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Bernstein functions and parabolic equations in $BUC(\mathbf{R}^n, \mathbf{R})$

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ABSTRACT. In this paper we give an application of quasimonotonicity methods and Bernstein functions to parabolic differential-functional equations in $BUC(\mathbf{R}^n, \mathbf{R})$, and a new proof of the known result in Theorem 1 on solvability and asymptotic behaviour of its solutions.

1. Introduction

Let $D_1, \dots, D_n$ denote the differential operators $D_j u = u_{x_j}$, and for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$ let $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $y \in \mathbf{R}^n$ let S_y denote the translation operator $(S_y u)(x) = u(x + y)$.

Let $T \in (0, \infty]$, $a_{j,k}, b_j : [0, T) \to \mathbf{R}$ $(j, k = 1, \dots, n)$ and $c_j : [0, T) \to \mathbf{R}$, $g_j : [0, T) \to \mathbf{R}^n$ $(j = 1, \dots, m)$ be continuous such that $(a_{j,k}(t))$ is positive semi-definite $(t \in [0, T))$.

In this paper we will consider the Cauchy problem for the parabolic differential functional equation

$$(1) \qquad u_t = \sum_{j,k=1}^{n} a_{j,k}(t) D_j D_k u + \sum_{j=1}^{n} b_j(t) D_j u + \sum_{j=1}^{m} c_j(t) S_{g_j(t)} u, \qquad u(0) = u_0$$

in $BUC(\mathbf{R}^n, \mathbf{R})$, the Banach space of all bounded, uniformly continuous functions on $\mathbf{R}^n$ endowed with the supremum norm $\|\cdot\|_\infty$.

We will obtain results on solvability and asymptotic behaviour of the solution of problem (1) by applying quasimonotonicity methods and Bernstein functions. For application of Bernstein functions to parabolic problems see also [8].

For $\tau > 0$ let B_τ denote the following class of Bernstein functions:

$$B_\tau := \{u \in C^\infty(\mathbf{R}^n, \mathbf{R}) : \exists M > 0 \ \forall x \in \mathbf{R}^n \ \forall \alpha \in \mathbf{N}_0^n : |(D^\alpha u)(x)| \le M \tau^{|\alpha|}\}.$$

Note that obviously $B_\tau \subseteq BUC(\mathbf{R}^n, \mathbf{R})$ $(\tau > 0)$.

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The vector space B_τ can be normed by

$$\|u\|_\tau := \sup_{\alpha \in \mathbf{N}_0^n} \frac{\|D^\alpha u\|_\infty}{\tau^{|\alpha|}},$$

and by standard reasoning $(B_\tau, \|\cdot\|_\tau)$ is a Banach space. We will see later that in fact $\|\cdot\|_\tau = \|\cdot\|_\infty$. In particular, each operator $S_y : B_\tau \to B_\tau$ is an isometry on B_τ.

Let $\mathscr{L}(E)$ be the algebra of all continuous linear operators on a Banach space E. Obviously $D^\alpha \in \mathscr{L}(B_\tau)$ and $\|D^\alpha u\|_\tau \le \tau^{|\alpha|}\|u\|_\tau$ $(\alpha \in \mathbf{N}_0^n, u \in B_\tau)$. Hence, we can define $L : [0,\infty) \to \mathscr{L}(B_\tau)$ by

$$L(t)u = \sum_{j,k=1}^n a_{j,k}(t) D_j D_k u + \sum_{j=1}^n b_j(t) D_j u + \sum_{j=1}^m c_j(t) S_{g_j(t)} u.$$

We will see that L is continuous. Therefore, for $t_0 \in [0,T)$ the Cauchy problem

$$u'(t) = L(t)u(t), \qquad u(t_0) = u_0 \in B_\tau \tag{2}$$

is uniquely solvable on $[t_0, T)$ in B_τ.

We will prove the following result, which for the classical case $g_j(t) = 0$ $(j = 1, \dots, m)$ is a consequence of [4], p. 43, Theorem 9.

