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On the conjugacy theorem of Cartan subalgebras

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Abstract. We shall give a new elementary proof for the conjugacy theorem of Cartan subalgebras of a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero.

1. Introduction

The aim of this paper is to give a new proof for the conjugacy theorem of Cartan subalgebras in a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero.

The known proofs of this theorem use geometric methods (algebraic geometry or analytic methods in the case the base field is the complex number field) or elementary but complicated inductive arguments (see [1], [4], [5], [6], [7], [8], [9]). In this paper we shall give an elementary, direct proof for this theorem which fits into the theme of the presentation in Bourbaki [4]. From our arguments the conjugacy theorem of Borel subalgebras is also deduced.

Let us go into the detail. For the basic theory of Lie algebras we refer to Bourbaki [2], [3], [4]. We fix the following notations and definitions which will be used throughout the paper. We fix a (commutative) field F which is algebraically closed and of characteristic zero. All Lie algebras are finite-dimensional and are defined over F. For a Lie algebra L its center is denoted by $Z(L)$. For a subalgebra C of L we denote its normalizer in L by $N_L(C)$. A subalgebra of a Lie algebra is called a Cartan subalgebra if it is a self normalizing nilpotent Lie algebra, and is called a Borel subalgebra if it is a maximal solvable subalgebra. For a Lie algebra L we denote its automorphism group by $\mathrm{Aut}(L)$. We denote by $\mathrm{Aut}_e(L)$ the subgroup of $\mathrm{Aut}(L)$ generated by the elements of the form $\exp(\mathrm{ad}(x))$ where $x \in L$ is such that $\mathrm{ad}(x) \in \mathrm{End}(L)$ is nilpotent.

Suppose that H is a nilpotent subalgebra of a Lie algebra L and let

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$$L = \sum_{\alpha \in H^*} L^{\alpha}(H)$$

be the primary decomposition (the weight space decomposition) of L with respect to H. We denote by $E_L(H)$ the subgroup of $\mathrm{Aut}_e(L)$ generated by the elements of the form $\exp(\mathrm{ad}(x))$ where $x \in L^{\alpha}(H)$ for some $\alpha \neq 0$.

Our main results are as follows.

Proposition 1. *Let L be Lie algebra, and let C_1, C_2 be Cartan subalgebras of L. Then we have $E_L(C_1) = E_L(C_2)$.*

Hence the subgroup $E_L(C)$ of $\mathrm{Aut}(L)$ do not depend on the choice of a Cartan subalgebra C of a Lie algebra. We shall denote it by E_L in the following.

Theorem 2. *Let L be a Lie algebra. Then the group E_L acts transitively on the set of Cartan subalgebras of L.*

Theorem 3. *Let L be a Lie algebra. Then the intersection of two Borel subalgebras of L contains a Cartan subalgebra of L.*

In particular, any Borel subalgebra of a Lie algebra L contains a Cartan subalgebra of L. By Theorem 2 and a standard argument involving the Weyl group we also obtain the following.

Corollary 4. *Let L be a Lie algebra. Then the group E_L acts transitively on the set of Borel subalgebras of L.*

2. Cartan subalgebras

Let L be a Lie algebra.

For $x \in L$ set

$$L^0(x) = L^0(Fx) = \{a \in L : \mathrm{ad}(x)^n(a) = 0 \text{ for a sufficiently large } n\}.$$

We define the rank $\mathrm{Rk}(L)$ of L by

$$\mathrm{Rk}(L) = \min\{\dim L^0(x) : x \in L\}.$$

We call an element $x \in L$ regular if $\dim L^0(x) = \mathrm{Rk}(L)$.

We shall use the following fundamental results on the Cartan subalgebras (see [4, Ch. VII, §2]).

