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Singularities of non-degenerate 2-ruled hypersurfaces in 4-space

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ABSTRACT. We study singularities of 2-ruled hypersurfaces in Euclidian 4-space. After defining a non-degenerate 2-ruled hypersurface we will give a necessary and sufficient condition for such a map germ to be right-left equivalent to the cross cap $\times$ interval. The behavior of a generic 2-ruled hypersurface map is also discussed.

1. Introduction

The study of ruled surfaces in $\mathbf{R}^3$ is a classical subject in differential geometry and ruled hypersurfaces in higher dimensions have also been studied by many authors. Although ruled hypersurfaces have singularities in general, there have been very few studies of ruled hypersurfaces with singularities. Recently Izumiya and Takeuchi [3] showed that every singularity that appears for some generic C^∞-map of a surface into 3-space occurs for some generic ruled surface in $\mathbf{R}^3$, and vice versa.

A 2-ruled hypersurface in $\mathbf{R}^4$ is a one-parameter family of planes in $\mathbf{R}^4$. This is a generalization of ruled surfaces in $\mathbf{R}^3$. In this paper, we first define non-degenerate 2-ruled hypersurfaces in $\mathbf{R}^4$ and give a necessary and sufficient condition for a non-degenerate 2-ruled hypersurface germ in $\mathbf{R}^4$ to be right-left equivalent to the cross cap $\times$ interval (Theorem 2.5). Furthermore, we show that the singularities of generic 2-ruled hypersurfaces are cross cap $\times$ interval (Theorem 5.3). Since any singularity of a generic smooth map of a 3-manifold into $\mathbf{R}^4$ is the cross cap $\times$ interval, the singularities of generic 2-ruled hypersurfaces are the same as those of generic C^∞-maps of 3-manifolds into $\mathbf{R}^4$.

The paper is organized as follows. In §2 we define non-degenerate 2-ruled hypersurfaces as an analogue of classical noncylindrical ruled surfaces. Classical noncylindrical ruled surfaces are those whose rulings always change directions and non-degenerate 2-ruled hypersurfaces will be defined in the same way. Then we present the main theorem (Theorem 2.5). In §3 we briefly review the properties of the classical striction curve and generalize them to non-degenerate 2-ruled hypersurfaces. It is quite remarkable that the striction curve

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coincides with the singluarity set in our case, while the set of singular points of a noncylindrical ruled surface is contained in its striction curve but may not coincide. In §4 the proof of our main theorem is completed. In §5 we discuss generic 2-ruled hypersurfaces. We will define almost non-degenerate 2-ruled hypersurfaces which are generic in the usual sense and are non-degenerate almost everywhere. We prove that the set of 2-ruled hypersurfaces whose map germ at any point is right-left equivalent to the cross cap $\times$ interval or an immersion germ contains an open and dense subset of the space of 2-ruled hypersurfaces.

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2. Preliminaries and statement of the main theorem

In this section we give the definition of 2-ruled hypersurfaces and state our main theorem.

Let S^3 be the unit sphere of $\mathbf{R}^4$ and I, J_1, J_2 open intervals.

DEFINITION 2.1. A 2-*ruled hypersurface in* $\mathbf{R}^4$ means (the image of) a map $F_{(\gamma,\delta,\varepsilon)} : I \times J_1 \times J_2 \to \mathbf{R}^4$ of the form

$$F_{(\gamma,\delta,\varepsilon)}(t,u,v) = \gamma(t) + u\delta(t) + v\varepsilon(t),$$

where $\gamma : I \to \mathbf{R}^4$, $\delta : I \to S^3$ and $\varepsilon : I \to S^3$ are smooth maps. We assume that the dimension of the vector space $\langle \delta(t), \varepsilon(t) \rangle$ spanned by δ and ε is always equal to 2 for any $t \in I$. We call γ a *base curve* and two curves δ and ε *director curves*. The planes $(u,v) \mapsto \gamma(t) + u\delta(t) + v\varepsilon(t)$ are called *rulings*.

We consider $(\gamma,\delta,\varepsilon) \in C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$ and we regard $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$ equipped with the Whitney C^∞-topology as a space of 2-ruled hypersurfaces. A non-degenerate 2-ruled hypersurface in $\mathbf{R}^4$ satisfies a condition analogous to a noncylindrical ruled surface in $\mathbf{R}^3$.