THEOREM 1. *Let* $a_{j,k}, b_j : [0,T) \to \mathbf{R}$ $(j,k = 1,\dots,n)$ *and* $c_j : [0,T) \to \mathbf{R}$, $g_j : [0,T) \to \mathbf{R}^n$ $(j = 1,\dots,m)$ *be continuous such that* $(a_{j,k}(t))$ *is positive semi-definite* $(t \in [0,T))$, *and* $g_j(t) \ne 0 \Rightarrow c_j(t) \ge 0$ $(j = 1,\dots,m, t \in [0,T))$. *Then the solution* $u : [t_0, T) \to B_\tau$ *of problem* (2) *satisfies*

$$\|u(t)\|_\infty \le \exp\left(\int_{t_0}^t c(s)\,ds\right)\|u_0\|_\infty \qquad (t \in [t_0, T)),$$

where $c(t) = \sum_{j=1}^m c_j(t)$.

The following approximation theorem, due to Bernstein for the one dimensional case (see [2], [9] p. 14), will be the main tool to construct generalized solutions of our problem in $BUC(\mathbf{R}^n, \mathbf{R})$, i.e., continuous functions u with $u(t_0) = u_0$ and which are locally (in $[t_0, T)$) uniform limits of solutions of (2).

THEOREM 2. *The set* $B_\infty := \bigcup_{\tau > 0} B_\tau$ *is dense in* $BUC(\mathbf{R}^n, \mathbf{R})$.

Next, let $|||\cdot|||$ denote the the operator norm on $\mathscr{L}(BUC(\mathbf{R}^n, \mathbf{R}))$ and $C := \{(t, t_0) \in \mathbf{R}^2 : 0 \le t_0 \le t < T\}$. We prove

THEOREM 3. *Let the functions* $a_{j,k}, b_j, c_j, g_j$ *be as in Theorem* 1. *Then there exists a solution operator* $U : C \times BUC(\mathbf{R}^n, \mathbf{R}) \to BUC(\mathbf{R}^n, \mathbf{R})$ *with the following properties:*

1) *The function U is continuous, $U(\cdot, t_0)u_0$ is a generalized solution of* (2), *and for* $(t, t_0) \in C$, $u_0 \in BUC(\mathbf{R}^n, \mathbf{R})$

$$U(t, 0)u_0 = U(t, t_0)U(t_0, 0)u_0.$$

2) *For each $\tau > 0$ and $u_0 \in B_\tau$ the function $t \mapsto U(t, t_0)u_0$ is the solution of problem* (2) *on* $[t_0, T)$.

3) *If $u_0 \in BUC(\mathbf{R}^n, \mathbf{R})$ is such that $u_0(x) \geq 0$ $(x \in \mathbf{R}^n)$ then* $(U(t, t_0)u_0)(x) \geq 0$ $(x \in \mathbf{R}^n, t_0 \leq t < T)$.

Remarks.

1. By 2), the function U is uniquely determined and may be considered a generalized fundamental system in $BUC(\mathbf{R}^n, \mathbf{R})$, associated to L. It will in general neither be differentiable with respect to t nor will $U(t, t_0)u_0$ be differentiable with respect to x, as is seen from the example $n = m = 1$, $a_{1,1} = 0$, $c_1 = 0$, where we have

$$(U(t, t_0)u_0)(x) = u_0\left(x + \int_{t_0}^{t} b_1(s)ds\right) \qquad (t_0 \leq t < T).$$

2. From 3) we will get the estimate

$$|||U(t, t_0)||| \leq \exp\left(\int_{t_0}^{t} c(s)ds\right) \qquad (t_0 \leq t < T),$$

where the functions $a_{j,k}$ and b_j do not appear. Related results for initial boundary value problems for parabolic equations are known, see for example [7]. The above estimate gives an information on the asymptotic behavior; since, if u_0 is constant, $U(t, 0)u_0 = \exp(\int_{t_0}^{t} c(s)ds)u_0$, we have

$$\lim_{t \to T-} U(t, 0)u_0 = 0 \qquad \textit{for each } u_0 \in BUC(\mathbf{R}^n, \mathbf{R})$$

$$\Leftrightarrow \lim_{t \to T-} \int_{t_0}^{t} c(s)ds = -\infty.$$

3. For asymptotic estimates of solutions for the Cauchy problem for parabolic equations with bounded initial function see [4], p. 56. There $L(t)u(t)$ is assumed to be bounded and the estimate for $u(t)$ depends on the bound of $L(t)u(t)$, whereas our bound depends only on c and u_0.
4. Classical parabolic equations with uniformly elliptic differential operator in $BUC(\mathbf{R}^n, \mathbf{R})$ are studied in [6] by semigroup methods. A cen-

tral tool in this paper are estimates for the fundamental solution of the parabolic equation.