Proposition 5. (i) *Let $\varphi : L \to L'$ be a surjective homomorphism of Lie algebras. If C is a Cartan subalgebra of L, then $\varphi(C)$ is a Cartan subalgebra of L'. If C' is a Cartan subalgebra of L', then there exists a Cartan subalgebra C of L such that $\varphi(C) = C'$.*

(ii) *Let x be a regular element of L. Then $L^0(x)$ is a Cartan subalgebra of L.*

(iii) *If C is a Cartan subalgebra of L, then there exists some $x \in C$ such that $C = L^0(x)$.*
(iv) *Assume that L is semisimple. If C is a Cartan subalgebra of L, then C is commutative and consists of semisimple elements.*

We note the following facts on the group $E_L(H)$.

Lemma 6. *Let H be a nilpotent subalgebra of L*
(i) *If L_1 is a subalgebra of L containing H, then any $u \in E_{L_1}(H)$ can be extended to some $v \in E_L(H)$.*
(ii) *Let $\varphi : L \to L_2$ be a surjective homomorphism of Lie algebras. By restricting the canonical group homomorphism $\mathrm{Aut}(L) \to \mathrm{Aut}(L_2)$ we obtain a surjective group homomorphism $E_L(H) \to E_{L_2}(\varphi(H))$.*

Proof. The statement (i) is obvious from the definition, and the statement (ii) follows from [4, Ch. VII, §1, Proposition 9(iv)]. □

The following result is a refinement of [4, Ch. VII, §3, Théorème 3] although the proof is almost identical. We include the proof here for the convenience of readers.

Proposition 7. *Assume that L is solvable. Let C_1, C_2 be Cartan subalgebra of L. Then there exist $u_i \in E_L(C_i)$ $(i = 1, 2)$ such that $u_1(C_1) = u_2(C_2)$.*

Proof. We use induction on $\dim L$. The case $\dim L = 0$ is trivial. Assume that $\dim L > 0$. Since L is solvable, it contains a non-zero ideal. Fix a non-zero ideal N of L with minimal dimension, and let $\varphi : L \to L/N$ be the canonical homomorphism. By Proposition 5 $\varphi(C_i)$ for $i = 1, 2$ are Cartan subalgebras of L/N. By the hypothesis of induction there exist $\tilde{u}_i \in E_{L/N}(\varphi(C_i))$ satisfying $\tilde{u}_1(\varphi(C_1)) = \tilde{u}_2(\varphi(C_2))$. By the surjectivity of $E_L(C_i) \to E_{L/N}(\varphi(C_i))$ we conclude that there exist $v_i \in E_L(C_i)$ satisfying $v_1(C_1) + N = v_2(C_2) + N$. Hence we may assume $C_1 + N = C_2 + N$. If $C_1 + N \subsetneq L$ we can apply the hypothesis of induction for the Lie subalgebra $L' = C_1 + N$. Hence we may assume that $L = C_1 + N = C_2 + N$.

Since $[L, N]$ is an ideal of L contained in N we have $[L, N] = \{0\}$ or $[L, N] = N$ by the minimality of N. If $[L, N] = 0$, we have $[N, C_i] \subseteq [L, N] = \{0\} \subseteq C_i$, and hence $N \subseteq N_L(C_i) = C_i$. In this case we have $C_1 = L = C_2$. Hence we may assume that $[L, N] = N$.

Regard N as an L-module via the adjoint action. It is an irreducible module by the choice of N. By $L = C_i + N$ and $[N, N] = \{0\}$, N is even an irreducible C_i-module. Since $N \cap C_i$ is a C_i-submodule of N, we have $N \cap C_i = N$ or $N \cap C_i = \{0\}$. If $N \cap C_i = N$, then we have $L = C_i + N = C_i$, and hence L is a nilpotent Lie algebra. In this case we have $C_1 = C_2 = L$. Hence we may assume that $N \cap C_1 = N \cap C_2 = \{0\}$.

Define $f : C_1 \to N$ so that $x - f(x) \in C_2$. We see easily that $f([x, y]) = [x, f(y)] - [y, f(x)]$ for any $x, y \in C_1$. Moreover, we have $N \cap L^0(C_1) = N \cap C_1 = \{0\}$. Hence there exists $a \in N$ such that $f(x) = [x, a]$ by [4, Ch. VII, §1, Corollaire to Proposition 9]. By $\mathrm{ad}(a)^2(L) \subseteq [N, [N, L]] \subseteq [N, N] = \{0\}$, we obtain $\exp(\mathrm{ad}(a))(x) = x - [x, a] \in C_2$ for any $x \in C_1$. Hence $(\exp(\mathrm{ad}(a))) \cdot (C_1) \subseteq C_2$. It implies in particular that $\dim C_1 \le \dim C_2$. Hence we have $\dim C_1 = \dim C_2$ by the symmetry, and it follows that $(\exp(\mathrm{ad}(a)))(C_1) = C_2$.

