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The Hausdorff dimension of deformed self-similar sets

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Abstract. We define deformed self-similar sets which are generated by a sequence of similar contraction mappings $\{\phi_\sigma : \sigma \in S^*\}$ on $\mathbf{R}^\mathbf{d}$, ϕ_σ having its contraction ratio r_σ, and calculate thier Hausdorff dimension.

1. Introduction

Hutchinson [4] proved that there exists a unique compact set $F \subset \mathbf{R}^d$ such that $F = \bigcup_{i=1}^n \phi_i(F)$ for any given finite set $\{\phi_i\}_{i=1}^n$ of similarities in $\mathbf{R}^d$ with ratio r_i, $1 \le i \le n$. He also showed that $\dim_H F = \dim_B F = \dim_p F = s$ and $\sum_{i=1}^n r_i^s = 1$ if $\{\phi_i\}_{i=1}^n$ satisfies the open set condition, that is, there exists a bounded non-empty open set O such that $\bigcup_{i=1}^n \phi_i(O) \subset O$ and $\phi_i(O) \cap \phi_j(O) = \emptyset$ if $i \ne j$.

Recently, S. Ikeda [5] defined the loosely self-similar set F which is generated by a sequence of mappings $\{\phi_{i_1 i_2 \dots i_k}\}$ ($i_j \in \{1, 2, \dots, n\}$), $\phi_{i_1 i_2 \dots i_k}$ having its contraction ratio r_{i_k}, and showed that $\dim_H F = s$ and $\sum_{i=1}^n r_i^s = 1$ if $\{\phi_{i_1 i_2 \dots i_k}\}$ satisfies the disjoint condition.

In this paper, we will generalize loosely self-similar sets [5] and perturbed Cantor sets [1]. The construction is as follows.

Fix $m \ge 2$, write $S_k = \{1, 2, \dots, m\}^k$ and $S^* = \bigcup_{k=1}^\infty S_k$. Consider a sequence of similar contraction mappings $\{\phi_\sigma : \sigma \in S^*\}$ on $\mathbf{R}^\mathbf{d}$. Suppose that each ϕ_σ has a contraction ratio r_σ, that is, $|\phi_\sigma(x) - \phi_\sigma(y)| = r_\sigma |x - y|$ for any $x, y \in \mathbf{R}^\mathbf{d}$, where $|\cdot|$ is the Euclidean norm. We further assume there exists $0 < \alpha, \beta < 1$ such that $\alpha < r_\sigma < \beta$ for any $\sigma \in S^*$ and there exists a bounded open set $V \subset \mathbf{R}^\mathbf{d}$ such that

(1) $\phi_\sigma(V) \subset V$ for any $\sigma \in S^*$

(2) $\phi_{i_1 i_2 \dots i_{k-1} i_k}(V) \cap \phi_{i_1 i_2 \dots i_{k-1} i_k'}(V) = \emptyset$, $i_k \ne i_k'$.

It is obvious that there exists a non-empty compact set $X \subset V$ such that the properties (1) and (2) are satisfied when V is replaced by X.

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For brevity, we write

$$\Phi_\sigma \equiv \phi_{i_1} \circ \phi_{i_1 i_2} \circ \cdots \circ \phi_{i_1 i_2 \dots i_k}$$

$$R_\sigma \equiv r_{i_1} r_{i_1 i_2} \dots r_{i_1 i_2 \dots i_k}$$

for any $\sigma = i_1 i_2 \dots i_k \in S_k$. Then using a compact set X given above we obtain a unique compact set K,

$$K = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in S_k} \Phi_\sigma(X).$$

We call this K *a deformed self-similar set*. If we take $r_\sigma = r_{i_k}$ for any $\sigma = i_1 i_2 \dots i_k$, the obtained set K becomes a loosely self-similar set in S. Ikeda's sense. Moreover, if we take $\phi_{i_1 i_2 \dots i_k} = \phi_{i_k}$ for all k, the obtained set K is a self-similar set in Hutchinson's sense.

2. Preliminaries and main theorem

We begin to recall the well-known s-dimensional Hausdorff measure and dimension: Let E be a bounded subset of $\mathbf{R^d}$ and $s \geq 0$. For every $\delta > 0$ we define

$$H^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : E \subset \bigcup_i U_i, |U_i| \leq \delta\right\},$$

where $|A|$ denotes the diameter of A. Then we obtain the s-dimensional Hausdorff measure of E by

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E).$$

The Hausdorff dimension of E is defined by

$$\dim_H E = \sup\{s \geq 0 : H^s(E) = \infty\}.$$

Then we see that

$$\dim_H E = \inf\{s \geq 0 : H^s(E) = 0\}.$$

To calculate the Hausdorff dimension of K we recall a measure on subsets of K due to Rogers [7]. For any $s \geq 0$, $n \in \mathbf{N}$ and $E \subset K$, let

$$M^s_n(E) = \inf\left\{\sum |\Phi_\sigma(X)|^s : E \subset \bigcup_\sigma \Phi_\sigma(X), \sigma \in \bigcup_{k \geq n} S_k\right\}.$$