5. Our main tool for proving Theorem 1 will be the theory of continuous quasimonotone increasing operators in ordered Banach spaces.

Using U, nonhomogenous and nonlinear problems can be handled. Since, for each fixed $\tau > 0$ and $u_0 \in B_\tau$, $r \in C([0,T), B_\tau)$, the solution of

$$u'(t) = L(t)u(t) + r(t), \qquad (t \in [0,T)), \qquad u(0) = u_0 \tag{3}$$

is given by

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)r(s)ds,$$

we call this expression the generalized solution of (3) if $u_0 \in BUC(\mathbf{R}^n, \mathbf{R})$ and $r \in C([0,T), BUC(\mathbf{R}^n, \mathbf{R}))$. Here, the integral is the Riemann integral for continuous functions with values in the Banach space $BUC(\mathbf{R}^n, \mathbf{R})$. Approximating u_0 and r according to Theorem 2, we see that this generalized solution is the (in $[0,T)$) locally uniform limit of solutions of (3). Finally, if $l \in C([0,T), \mathbf{R})$, the generalized fundamental system $\tilde{U}$ which belongs to $\tilde{L}(t) := L(t) - l(t)I$ (I the identity) is given by

$$\tilde{U}(t,t_0) = \exp\left(-\int_{t_0}^t l(s)ds\right)U(t,t_0).$$

With this at hand, as an application of Theorem 3 we will prove:

Theorem 4. *Let U be as in Theorem 3 and let $f : [0,T) \times \mathbf{R} \to \mathbf{R}$ be a continuous function such that there exists an $l \in C([0,T), \mathbf{R})$ with*

$$|f(t,x) - f(t,\bar{x})| \le l(t)|x - \bar{x}|, \qquad (t \in [0,T), x, \bar{x} \in \mathbf{R}).$$

Then for each $u_0 \in BUC(\mathbf{R}^n, \mathbf{R})$, the problem

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s,u(s))ds \qquad (t \in [0,T)) \tag{4}$$

has a unique solution u in $C([0,T), BUC(\mathbf{R}^n, \mathbf{R}))$.

This solution depends monotonic and continuously (with respect to uniform convergence on compact subintervals of $[0,T)$) on u_0.

For $s \in [0,T)$, $u \in BUC(\mathbf{R}^n, \mathbf{R})$, $f(s,u)$ denotes the function $f(s,u(\cdot)) \in BUC(\mathbf{R}^n, \mathbf{R})$.

2. Bernstein classes and quasimonotonicity

To discuss some properties of the functions in B_τ we consider first the case $n = 1$. Then each function $f \in B_\tau$ is the restriction of the entire function

$$F(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

to $\mathbf{R}$, and F is of exponential growth and type τ, and bounded on $\mathbf{R}$. According to a theorem of Bernstein, such a function satisfies the inequality

$$\|f'\|_\infty \le \tau \|f\|_\infty \qquad (f \in B_\tau),$$

see [1], p. 206, Theorem 11.1.2. This proves $\|\cdot\|_\tau = \|\cdot\|_\infty$ in case $n = 1$. For the general case of a function $u = u(x_1, \dots, x_n)$ in B_τ just note that $f(x_j) = u(x_1, \dots, x_n)$ has all properties discribed above if $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$ are fixed. Hence $\|D_j u\|_\infty \le \tau \|u\|_\infty$ $(j = 1, \dots, n)$ and therefore $\|\cdot\|_\tau = \|\cdot\|_\infty$ on B_τ. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbf{R}^n$. By the Mean Value Theorem

$$|u(x) - u(\tilde{x})| \le \sup_{\xi \in \mathbf{R}^n} \|\nabla u(\xi)\| \, \|x - \tilde{x}\| \le \tau\sqrt{n} \|u\|_\infty \|x - \tilde{x}\| \qquad (x, \tilde{x} \in \mathbf{R}^n)$$

for each $u \in B_\tau$. Hence for $y, z \in \mathbf{R}^n$ and $u, v \in B_\tau$

$$\begin{aligned} \|S_y u - S_z v\|_\infty &\le \|S_y u - S_z u\|_\infty + \|S_z u - S_z v\|_\infty \\ &\le \tau\sqrt{n} \|u\|_\infty \|y - z\| + \|u - v\|_\infty. \end{aligned}$$