It remains to show $\exp(\mathrm{ad}(a)) \in E_L(C_1)$. Since N is $\mathrm{ad}(C_1)$-stable, we have $N = \sum_\alpha (N \cap L^\alpha(C_1))$. Hence by $L^0(C_1) = C_1$ and $L = C_1 \oplus N$ we obtain $N = \sum_{\alpha \neq 0}(L^\alpha(C_1))$. Write $a = \sum_{\alpha \neq 0} a_\alpha$ where $a_\alpha \in L^\alpha(C_1)$. Then we have

$$\exp(\mathrm{ad}(a)) = \prod_{\alpha \neq 0} \exp(\mathrm{ad}(a_\alpha)) \in E_L(C_1)$$

by $[N, N] = \{0\}$. □

Corollary 8. *Assume that L is solvable. If C_1, C_2 are Cartan subalgebras of L, then we have $E_L(C_1) = E_L(C_2)$.*

Proof. Take u_1, u_2 as in Proposition 7. Then we have

$$\begin{aligned} E_L(C_1) &= u_1 E_L(C_1) u_1^{-1} = E_L(u_1(C_1)) = E_L(u_2(C_2)) = u_2 E_L(C_2) u_2^{-1} \\ &= E_L(C_2). \end{aligned}$$ □

In particular, if L is solvable, then the subgroup $E_L(C)$ of $\mathrm{Aut}_e(L)$, where C is a Cartan subalgebra of L, does not depend on the choice of C. We shall denote this group by E_L (when L is solvable).

Corollary 9. *Assume that L is solvable.*
(i) *The group E_L acts transitively on the set of Cartan subalgebras of L.*
(ii) *Any Cartan subalgebra of L has dimension* $\mathrm{Rk}(L)$.
(iii) *An element $x \in L$ is regular if and only if $L^0(x)$ is a Cartan subalgebra of L, and any Cartan subalgebra is obtained in this manner.*

Proof. The statement (i) is a consequence of Corollary 8 and Proposition 7(ii). The statements (ii) and (iii) follow from (i) and Proposition 5(ii). □

Proposition 10. *Let R be the radical of L, and let $\varphi : L \to L/R$ be the canonical homomorphism. The following statements* (i), (ii) *for Cartan subalgebras C_1, C_2 of L are equivalent.*
(i) *There exist $u_i \in E_L(C_i)$ $(i = 1, 2)$ such that $u_1(C_1) = u_2(C_2)$.*
(ii) *There exist $v_i \in E_{L/R}(\varphi(C_i))$ $(i = 1, 2)$ such that $v_1(\varphi(C_1)) = v_2(\varphi(C_2))$.*

Proof. The implication (i) ⇒ (ii) is obvious. Let us show (ii) ⇒ (i). Since $E_L(C_i) \to E_{L/R}(\varphi(C_i))$ is surjective, we can take $u_i \in E_L(C_i)$ such that

$\varphi(u_1(C_1)) = \varphi(u_2(C_2))$. Set $T = u_1(C_1) + R = u_2(C_2) + R$. Since T is a solvable Lie algebra and since $u_i(C_i)$ are Cartan subalgebras of T, we can take $u_i' \in E_T(u_i(C_i))$ such that $u_1'u_1(C_1) = u_2'u_2(C_2)$. Then we have $u_i'u_i \in E_L(C_i)$ by

$$u_i'u_i \in E_L(u_i(C_i))u_i = u_iE_L(C_i) = E_L(C_i). \qquad \square$$

Lemma 11. *Assume that L is semisimple. Let C be a Cartan subalgebra of L and let B be a solvable subalgebra of L containing C.*

