\tilde{A}_1'\tilde{A}_1)^{-1}.
$$

Proof. From (2.2), (3.1) and $\tilde{A}_1'\tilde{A}_1 = L_{11}'L_{11}$, we obtain the result on $\hat{\boldsymbol{\xi}}_1$. It can be shown that for any positive definite covariance matrix Σ,

$$
E(\tilde{\boldsymbol{\xi}}_1) = \boldsymbol{\xi}_1 \qquad \text{and} \qquad \mathrm{Var}(\tilde{\boldsymbol{\xi}}_1) = (\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)^{-1}\left(1+\frac{p-1}{N-k-p}\right)(\tilde{A}_1'\tilde{A}_1)^{-1}.
$$

Under the assumption that $\Sigma = \delta^2\mathbf{1}_p\mathbf{1}_p' + \sigma^2 I_p$, it holds that

$$
(\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)^{-1} = \frac{1}{p}(p\delta^2+\sigma^2),
$$

which proves the desired result on $\tilde{\boldsymbol{\xi}}_1$.

From Theorem 4.1, we obtain

$$\mathrm{Var}(\tilde{\boldsymbol{\xi}}_1) - \mathrm{Var}(\hat{\boldsymbol{\xi}}_1) = (p\delta^2 + \sigma^2)\frac{p-1}{p(N-k-p)}(\tilde{A}_1'\tilde{A}_1)^{-1} > 0, \tag{4.2}$$

which implies that $\hat{\boldsymbol{\xi}}_1$ is more efficient than $\tilde{\boldsymbol{\xi}}_1$ in the model (1.7). This shows that we can get a more efficient estimator for $\boldsymbol{\xi}_1$ by assuming a random effects covariance structure. Especially, when p is large relative to N, we can obtain greater gains.

5. Numerical example

In this section we give a numerical example to illustrate the efficiency of $\hat{\boldsymbol{\xi}}_1$ by assuming a random effects covariance structure. We apply the results of §4 to the data (see, e.g., Srivastava and Carter [8, p. 227]) of the price indices of hand soaps packaged in four ways, estimated by twelve consumers. For six of the consumers, the packages have been labeled with a well-known brand name. For the remaining six consumers, no label is used. Then, from the data we obtain

$$\bar{\boldsymbol{x}}^{(1)} = [.31667, \quad .45833, \quad .47500, \quad .64167]',$$

$$\bar{\boldsymbol{x}}^{(2)} = [.60000, \quad .66667, \quad .85000, \quad .96667]',$$

$$\bar{\boldsymbol{x}} = [.45833, \quad .56250, \quad .66250, \quad .80417]',$$

$$S_w = \begin{bmatrix} .21833 & .15167 & .20500 & .08333 \\ & .25542 & .16375 & .21375 \\ & & .30375 & .17875 \\ & & & .29542 \end{bmatrix},$$

$$S_t = \begin{bmatrix} .45917 & .32875 & .52375 & .35958 \\ & .38563 & .39813 & .41688 \\ & & .72563 & .54438 \\ & & & .61229 \end{bmatrix}.$$

For the observation matrix $X : 12 \times 4$, we assume the model (1.7) with

$$E(X) = \begin{bmatrix} \mathbf{1}_6 \\ \mathbf{0} \end{bmatrix} \xi_{11}\mathbf{1}_4' + \mathbf{1}_{12}[\xi_{21}, \quad \xi_{22}] \begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \end{bmatrix} \tag{5.1}$$

$$= \begin{bmatrix} \mathbf{1}_6 & \mathbf{1}_6 \\ \mathbf{0} & \mathbf{1}_6 \end{bmatrix} \begin{bmatrix} \xi_{11} & 0 \\ \xi_{21} & \xi_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \end{bmatrix}$$

and

$$\mathrm{Var}(\mathrm{vec}(X)) = (\delta^2\mathbf{1}_4\mathbf{1}_4' + \sigma^2 I_4) \otimes I_{12}. \tag{5.2}$$

Now we estimate how much gains can be obtained for the maximum likelihood estimation of ξ_{11} by assuming the covariance structure (5.2). Since $p = 4$, $N = 12$, $k = 2$, $\tilde{A}_1'\tilde{A}_1 = 3$, $\hat{\delta}^2 = .01353$ and $\hat{\sigma}^2 = .00976$, it follows from Theorem 4.1 and (4.2) that

$$\hat{\mathrm{V}}\mathrm{ar}(\tilde{\xi}_{11}) - \hat{\mathrm{V}}\mathrm{ar}(\hat{\xi}_{11}) = \frac{1}{24}(4\hat{\delta}^2 + \hat{\sigma}^2) = .00266.$$

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