Therefore $(y, u) \mapsto S_y u$ is a continuous mapping from $\mathbf{R}^n \times B_\tau$ to B_τ. This, together with the fact that the functions $a_{j,k}, b_j, c_j, g_j$ are continuous, implies that L is continuous on $[0, T)$.

We now consider $(B_\tau, \|\cdot\|_\infty)$ ordered by the cone

$$K = \{u \in B_\tau : u(x) \ge 0 \ \ (x \in \mathbf{R}^n)\}.$$

As usual $u \le v :\Leftrightarrow y - x \in K$. The cone K is solid (that is K has nonempty interior), since $1 \in \operatorname{Int} K$.

The dual cone of K, denoted by K^*, is the set of all continuous linear functionals $\varphi \in B_\tau^*$ such that $\varphi(u) \ge 0$ $(u \ge 0)$.

A linear operator $R \in \mathscr{L}(B_\tau)$ is called quasimonotone increasing (in the sense of Volkmann [10]) if

$$u \in B_\tau, \qquad u \ge 0, \qquad \varphi \in K^*, \qquad \varphi(u) = 0 \Rightarrow \varphi(Ru) \ge 0.$$

Let $Q_+ := \{R \in \mathscr{L}(B_\tau) : R$ is quasimonotone increasing$\}$. As a consequence of results on differential inequalities [10] it is well known that $R \in Q_+ \Leftrightarrow \exp(tR)$ is increasing $(t \ge 0)$. Let $Q_\pm$ denote all $R \in \mathscr{L}(B_\tau)$ with $\pm R \in Q_+$ (sometimes called the quasimonotone constant operators), and note that $R \in Q_{\pm} \Rightarrow R^2 \in Q_+$ (see [5]). Moreover Q_+ is a wedge, that is Q_+ is closed, convex and $T \in Q_+ \Rightarrow \lambda T \in Q_+$ $(\lambda \geq 0)$.

As a central tool for handling problem (1) we prove

THEOREM 5. *Under the assumptions of Theorem* 1, $L(t) \in Q_+$ $(0 \leq t < T)$.

PROOF. We fix $t_0 \in [0, T)$. Since the identity I on B_τ is in $Q_{\pm}$ and S_y is increasing for each $y \in \mathbf{R}^n$ we have

$$\sum_{j=1}^{m} c_j(t_0) S_{g_j(t_0)} \in Q_+,$$

since $c_j(t_0) \geq 0$ if $g_j(t_0) \neq 0$ $(j = 1, \dots, m)$. Next, for each $j = 1, \dots, n$ we have $D_j \in Q_{\pm}$ since (by Taylor's theorem)

$$(\exp(tD_j)u)(x) = u(x_1, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_n) \qquad (t \in \mathbf{R}).$$

Hence

$$\sum_{j=1}^{n} b_j(t_0) D_j \in Q_{\pm}.$$

Finally, to prove that the second order part of $L(t)$ is in Q_+ it is sufficient to prove that this part is the sum of squares of operators in $Q_{\pm}$. Consider the matrix $A = (a_{j,k}(t_0))$. Since A is positive semidefinite it has a positive semidefinite square root $B = (b_{j,l})$. By setting

$$R_l = \sum_{j=1}^{n} b_{j,l} D_j \qquad (l = 1, \dots, n)$$

we obtain, since each R_l is in $Q_{\pm}$, that

$$R_l^2 = \sum_{j,k=1}^{n} b_{j,l} b_{l,k} D_j D_k \in Q_+,$$

and therefore

$$\sum_{l=1}^{n} R_l^2 = \sum_{j,k=1}^{n} \sum_{l=1}^{n} b_{j,l} b_{l,k} D_j D_k = \sum_{j,k=1}^{n} a_{j,k}(t_0) D_j D_k \in Q_+.$$

Alltogether $L(t_0) \in Q_+$. □

We now know that L is continuous and quasimonotone increasing in u. Hence problem (1) is uniquely solvable in B_τ and the solution depends monotone increasing on u_0.