(i) *We have $B = C \oplus [B, B]$.*

(ii) *The set $[B, B]$ coincides with the set of nilpotent elements in L contained in B.*

Proof. By $L^0(C) = C$ we have $B^0(C) = C$, and hence $B = C \oplus \sum_{\alpha \neq 0} B^{\alpha}(C)$. Set $B^+(C) = \sum_{\alpha \neq 0} B^{\alpha}(C)$. We can take $s \in C$ satisfying $C = L^0(s)$ by Proposition 5(iii). Since $\mathrm{ad}(s) \,|\, B^+(C)$ is bijective, we have $B^+(C) \subseteq [B, B]$.

Now $[B, B]$ consists of nilpotent elements by Lie's theorem, and C consists of semisimple elements by Proposition 5(iv). In particular, we have $[B, B] \cap C = \{0\}$. Hence (i) follows from $B = C \oplus B^+(C)$ and $B^+(C) \subseteq [B, B]$. By Lie's theorem the set of nilpotent elements in B is a subspace of B. Hence (ii) follows from (i) and the facts that elements of $[B, B]$ are nilpotent and elements of C are semisimple. $\square$

If L is a semisimple Lie algebra, we denote by κ the Killing form on L, and for any vector subspace S of L we denote by $S^{\perp}$ the vector subspace of L which is orthogonal to S with respect to κ.

Lemma 12. *Assume that L is semisimple. Then a subalgebra B of L is a Borel subalgebra of L if and only if $[B, B] = B^{\perp}$.*

Proof. Note that a subalgebra H of L is solvable if and only if $[H, H] \subseteq H^{\perp}$ by Cartan's criterion [2, Ch. I, §5, Théorème 2].

Suppose that B is a Borel subalgebra, and assume that $[B, B] \subsetneq B^{\perp}$. Then we have $B \subsetneq [B, B]^{\perp}$. By applying Lie's theorem to the B-module $[B, B]^{\perp}/B$ we see that there exists $x \in [B, B]^{\perp} \backslash B$ such that $[B, x] \subseteq B + Fx$. Set $B_1 = B + Fx$. We see easily that B_1 is a subalgebra of L. Let us show $[B_1, B_1] \subseteq B_1^{\perp}$. Note $B_1^{\perp} = (B + Fx)^{\perp} = B^{\perp} \cap (Fx)^{\perp}$ and $[B_1, B_1] = [B, B] + [B, x]$. Since B is solvable, we have $[B, B] \subseteq B^{\perp}$. Since $x \in [B, B]^{\perp}$, we have $[B, B] \subseteq (Fx)^{\perp}$. For $a, b \in B$ we have $\kappa([a, x], b) = -\kappa(x, [a, b]) = 0$ by $x \in [B, B]^{\perp}$, and hence $[B, x] \subseteq B^{\perp}$. For $a \in B$ we have $\kappa([a, x], x) = \kappa(a, [x, x]) = 0$, and hence $[B, x] \subseteq (Fx)^{\perp}$. The statement $[B_1, B_1] \subseteq B_1^{\perp}$ is proved. Hence B_1 is a solvable subalgebra of L. This contradicts the assumption that B is a Borel subalgebra. Hence $[B, B] = B^{\perp}$.

Suppose that $[B, B] = B^{\perp}$. Let B_1 be a solvable subalgebra of L containing B. Then we have

$$[B_1, B_1] \subseteq B_1^{\perp} \subseteq B^{\perp} = [B, B] \subseteq [B_1, B_1].$$

It follows that $B_1^{\perp} = B^{\perp}$ and hence $B_1 = B$. Hence B is a Borel subalgebra. □

Lemma 13. *Assume that L is semisimple. Let C_1, C_2 be Cartan subalgebras of L, and let B_1, B_2 be Borel subalgebras of L containing C_1, C_2 respectively.*
(i) *$B_1 \cap B_2$ contains a Cartan subalgebra of L.*
(ii) *There exist $u_i \in E_L(C_i)$ $(i = 1, 2)$ such that $u_1(C_1) = u_2(C_2)$.*

Proof. Set $N_i = [B_i, B_i]$. By Lemma 11 and Lemma 12 we have $B_i = C_i \oplus N_i$ and $B_i = N_i^{\perp}$. Moreover, we have $B_1 \cap N_2 = N_1 \cap N_2 = N_1 \cap B_2$ by Lemma 11 (ii). By

$$B_1 = N_1^{\perp} \subseteq (N_1 \cap N_2)^{\perp} = (B_1 \cap N_2)^{\perp} = B_1^{\perp} + N_2^{\perp} = N_1 + B_2$$

we have $B_1 = N_1 + (B_1 \cap B_2)$.