DEFINITION 2.2. A 2-ruled hypersurface $F_{(\gamma,\delta,\varepsilon)}(t,u,v) = \gamma(t) + u\delta(t) + v\varepsilon(t)$ is said to be *non-degenerate at* $t \in I$, if the four vectors $\delta(t)$, $\delta'(t)$, $\varepsilon(t)$ and $\varepsilon'(t)$ span $\mathbf{R}^4$, that is, if

$$\dim\langle \delta(t), \delta'(t), \varepsilon(t), \varepsilon'(t) \rangle = 4.$$

DEFINITION 2.3. A 2-ruled hypersurface $F_{(\gamma,\delta,\varepsilon)}(t,u,v) = \gamma(t) + u\delta(t) + v\varepsilon(t)$ is said to be *non-degenerate*, if $F_{(\gamma,\delta,\varepsilon)}(t,u,v)$ is globally non-degenerate, that is, if it is non-degenerate at any $t \in I$.

Note that the non-degeneracy condition is not generic in the usual sense. The generic condition will be discussed in §5.

LEMMA 2.4. *The non-degeneracy does not depend on the choice of director curves δ and ε.*

PROOF. Suppose that δ, δ', ε and ε' are linearly independent. Put

$$\begin{cases} \delta_1(t) = a(t)\delta(t) + b(t)\varepsilon(t), \\ \varepsilon_1(t) = c(t)\delta(t) + d(t)\varepsilon(t), \end{cases}$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are smooth real valued functions with $a(t)d(t) - b(t)c(t) \neq 0$. We prove that δ_1, δ_1', ε_1 and ε_1' are linearly independent. Suppose that $\lambda_1\delta_1 + \lambda_2\varepsilon_1 + \lambda_3\delta_1' + \lambda_4\varepsilon_1' = 0$ for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbf{R}$. Then we have $(\lambda_1 a + \lambda_2 c + \lambda_3 a' + \lambda_4 c')\delta + (\lambda_1 b + \lambda_2 d + \lambda_3 b' + \lambda_4 d')\varepsilon + (\lambda_3 a + \lambda_4 c)\delta' + (\lambda_3 b + \lambda_4 d)\varepsilon' = 0$. Since δ, δ', ε and ε' are linearly independent,

$$\begin{cases} \lambda_1 a + \lambda_2 c + \lambda_3 a' + \lambda_4 c' = 0, \\ \lambda_1 b + \lambda_2 d + \lambda_3 b' + \lambda_4 d' = 0, \\ \lambda_3 a + \lambda_4 c = 0, \\ \lambda_3 b + \lambda_4 d = 0 \end{cases}$$

holds. Since $ad - bc \neq 0$, it holds that $\lambda_3 = \lambda_4 = 0$ by the last two equations. Now, the first two equations become

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0.$$

So $\lambda_1 = \lambda_2 = 0$ holds. Hence δ_1, δ_1', ε_1 and ε_1' are linearly independent. $\square$

Recall that $x \in N$ is a *singular point* of a differentiable map $f : N \to P$ between manifolds if $\operatorname{rank}(df)_x < \min\{\dim N, \dim P\}$. The image of a singular point of a ruled surface map or a 2-ruled hypersurface map will also be called a singular point of a ruled surface or a 2-ruled hypersurface respectively.

Singular points of non-degenerate 2-ruled hypersurfaces are characterized by the following main theorem, by using the notion of the striction curve σ which will be defined in the following section.

THEOREM 2.5 (Main Theorem). *Let $F = F_{(\sigma,\delta,\varepsilon)}$ be the map germ of a non-degenerate 2-ruled hypersurface with striction curve $\sigma(t)$ at (t_0, u_0, v_0).*

(1) *The point $p_0 = F(t_0, u_0, v_0)$ does not lie on the striction curve (i.e., $(u_0, v_0) \neq (0,0)$) if and only if the map germ F is regular at (t_0, u_0, v_0).*

(2) *If p_0 lies on the striction curve (i.e., $(u_0, v_0) = (0,0)$), then the following two conditions are equivalent.*

(a) *The striction curve* $\sigma(t)$ *is an immersion near* $t = t_0$.