Then we get a measure of Hausdorff type on subsets of K by letting

$$M^s(E) = \lim_{n \to \infty} M^s_n(E).$$

We note that if $M^t(E) > 0$ then $M^s(E) = \infty$ for all $s < t$. Therefore we see that

$$\sup\{s \geq 0 : M^s(E) = \infty\} = \inf\{s \geq 0 : M^s(E) = 0\}.$$

Now we are ready to show a method to get the Hausdorff dimension of a deformed self-similar set.

THEOREM 2.1. *Let K be a deformed self-similar set as defined in the introduction. Put* $\alpha = \sup\{s \geq 0 \mid M^s(K) = \infty\}$. *Then we have* $0 < H^\alpha(K) < \infty$ *and*

$$\dim_H K = \alpha.$$

PROOF. By the definition of $\alpha, M^s(K) < \infty$ for any $s > \alpha$. Thus, for such an s, $H^s_\delta(K) \leq M^s_n(K) \leq M^s(K)$ if $\delta \geq \beta^n$, where β is an upper bound of contraction ratios r_σ. Letting $\delta \to 0$, we have

$$H^s(K) \leq M^s(K) < \infty \qquad \text{for } s > \alpha,$$

which implies $\dim_H K \leq \alpha$.

To prove that $\dim_H K \geq \alpha$, we claim that $s \leq \dim_H K$, for all $s < \alpha$. For $0 < s < \alpha$, $M^s(K) = \infty$ by the definition of α. Then there exists a compact subset $F \subset K$ and a constant $b > 0$ such that $0 < M^s(F) < \infty$ and $M^s(F \cap \Phi_\sigma(X)) \leq bR^s_\sigma$ for all $\sigma \in S^*$. (See Proposition 3.1 [3].) Define a Borel measure μ by

$$\mu(A) = M^s(F \cap A) \qquad \text{for } A \subset K.$$

Then $0 < \mu(K) < \infty$ and $\mu(\Phi_\sigma(X)) \leq bR^s_\sigma$ for all $\sigma \in S^*$.

Let V be the open set given in the introduction and let $U \subset \mathbf{R}^\mathbf{d}$ satisfy $0 < |U| \leq |V|$. Set

$$Q = \{\sigma \in S^* : |\Phi_\sigma(V)| < |U| \text{ and } |\Phi_{\sigma\,|\,(|\sigma|-1)}(V)| \geq |U|\},$$

where $\sigma|l = i_1 i_2 \dots i_l$ for $\sigma = i_1 i_2 \dots i_k$ and $l \leq k$. We see that $\alpha|U| < |\Phi_\sigma(V)| < |U|$ for $\sigma \in Q$. Then $Q_o = \{\sigma \in Q : U \cap \Phi_\sigma(\overline{V}) \neq \emptyset\}$ has at most some finite m_o elements which is independent of U since $\{\Phi_\sigma(V) \mid \sigma \in Q_o\}$ is disjoint. (See Lemma 9.2 [2].)

Hence

$$\begin{aligned}\mu(U) &\leq \sum_{\sigma \in Q_o} \mu(\Phi_\sigma(X)) \\ &\leq b \sum_{\sigma \in Q_o} R^s_\sigma \\ &\leq bm_o|V|^{-s}|U|^s\end{aligned}$$

which, by Mass distribution principle 4.2 [2], implies that $H^s(K) > 0$. That is, $\dim_H K \geq s$ for any $s < \alpha$. Therefore $\dim_H K \geq \alpha$. □

Remark (cf. [3]). Let K be a non-empty compact set generated by relaxing equality (3) to inclusion, that is,

$$K \subset \bigcap_{k=1}^{\infty} \bigcup \Phi_{\sigma}(K).$$

Then it can be proved that

$$\dim_H K = \sup\{s \geq 0 : M^s(K) = \infty\}$$

by the same proof as in Theorem 2.1.