Set $r_i = \dim C_i$. By the symmetry we may assume $r_1 \leq r_2$. Moreover, we have $r_i = \mathrm{Rk}(B_i)$ by Corollary 9. We can take some $z \in C_1$ such that $C_1 = L^0(z)$ by Proposition 5 (iii). Take $n \in N_1$ such that $w = z + n \in B_1 \cap B_2$. Then we have

$$r_1 = \dim L^0(z) = \dim L^0(w) \geq \dim B_i^0(w) \geq \mathrm{Rk}(B_i) = r_i \geq r_1,$$

where the equality $\dim L^0(z) = \dim L^0(w)$ follows from Lie's theorem. Hence we obtain

$$r_1 = r_2 = \dim L^0(z) = \dim L^0(w) = \dim B_i^0(w) = \mathrm{Rk}(B_i).$$

It follows that all Cartan subalgebras of L have the same dimension $r_1 = r_2 = \mathrm{Rk}(L)$. Hence we see from Proposition 5(ii) that a subalgebra C of L is a Cartan subalgebra of L if and only if $C = L^0(x)$ for some $x \in L$ and $\dim C = \mathrm{Rk}(L)$. In particular, $L^0(w)$ is a Cartan subalgebra of L. Moreover, by $\dim L^0(w) = \dim B_i^0(w)$ we have $L^0(w) = B_i^0(w) \subseteq B_1 \cap B_2$. The statement (i) is proved. Since $B_i^0(w)$ is a Cartan subalgebra of B_i, we can take some $u_i \in E_{B_i}(C_i)$ such that $u_1(C_1) = L^0(w) = u_2(C_2)$ by Corollary 9. We can regard u_i as an element of $E_L(C_i)$ since any $u \in E_{B_i}(C_i)$ can be extended to an element of $E_L(C_i)$. The statement (ii) is proved. □

By Proposition 10 and Lemma 13 (ii) we obtain the following.

Corollary 14. *Let C_1, C_2 be Cartan subalgebra of L. Then there exist $u_i \in E_L(C_i)$ $(i = 1, 2)$ such that $u_1(C_1) = u_2(C_2)$.*

Similarly to Corollary 8, Corollary 9 we have the following.

THEOREM 15. (i) *The subgroup* $E_L(C)$ *of* $\mathrm{Aut}_e(L)$, *where* C *is a Cartan subalgebra of* L, *does not depend on the choice of* C *(We shall denote it by* E_L *in the following).*
(ii) *The group* E_L *acts transitively on the set of Cartan subalgebras of* L.
(iii) *Any Cartan subalgebra of* L *has dimension* $\mathrm{Rk}(L)$.
(iv) *An element* $x \in L$ *is regular if and only if* $L^0(x)$ *is a Cartan subalgebra of* L, *and any Cartan subalgebra is obtained in this manner.*

A subalgebra S of a Lie algebra L is called an Engel subalgebra if $S = L^0(x)$ for some $x \in L$. An Engel subalgebra S of L is called a minimal Engel subalgebra if it does not properly contain any Engel subalgebra of L.

The following corollary relates our result with the work [1].

COROLLARY 16. *Cartan subalgebras of* L *are precisely minimal Engel subalgebras of* L.

PROOF. By the definition of $\mathrm{Rk}(L)$ any Engel subalgebra has dimension $\geq \mathrm{Rk}(L)$. Hence any Cartan subalgebra is a minimal Engel subalgebra by Theorem 15 (iv). Assume that S is a minimal Engel subalgebra. Then S contains a Cartan subalgebra C of L by [4, Ch. VII, §2, Proposition 11]. Since C is an Engel subalgebra, we have $S = C$ by the minimality of S. □

3. Borel subalgebras

We first establish the following.