(b) *The map germ* F *at* (t_0, u_0, v_0) *is right-left equivalent to the cross cap* $\times$ *interval.*

Here, a *cross cap* $\times$ *interval* means the map germ at the origin of the map defined by

$$(x_1, x_2, x_3) \to (x_1^2, x_2, x_3, x_1 x_2)$$

and the right-left equivalence is defined as follows.

DEFINITION 2.6. Let $f_i : (N_i, x_i) \to (P_i, y_i)$, $i = 1, 2$, be C^∞-map germs. We say that f_1 and f_2 are *right-left equivalent* if there exist diffeomorphism germs $\phi : (N_1, x_1) \to (N_2, x_2)$ and $\psi : (P_1, y_1) \to (P_2, y_2)$ such that

$$\psi \circ f_1 = f_2 \circ \phi.$$

3. Striction curve of a non-degenerate 2-ruled hypersurface

Before defining the striction curve for a non-degenerate 2-ruled hypersurface, we review the case of ruled surfaces [3]. A *ruled surface* in $\mathbf{R}^3$ is (the image of) a map $F_{(\gamma,\delta)} : I \times J \to \mathbf{R}^3$ of the form $F_{(\gamma,\delta)}(t, u) = \gamma(t) + u\delta(t)$, where $\gamma : I \to \mathbf{R}^3$ and $\delta : I \to S^2$ are smooth maps, and I and J are open intervals. A ruled surface $F_{(\gamma,\delta)}$ is said to be *noncylindrical* if $\delta \times \delta'$ never vanishes. For any noncylindrical ruled surface, its striction curve is defined as a special base curve as follows.

LEMMA 3.1 ([3], Lemmas 2.1 and 2.2). (1) *Let* $F_{(\gamma,\delta)}(t, u)$ *be a noncylindrical ruled surface. Then there exists a smooth curve* $\sigma : I \to \mathbf{R}^3$ *such that*

$$\text{Image } F_{(\gamma,\delta)} = \text{Image } F_{(\sigma,\delta)} \qquad \textit{and} \qquad \sigma'(t) \cdot \delta'(t) = 0 \qquad \textit{for all } t \in I.$$

The curve $\sigma(t)$ *is called the striction curve of* $F_{(\gamma,\delta)}(t, u)$.

(2) *The striction curve of a noncylindrical ruled surface* $F_{(\gamma,\delta)}(t, u)$ *does not depend on the choice of the base curve* γ.

(3) *Every singular point of a noncylindrical ruled surface is contained in the image of the striction curve* σ. *Moreover, at every singular point* $p_0 = F_{(\sigma,\delta)}(t_0, u_0)$, *the ruling through* $\sigma(t_0)$ *of* $F_{(\sigma,\delta)}$ *is tangent to* σ.

We will define the striction curve of a non-degenerate 2-ruled hypersurface after preparing Lemmas 3.2 and 3.3.

LEMMA 3.2. *For any 2-ruled hypersurface* $F_{(\gamma,\delta,\varepsilon)}(t, u, v) = \gamma(t) + u\delta(t) + v\varepsilon(t)$, *we can choose director curves* δ *and* ε *such that not only* $\|\delta\| = \|\varepsilon\| = 1$, *but also* $\delta \cdot \varepsilon = 0$ *and* $\delta' \cdot \varepsilon = \delta \cdot \varepsilon' = 0$ *hold.*

We say that the director curves δ and ε are *constrictively adapted* if they satisfy the above conditions.

PROOF. We may suppose that the director curves δ and ε satisfy the conditions that $\|\delta\| = \|\varepsilon\| = 1$ and $\delta \cdot \varepsilon = 0$. Now, we put

$$\begin{cases} \delta_1(t) = (\cos\theta(t))\delta(t) + (\sin\theta(t))\varepsilon(t), \\ \varepsilon_1(t) = -(\sin\theta(t))\delta(t) + (\cos\theta(t))\varepsilon(t) \end{cases}$$

for a smooth function $\theta(t)$. We see that $\|\delta_1\| = \|\varepsilon_1\| = 1$ and $\delta_1 \cdot \varepsilon_1 = 0$. On the other hand, we have

$$\begin{aligned} \delta_1'(t) \cdot \varepsilon_1(t) &= (-(\sin\theta(t))\theta'(t)\delta(t) + (\cos\theta(t))\delta'(t) \\ &\qquad + (\cos\theta(t))\theta'(t)\varepsilon(t) + (\sin\theta(t))\varepsilon'(t)) \\ &\qquad \cdot (-(\sin\theta(t))\delta(t) + (\cos\theta(t))\varepsilon(t)) \\ &= \theta'(t) + \delta'(t) \cdot \varepsilon(t). \end{aligned}$$