Remark. The monotonicity of the solution of the parabolic Cauchy problem is not new, see for example [4], p. 56. Theorem 3 gives a different proof of this fact in our framework.

3. Proofs

Proof of Theorem 1.

Let η be a constant; then $\exp(\int_{t_0}^t c(s)ds)\eta$ is the solution of (2) with initial value η, and by monotonicity,

$$u_0 \le \eta \Rightarrow u(t) \le \exp\left(\int_{t_0}^t c(s)ds\right)\eta \qquad (t \in [t_0, T)).$$

From this, the proposition readily follows. □

Proof of Theorem 2.

Consider the entire function

$$\varphi(z) = \frac{\sin^2(z)}{\pi z^2} \ \ (z \neq 0), \qquad \varphi(0) = \frac{1}{\pi}.$$

The function φ is of exponential type 2 and $\int_{-\infty}^{\infty} \varphi(x)dx = 1$ (see for example [3], p. 641). For any $\tau > 0$ let φ_τ be defined by $\varphi_\tau(z) = \tau\varphi(\tau z)$, and note that φ_τ is of exponential type 2τ and $\int_{-\infty}^{\infty} \varphi_\tau(x)dx = 1$. Hence Φ_τ defined by

$$\Phi_\tau(x_1, \dots, x_n) = \prod_{j=1}^n \varphi_\tau(x_j)$$

is a function in $B_{2\tau}$ and $\int_{\mathbf{R}^n} \Phi_\tau(x)dx = 1$. For a function $u \in BUC(\mathbf{R}^n, \mathbf{R})$ we set

$$u_\tau(x) = \int_{\mathbf{R}^n} u(\xi)\Phi_\tau(x - \xi)d\xi \qquad (x \in \mathbf{R}^n).$$

First we will prove $u_\tau \in B_{2\tau}$. Obviously $|u_\tau(x)| \le \|u\|_\infty$ $(x \in \mathbf{R}^n)$. We fix $n-1$ real variables and consider

$$h(z) = \int_{\mathbf{R}^n} u(\xi)\varphi_\tau(x_1 - \xi_1) \cdot \cdots \cdot \varphi_\tau(z - \xi_k) \cdot \cdots \cdot \varphi_\tau(x_n - \xi_n)d\xi$$

The function h is an entire function, bounded on $\mathbf{R}$ (by $\|u\|_\infty$), and for $z = a + ib$, $b \neq 0$ we have

$$|h(z)| \le \tau\|u\|_\infty \int_{\mathbf{R}} |\varphi(\tau(z - \xi_k))|d\xi_k = \frac{\|u\|_\infty}{\pi}\int_{\mathbf{R}} \frac{\sin^2\eta + \sinh^2(\tau b)}{\eta^2 + \tau^2 b^2}\, d\eta$$

$$\le \|u\|_\infty \left(1 + \frac{\sinh^2(\tau b)}{|\tau b|}\right).$$

Hence h is of exponential type 2τ, bounded on $\mathbf{R}$, and according to Bernstein's Theorem we have

$$|D_j^{\alpha_j} u_\tau(x)| \le (2\tau)^{\alpha_j} \|u\|_\infty \qquad (x \in \mathbf{R}^n, \alpha_j \in \mathbf{N}_0).$$

Hence $u_\tau \in B_{2\tau}$.

To show that u_τ approximates u, let $\varepsilon > 0$ and choose $\delta > 0$ such that $\|x - y\| < \delta$ implies $|u(x) - u(y)| \le \varepsilon/2$. For $x \in \mathbf{R}^n$ we have

$$\begin{aligned}
|u(x) - u_\tau(x)| &= \left| \int_{\mathbf{R}^n} (u(x) - u(\xi)) \Phi_\tau(x - \xi) d\xi \right| \\
&\le \frac{\varepsilon}{2} + 2\|u\|_\infty \int_{\|x-\xi\| \ge \delta} \tau^n \varphi(\tau(x_1 - \xi_1)) \cdot \dots \cdot \varphi(\tau(x_n - \xi_n)) d\xi \\
&= \frac{\varepsilon}{2} + 2\|u\|_\infty \int_{\|\eta\| \ge \delta\tau} \varphi(\eta_1) \cdot \dots \cdot \varphi(\eta_n) d\eta < \varepsilon
\end{aligned}$$

for $\tau \ge \tau_1$ if τ_1 is such that

$$\int_{\|\eta\| \ge \delta\tau_1} \varphi(\eta_1) \cdot \dots \cdot \varphi(\eta_n) d\eta < \frac{\varepsilon}{4\|u\|_\infty}.$$

This is possible since $|\varphi|$ is integrable. □

PROOF OF THEOREM 3.