THEOREM 17. *Any Borel subalgebra of* L *contains a Cartan subalgebra of* L.

PROOF. Let R be the radical of L, and let $\varphi : L \to L/R$ be the canonical homomorphism. For any Cartan subalgebra C' of L/R there exists a Cartan subalgebra C of L such that $\varphi(C) = C'$ by Proposition 5. In particular, the natural group homomorphism $E_L \to E_{L/R}$ is surjective. Taking into account of these facts and the fact that Borel subalgebras of L are in one-to-one correspondence with the Borel subalgebras of L/R we see easily that we have only to show the corresponding statement for L/R. Hence we assume that L is semisimple in the rest of the proof.

Let B be a Borel subalgebra of L and let C be a Cartan subalgebra of B. We shall prove that C is a Cartan subalgebra of L by showing $N_L(C) = C$.

We first show that for any $c \in C$ its semisimple part c_s and its nilpotent part c_n (as an element of L) are elements of C and that $c_s \in C \cap Z(L^0(C))$. Note that $\mathrm{ad}(c_s)$ acts on each weight space $L^\alpha(C)$ by the scalar multiplication of $\alpha(c)$. In particular, we have $[c_s, L^0(C)] = 0$. By $C \subseteq L^0(C)$ we have only to show $c_s \in C$. Since $\mathrm{ad}(c_s)$ is a polynomial in $\mathrm{ad}(c)$, we have $c_s \in N_L(C) \cap N_L(B)$. We have $N_L(B) = B$ by Lie's theorem, and we have $B \cap N_L(C) = C$ since C is a Cartan subalgebra of B. We obtain $c_s \in N_L(C) \cap N_L(B) = N_L(C) \cap B = C$.

Observe that $L^0(C)$ is reductive by [4, Ch. VII, §1, Proposition 112], so that $L^0(C) = Z(L^0(C)) \oplus [L^0(C), L^0(C)]$ by [2, Ch. 1, §6, Proposition 5]. By $C \subseteq L^0(C)$ we have $[Z(L^0(C)), C] = 0$ and hence $Z(L^0(C)) \subseteq N_L(C)$. Moreover, we have $N_L(C) \subseteq L^0(C)$. In fact, for $x \in N_L(C)$, $c \in C$ we have $\mathrm{ad}(c)(x) \in C$. Since C is nilpotent, we have $\mathrm{ad}(c)^n(\mathrm{ad}(c)(x)) = 0$ for a sufficiently large n, and we obtain $x \in L^0(C)$. It follows from $Z(L^0(C)) \subseteq N_L(C) \subseteq L^0(C)$ that

$$N_L(C) = Z(L^0(C)) \oplus (N_L(C) \cap [L^0(C), L^0(C)]). \tag{1}$$

Hence we have only to show

$$Z(L^0(C)) \subseteq C, \tag{2}$$

$$N_L(C) \cap [L^0(C), L^0(C)] \subseteq C. \tag{3}$$

Let us show

$$[B, B] \subseteq (Z(L^0(C)))^{\perp}. \tag{4}$$

Set $B^+(C) = \sum_{\alpha \neq 0} B^{\alpha}(C)$. Since C is a Cartan subalgebra of B, we have $B = C \oplus B^+(C)$. We can take $c \in C$ satisfying $\alpha(c) \neq 0$ for any $\alpha \in C^* \setminus \{0\}$ such that $B^{\alpha}(C) \neq \{0\}$. Since $\mathrm{ad}(c)$ acts bijectively on $B^+(C)$, we have $B^+(C) \subseteq [B, B]$, and hence $[B, B] = (C \cap [B, B]) \oplus B^+(C)$. By a standard argument we see that $\kappa(L^{\alpha}(C), L^{\beta}(C)) = 0$ unless $\alpha + \beta = 0$, and in particular, we have $B^+(C) \subseteq (Z(L^0(C)))^{\perp}$. Hence we have to show $C \cap [B, B] \subseteq (Z(L^0(C)))^{\perp}$. We have $[Z(L^0(C)), C \cap [B, B]] = \{0\}$ by $C \cap [B, B] \subseteq C \subseteq L^0(C)$. Moreover, $C \cap [B, B]$ consists of nilpotent elements by Lie's theorem. It follows that $\mathrm{ad}(x)\,\mathrm{ad}(y)$ is nilpotent for any $x \in Z(L^0(C))$, $y \in C \cap [B, B]$. Hence we have $C \cap [B, B] \subseteq (Z(L^0(C)))^{\perp}$ by the definition of the Killing form κ. (4) is proved.