Since $\delta_1 \cdot \varepsilon_1 = 0$, we have $\delta_1' \cdot \varepsilon_1 + \delta_1 \cdot \varepsilon_1' = 0$. So any solution θ of the differential equation

$$\theta'(t) + \delta'(t) \cdot \varepsilon(t) = 0$$

gives a desired pair $(\delta_1, \varepsilon_1)$ of director curves. □

LEMMA 3.3. *Let $F_{(\gamma,\delta,\varepsilon)}(t,u,v) = \gamma(t) + u\delta(t) + v\varepsilon(t)$, $t \in I$, be a non-degenerate 2-ruled hypersurface whose director curves δ and ε are constrictively adapted. Then, there exists a smooth curve $\sigma : I \to \mathbf{R}^4$ such that*

$$\text{Image } F_{(\gamma,\delta,\varepsilon)} = \text{Image } F_{(\sigma,\delta,\varepsilon)} \qquad \text{and} \qquad \sigma' \cdot \delta' = \sigma' \cdot \varepsilon' = 0.$$

PROOF. Since δ' and ε' are linearly independent by the non-degeneracy of the 2-ruled hypersurface $F_{(\gamma,\delta,\varepsilon)}$, we see easily that

$$\det\begin{pmatrix} \delta' \cdot \delta' & \varepsilon' \cdot \delta' \\ \delta' \cdot \varepsilon' & \varepsilon' \cdot \varepsilon' \end{pmatrix} \neq 0.$$

So, we can put

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \delta' \cdot \delta' & \varepsilon' \cdot \delta' \\ \delta' \cdot \varepsilon' & \varepsilon' \cdot \varepsilon' \end{pmatrix}^{-1} \begin{pmatrix} -\gamma' \cdot \delta' \\ -\gamma' \cdot \varepsilon' \end{pmatrix}.$$

Then, $\sigma(t) = \gamma(t) + f(t)\delta(t) + g(t)\varepsilon(t)$ satisfies the conditions $\sigma' \cdot \delta' = \sigma' \cdot \varepsilon' = 0$. □

DEFINITION 3.4. A curve $\sigma(t)$ which satisfies the condition in Lemma 3.3 is called a *striction curve* of a non-degenerate 2-ruled hypersurface $F_{(\gamma,\delta,\varepsilon)}(t,u,v)$.

Since $\sigma' \perp \delta'$, $\sigma' \perp \varepsilon'$, $\delta' \perp \delta$, $\delta' \perp \varepsilon$, $\delta \perp \varepsilon'$, $\varepsilon \perp \varepsilon'$ and $\dim\langle \delta, \delta', \varepsilon, \varepsilon' \rangle = 4$, we have $\sigma' \in \langle \delta, \varepsilon \rangle$. This means that the striction curve is tangent to the ruling at any t.

Lemma 3.5. *Let $F_{(\sigma,\delta,\varepsilon)}(t,u,v) = \sigma(t) + u\delta(t) + v\varepsilon(t)$ be a non-degenerate 2-ruled hypersurface with the striction curve $\sigma(t)$. Then the set of the singular points of the 2-ruled hypersurface $F_{(\sigma,\delta,\varepsilon)}$ coincides with the image of the striction curve $\sigma(t)$.*

Proof. By definition (t,u,v) is a singular point of $F = F_{(\sigma,\delta,\varepsilon)}$ if and only if the Jacobian matrix

$$\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right)(t,u,v) = (\sigma'(t) + u\delta'(t) + v\varepsilon'(t), \delta(t), \varepsilon(t))$$

of F is not of full rank. Note that $\sigma' \in \langle \delta, \varepsilon \rangle$ as remarked just after Definition 3.4, and the four vectors δ, δ', ε and ε' are linearly independent. Then, we see easily that the above matrix is not of full rank if and only if $u = v = 0$. $\square$

Corollary 3.6. *The striction curve of a non-degenerate 2-ruled hypersurface in $\mathbf{R}^4$ does not depend on the choice of a solution θ of the differential equation $\theta' + \delta' \cdot \varepsilon = 0$ in the proof of Lemma 3.2 or on the choice of director curves δ and ε.*