Corollary 2.2 (I. S. Baek [1]). *Put $X = [0,1]$. Define a sequence of contraction maps $\{\phi_{i_1 i_2 \dots i_k}\}$ on X for $i_1 i_2 \dots i_k \in \{1,2\}^k$, $k = 1,2,\dots,$ such that*

$$\phi_{i_1 i_2 \dots i_k}(x) = \begin{cases} r_1^{(k)} x, & \text{if } i_k = 1 \\ r_2^{(k)} x + (1 - r_2^{(k)}), & \text{if } i_k = 2 \end{cases}$$

and there exist $0 < \alpha, \beta < 1$ such that

$$\alpha < r_i^{(k)} < \beta, \qquad \text{and} \qquad \alpha < 1 - \sum_{i=1}^{2} r_i^{(k)} < \beta$$

for all k. Put

$$K = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in S_k} \Phi_{\sigma}(X).$$

Then K is a perturbed Cantor set and

$$\begin{aligned} \dim_H K &= \sup\{s \geq 0 : M^s(K) = \infty\} \\ &= \sup\left\{ s \geq 0 : \liminf_{n \to \infty} \sum_{i_1 i_2 \dots i_n \in \{1,2\}^n} (r_{i_1}^{(1)} r_{i_2}^{(2)} \dots r_{i_n}^{(n)})^s = \infty \right\}. \end{aligned}$$

Proof. Let

$$\begin{aligned} b &= \sup\left\{ s \geq 0 : \liminf_{n \to \infty} \sum_{i_1 i_2 \dots i_n \in \{1,2\}^n} (r_{i_1}^{(1)} r_{i_2}^{(2)} \dots r_{i_n}^{(n)})^s = \infty \right\} \\ &= \sup\left\{ s \geq 0 : \liminf_{n \to \infty} \sum_{\sigma \in \{1,2\}^n} |\Phi_{\sigma}(X)|^s = \infty \right\}. \end{aligned}$$

To show that $\dim_H K \leq b$, suppose that $0 < s < \dim_H K$. Then $\lim_{n \to \infty} M_n^s(K) = \infty$, that is, for given L, there exists n_o such that $M_n^s(K) \geq L$ for $n \geq n_o$. Hence

$\sum_{\sigma \in \{1,2\}^n} |\Phi_\sigma(X)|^s > L$ for $n \geq n_o$. Therefore $\liminf_{n\to\infty} \sum_{\sigma \in \{1,2\}^n} |\Phi_\sigma(X)|^s = \infty$, so $s \leq b$. Now suppose $s < b$. Then $\liminf_{n\to\infty} \sum_{\sigma \in \{1,2\}^n} |\Phi_\sigma(X)|^s = \infty$. Define a finite Borel measure μ on X such that

$$\mu(\Phi_\sigma(X)) = \frac{|\Phi_\sigma(X)|^s}{\sum_{\sigma \in \{1,2\}^n} |\Phi_\sigma(X)|^s}.$$

Then $\mu(\Phi_\sigma(X)) = \sum_{i=1}^{2} \mu(\Phi_{\sigma i}(X))$ and this measure μ is supported on K, thus, $\mu(K) = 1$. Since $s < b$, it is easy to see that from the definition of b that

$$\liminf_{n\to\infty} \frac{|\Phi_{i_1 i_2 \ldots i_n}(X)|^s}{\mu(\Phi_{i_1 i_2 \ldots i_n}(X))} = \liminf_{n\to\infty} \sum_{\sigma \in \{1,2\}^n} |\Phi_\sigma(X)|^s = \infty$$

for all $x \in K$, where $i_1 i_2 \ldots i_n \ldots$ is the uniquely obtained sequence such that $x = \lim_{n\to\infty} \Phi_{i_1 i_2 \ldots i_n}(X)$. Hence there exists $c > 0$ such that

$$H^s(K) \geq c \cdot \mu(K) \inf_{x \in K} \liminf_{n\to\infty} \frac{|\Phi_{i_1 i_2 \ldots i_n}(X)|^s}{\mu(\Phi_{i_1 i_2 \ldots i_n}(X))} = \infty.$$

(See Theorem 2.1 [8] and Lemma 2.2 [6].)

Therefore $M^s(K) \geq H^s(K) = \infty$. So, $M^s(K) = \infty$, which implies $b \leq \dim_H K$. □

Corollary 2.3 (S. Ikeda [5]). *Let K be a deformed self-similar set such that $r_{i_1,\ldots,i_k} = r_{i_k}$ for all $\sigma = i_1 \ldots i_k \in S^*$. Then K is a loosely self-similar set and*

$$\dim_H K = \sup\{s \geq 0 : M^s(K) = \infty\}$$

$$= d \text{ with } \sum_{i=1}^{m} r_i^d = 1.$$

Proof. The proof is similar to that of Corollary 2.2. □

Open question: *Let K be a deformed self-similar set. We wonder if the packing dimension* [8] *of K is equal to*

$$\sup\Big\{s \geq 0 : \lim_{n\to\infty} M_n^s(K) = \infty\Big\}$$

where $M_n^s(K) = \sup\Big\{\sum |\Phi_\sigma(X)|^s : \Phi_\sigma(X) \cap \Phi_{\sigma'}(X) = \emptyset,\ \sigma \neq \sigma' \text{ and } \sigma, \sigma' \in \bigcup_{k \geq n} S_k\Big\}$.

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