By (4) and Lemma 12 we have $[Z(L^0(C)), B] \subseteq B^{\perp} \subseteq B$, and hence $Z(L^0(C)) \subseteq N_L(B) \cap N_L(C) = B \cap N_L(C) = C$. (2) is proved.

Let us show (3). Let $z \in N_L(C) \cap [L^0(C), L^0(C)]$ and $c \in C$. Then we have $[C + Fz, C + Fz] \subseteq C$, and hence $C + Fz$ is a solvable subalgebra of L. Since c_n is a nilpotent element belonging to C, we have $\kappa(z, c_n) = 0$ by Lie's theorem. Since c_s belongs to $Z(L^0(C))$, we have

$$\begin{aligned}\kappa(z, c_s) &\subseteq \kappa([L^0(C), L^0(C)], Z(L^0(C))) \\ &= \kappa(L^0(C), [L^0(C), Z(L^0(C))]) = 0.\end{aligned}$$

Hence we have $\kappa(z,c)=0$. We have proved that $N_L(C)\cap[L^0(C),L^0(C)] \subseteq C^{\perp}$. We have also $N_L(C)\cap[L^0(C),L^0(C)]\subseteq L^0(C)\subseteq (B^+(C))^{\perp}$ since $B^+(C)\subseteq \sum_{\alpha\neq 0} L^{\alpha}(C)$. By $B=C\oplus B^+(C)$ and Lemma 12 we obtain

$$N_L(C)\cap[L^0(C),L^0(C)]\subseteq B^{\perp}\cap N_L(C)\subseteq B\cap N_L(C)=C.$$

(3) is proved. □

Corollary 18. *The intersection of two Borel subalgebras of L contains a Cartan subalgebra of L.*

Proof. We can assume that L is semisimple similarly to the proof of Theorem 17. Then the assertion is a consequence of Theorem 17 and Lemma 13. □

Corollary 19. *The group E_L acts transitively on the set of Borel subalgebras of L.*

Proof. We can again assume that L is semisimple. For Borel subalgebras B_1 and B_2 of L we have to find $u\in E_L$ satisfying $u(B_1)=B_2$. By Corollary 18 B_1 and B_2 contain a common Cartan subalgebra C of L. Then the assertion follows from a standard argument involving the Weyl group (see [3, Ch. VI, §1, Théorème 2] and [4, Ch. VIII, §2, Corollaire to Théorème 2]). □

References

[1] D. W. Barnes, On Cartan subalgebras of Lie algebras, Math. Z., **101** (1967), 350–355.
[2] N. Bourbaki, Groupes et algèbres de Lie, Ch. 1 Herman, Paris 1960.
[3] N. Bourbaki, Groupes et algèbres de Lie, Ch. 4, 5 et 6 Herman, Paris 1968.
[4] N. Bourbaki, Groupes et algèbres de Lie, Ch. 7, 8 Herman, Paris 1975.
[5] C. Chevalley, Théorie des groupes de Lie, Herman, Paris 1955.
[6] J. E. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, 1972.
[7] N. Jacobson, Lie algebras, Wiley Interscience, 1962.
[8] Séminaire "Sophus Lie" Théorie des algèbres de Lie, Topologie des groupes de Lie, Paris, École Norm. Sup., 1954–55.
[9] D. J. Winter, Abstract Lie algebras, Cambridge, MA, M.I.T. Press, 1972.

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