Remark 3.7. For a non-degenerate 2-ruled hypersurface $F_{(\gamma,\delta,\varepsilon)}(t,u,v) = \gamma(t) + u\delta(t) + v\varepsilon(t)$ whose director curves satisfy $\delta(t) \cdot \varepsilon(t) = 0$, a direct calculation gives the following formula for the striction curve:

$$\sigma(t) = \gamma(t) + \frac{A(t)}{C(t)}\delta(t) + \frac{B(t)}{C(t)}\varepsilon(t),$$

with

$$\begin{aligned}
A(t) &= -(\gamma' \cdot \varepsilon)(\delta' \cdot \varepsilon)^3 + (\gamma' \cdot \delta')(\delta' \cdot \varepsilon)^2 \\
&\quad + ((\delta' \cdot \varepsilon')(\gamma' \cdot \delta) + (\varepsilon' \cdot \varepsilon')(\gamma' \cdot \varepsilon))(\delta' \cdot \varepsilon) \\
&\quad + (\delta' \cdot \varepsilon')(\gamma' \cdot \varepsilon') - (\varepsilon' \cdot \varepsilon')(\gamma' \cdot \delta'), \\
B(t) &= (\gamma' \cdot \delta)(\delta' \cdot \varepsilon)^3 + (\gamma' \cdot \varepsilon')(\delta' \cdot \varepsilon)^2 \\
&\quad - ((\delta' \cdot \delta')(\gamma' \cdot \delta) + (\delta' \cdot \varepsilon')(\gamma' \cdot \varepsilon))(\delta' \cdot \varepsilon) \\
&\quad - (\delta' \cdot \delta')(\gamma' \cdot \varepsilon') + (\delta' \cdot \varepsilon')(\gamma' \cdot \delta'), \\
C(t) &= (\delta' \cdot \varepsilon)^4 - ((\delta' \cdot \delta') + (\varepsilon' \cdot \varepsilon'))(\delta' \cdot \varepsilon)^2 \\
&\quad + (\delta' \cdot \delta')(\varepsilon' \cdot \varepsilon') - (\delta' \cdot \varepsilon')^2.
\end{aligned}$$

By setting $\bar{\delta} = \delta' - (\delta' \cdot \varepsilon)\varepsilon$ and $\bar{\varepsilon} = \varepsilon' - (\delta \cdot \varepsilon')\delta$ we see that

$$C(t) = \det\begin{pmatrix} \bar{\delta}(t) \cdot \bar{\delta}(t) & \bar{\delta}(t) \cdot \bar{\varepsilon}(t) \\ \bar{\varepsilon}(t) \cdot \bar{\delta}(t) & \bar{\varepsilon}(t) \cdot \bar{\varepsilon}(t) \end{pmatrix} \neq 0.$$

Now, we give some examples. We can easily check that the striction curve coincides with the set of singular points in these examples by a simple calculation.

Example 3.8. We put $\gamma(t) = (t, 0, 0, 0)$, $\delta(t) = \left(0, \dfrac{1}{\sqrt{1+t^2}}, \dfrac{t}{\sqrt{1+t^2}}, 0\right)$, and $\varepsilon(t) = \left(\dfrac{1}{\sqrt{1+t^2}}, 0, 0, \dfrac{t}{\sqrt{1+t^2}}\right)$. This gives a non-degenerate 2-ruled hypersurface whose striction curve is

$$\sigma(t) = (2t, 0, 0, t^2).$$

Example 3.9. We put $\sigma(t) = (t^2, t^3, 0, 0)$, $\delta(t) = \left(\dfrac{2}{\sqrt{4+9t^2}}, \dfrac{3t}{\sqrt{4+9t^2}}, 0, 0\right)$, and $\varepsilon(t) = \left(0, 0, \dfrac{t^2+1}{\sqrt{t^4+3t^2+1}}, \dfrac{t}{\sqrt{t^4+3t^2+1}}\right)$, $-1 < t < 1$. This gives a non-degenerate 2-ruled hypersurface with the striction curve $\sigma(t)$. In this example, the striction curve has a $(2,3)$-cusp singularity at $t = 0$ and is not an immersion at $t = 0$.