1) To each $u_0 \in B_\infty$, according to Theorem 2 there exists a unique solution u of $u'(t) = L(t)u(t)$, $u(t_0) = u_0$; we define $U : C \times B_\infty \to B_\infty$ by

$$U(t, t_0)u_0 = u(t), \qquad (t, t_0) \in C, \qquad u_0 \in B_\infty.$$

Clearly U is continuous and linear with respect to the third variable, and according to Theorem 1 we have

$$\|U(t, t_0)u_0\|_\infty \le \exp\left(\int_{t_0}^t c(s) ds \right) \|u_0\|_\infty.$$

Hence $U(t, t_0)$ admits a unique continuous linear extension to $BUC(\mathbf{R}^n, \mathbf{R})$, also denoted by $U(t, t_0)$, such that

$$|||U(t, t_0)||| \le \exp\left(\int_{t_0}^t c(s) ds \right) \qquad (t_0 \le t < T).$$

Finally it is easy to see that U is continuous.

2) and 3) follow immediately from the definition of U and the properties of the solution of problem 2 in the spaces B_τ. □

Proof of Theorem 4.

As f is uniformly continuous on compact subsets of its domain of definition, $s \mapsto f(s, u(s))$ is continuous if $s \mapsto u(s)$ is a continuous function from $[0, T)$ to $BUC(\mathbf{R}^n, \mathbf{R})$. Under the given conditions, the right hand side of our integral equation defines a map J from $C([0, T), BUC(\mathbf{R}^n, \mathbf{R}))$ into itself which turns out to be a contraction on $C([0, T_1], BUC(\mathbf{R}^n, \mathbf{R}))$ for each $T_1 < T$ if a norm in this space is suitably chosen.

To see this, let $M = C([0, T_1], BUC(\mathbf{R}^n, \mathbf{R}))$ and

$$\|u\|_M = \sup_{t \in [0, T_1]} \|u(t)\|_\infty \exp\left(-\int_0^t (c(s) + l(s))ds\right).$$

It is easily seen that $(M, \|\cdot\|_M)$ is a Banach space. For $u, v \in M$ we get

$$\begin{aligned}
&\|J(u)(t) - J(v)(t)\|_\infty \\
&\quad \le \int_0^t \exp\left(\int_s^t c(\sigma)d\sigma\right) l(s)\|u(s) - v(s)\|_\infty ds \\
&\quad \le \int_0^t \exp\left(\int_s^t c(\sigma)d\sigma\right) l(s)\|u - v\|_M \exp\left(\int_0^s (c(\sigma) + l(\sigma))d\sigma\right) ds \\
&\quad = \exp\left(\int_0^t (c(s) + l(s))ds\right)\|u - v\|_M\left(1 - \exp\left(-\int_0^t l(s)ds\right)\right),
\end{aligned}$$

so

$$\|J(u) - J(v)\|_M \le \|u - v\|_M\left(1 - \exp\left(-\int_0^{T_1} l(s)ds\right)\right),$$

and Banach's Fixed Point Theorem gives exactly one solution in M. Since T_1 was arbitrary, we get a unique solution in $C([0, T), BUC(\mathbf{R}^n, \mathbf{R}))$, which a forteriori depends continuously on u_0. As to monotonicity, as a consequence of the considerations before Theorem 4, (4) is equivalent to

$$u(t) = \tilde{U}(t, t_0)u_0 + \int_0^t \tilde{U}(t, s)(l(s)u(s) + f(s, u(s)))ds.$$

The respective succesive approximations for this equation, starting with $u_0 \le v_0$ converge, and since this inequality is maintained for the iteration, we get $u \le v$ on $[0, T)$ for the respective solutions. □

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