4. Proof of the main theorem

Let $f : (\mathbf{R}^3, 0) \to (\mathbf{R}^4, 0)$ be a smooth map germ and we consider the Thom-Boardman singularity set $\Sigma^{1,0} \subset J^2(3,4)$ defined in [1]. Morin [4] proved the following lemma.

Lemma 4.1 ([4], Théorème). *Let $f : (\mathbf{R}^3, 0) \to (\mathbf{R}^4, 0)$ be a smooth map germ. Then the following two conditions are equivalent.*

(1) *$j^2 f(0) \in \Sigma^{1,0}$ and the map germ $j^2 f : (\mathbf{R}^3, 0) \to J^2(3,4)$ is transverse to $\Sigma^{1,0}$ at $j^2 f(0)$.*

(2) *f is right-left equivalent to the cross cap $\times$ interval, that is, there exist local coordinates (x_1, x_2, x_3) of $\mathbf{R}^3$ around 0 and local coordinates (y_1, y_2, y_3, y_4) of $\mathbf{R}^4$ around 0, such that $f = (y_1 \circ f, y_2 \circ f, y_3 \circ f, y_4 \circ f)$ is expressed as*

$$f(x_1, x_2, x_3) = (x_1^2, x_2, x_3, x_1 x_2).$$

Furthermore, he rewrites the above condition as follows. We use the notation $f(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3), f_4(x_1, x_2, x_3))$.

Lemma 4.2 ([4], Lemme). *Let $f : (\mathbf{R}^3, 0) \to (\mathbf{R}^4, 0)$ be a smooth map germ. $j^2 f(0) \in \Sigma^{1,0}$ and the map germ $j^2 f : (\mathbf{R}^3, 0) \to J^2(3,4)$ is transverse to*

$\Sigma^{1,0}$ *at* $j^2f(0)$ *if and only if for some local coordinates* (x_1, x_2, x_3) *of* $\mathbf{R}^3$ *and* (z_1, z_2, z_3, z_4) *of* $\mathbf{R}^4$ *satisfying* $f_2(x_1, x_2, x_3) = x_2$, $f_3(x_1, x_2, x_3) = x_3$,

$$\frac{\partial f_1}{\partial x_1}(0,0,0) = \frac{\partial f_1}{\partial x_2}(0,0,0) = \frac{\partial f_1}{\partial x_3}(0,0,0) = 0 \qquad \textit{and}$$

$$\frac{\partial f_4}{\partial x_1}(0,0,0) = \frac{\partial f_4}{\partial x_2}(0,0,0) = \frac{\partial f_4}{\partial x_3}(0,0,0) = 0,$$

(i) $\dfrac{\partial^2 f}{\partial x_1^2}(0,0,0) \neq 0$, *and*

(ii) $\operatorname{rank}\left(\dfrac{\partial^2 f}{\partial x_1^2}, \dfrac{\partial^2 f}{\partial x_1 \partial x_2}, \dfrac{\partial^2 f}{\partial x_1 \partial x_3}\right)(0,0,0) = 2.$

Proof of Theorem 2.5. The statement (1) follows directly from Lemma 3.5. So we prove (2) here.

Let $F_{(\sigma,\delta,\varepsilon)}(t,u,v) = \sigma(t) + u\delta(t) + v\varepsilon(t)$ be a non-degenerate 2-ruled hypersurface with the striction curve $\sigma(t)$. For any $t_0 \in I$, the point p_0 denotes $F_{(\sigma,\delta,\varepsilon)}(t_0, 0, 0)$. We put $F = F_{(\sigma,\delta,\varepsilon)}$ and suppose that the director curves δ and ε are constrictively adapted.

First, changing the coordinates (z_1, z_2, z_3, z_4) of $\mathbf{R}^4$ by an orthogonal transformation if necessary, we may assume $\delta(t_0) = (0,1,0,0)$ and $\varepsilon(t_0) = (0,0,1,0)$. Let us define the new coordinates (x_1, x_2, x_3) of $\mathbf{R}^3$ by

$$x_1 = t - t_0,$$

$$x_2 = (F(t,u,v) - F(t_0,0,0)) \cdot \delta(t_0),$$

$$x_3 = (F(t,u,v) - F(t_0,0,0)) \cdot \varepsilon(t_0).$$

Then, we get

$$\frac{\partial F}{\partial x_1}(0,0,0) = 0, \qquad \frac{\partial F}{\partial x_2}(0,0,0) = \delta(t_0), \qquad \frac{\partial F}{\partial x_3}(0,0,0) = \varepsilon(t_0).$$

So, the coordinates (x_1, x_2, x_3) and (z_1, z_2, z_3, z_4) are adapted coordinate systems in the sense of Morin [4].

We use the notation: $\sigma_0 = \sigma(t_0)$, $\sigma_0' = \sigma'(t_0)$, $\delta_0 = \delta(t_0)$, $\delta_0' = \delta'(t_0)$, $\varepsilon_0 = \varepsilon(t_0)$ and $\varepsilon_0' = \varepsilon'(t_0)$. We have

$$\begin{aligned}\sigma''(t_0) &= \frac{\partial^2 F}{\partial t^2}(t_0, 0, 0) \\ &= \left(\frac{\partial^2 F}{\partial x_1^2} + (\sigma_0' \cdot \delta_0)^2 \frac{\partial^2 F}{\partial x_2^2} + (\sigma_0' \cdot \varepsilon_0)^2 \frac{\partial^2 F}{\partial x_3^2} + 2(\sigma_0' \cdot \delta_0) \frac{\partial^2 F}{\partial x_1 \partial x_2}\right.\end{aligned}$$

$$
\begin{aligned}
&+ 2(\sigma_0' \cdot \varepsilon_0)\frac{\partial^2 F}{\partial x_1 \partial x_3} + 2(\sigma_0' \cdot \delta_0)(\sigma_0' \cdot \varepsilon_0)\frac{\partial^2 F}{\partial x_2 \partial x_3} \\
&+ (\sigma_0'' \cdot \delta_0)\delta_0 + (\sigma_0'' \cdot \varepsilon_0)\varepsilon_0\Bigg)(0,0,0),
\end{aligned}
$$

$$
\begin{aligned}
\delta'(t_0) &= \frac{\partial^2 F}{\partial t \partial u}(t_0, 0, 0) \\
&= \left((\sigma_0' \cdot \delta_0)\frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 F}{\partial x_1 \partial x_2} + (\sigma_0' \cdot \varepsilon_0)\frac{\partial^2 F}{\partial x_2 \partial x_3}\right)(0,0,0), \\
\varepsilon'(t_0) &= \frac{\partial^2 F}{\partial t \partial v}(t_0, 0, 0) \\
&= \left((\sigma_0' \cdot \varepsilon_0)\frac{\partial^2 F}{\partial x_3^2} + \frac{\partial^2 F}{\partial x_1 \partial x_3} + (\sigma_0' \cdot \delta_0)\frac{\partial^2 F}{\partial x_2 \partial x_3}\right)(0,0,0), \\
0 &= \frac{\partial^2 F}{\partial u^2}(t_0, 0, 0) = \frac{\partial^2 F}{\partial x_2^2}(0,0,0), \\
0 &= \frac{\partial^2 F}{\partial u \partial v}(t_0, 0, 0) = \frac{\partial^2 F}{\partial x_2 \partial x_3}(0,0,0), \\
0 &= \frac{\partial^2 F}{\partial v^2}(t_0, 0, 0) = \frac{\partial^2 F}{\partial x_3^2}(0,0,0).
\end{aligned}
$$

Since $\dim\langle \delta(t), \delta'(t), \varepsilon(t), \varepsilon'(t)\rangle = 4$ and $\sigma'(t) \cdot \delta'(t) = \sigma'(t) \cdot \varepsilon'(t) = 0$ for any t, we have

$$
\sigma'(t) = (\sigma'(t) \cdot \delta(t))\delta(t) + (\sigma'(t) \cdot \varepsilon(t))\varepsilon(t)
$$

and hence

$$
\sigma''(t) - (\sigma''(t) \cdot \delta(t))\delta(t) - (\sigma''(t) \cdot \varepsilon(t))\varepsilon(t) = (\sigma'(t) \cdot \delta(t))\delta'(t) + (\sigma'(t) \cdot \varepsilon(t))\varepsilon'(t).
$$

So we obtain

$$
\frac{\partial^2 F}{\partial x_1^2}(0,0,0) = -(\sigma_0' \cdot \delta_0)\delta_0' - (\sigma_0' \cdot \varepsilon_0)\varepsilon_0'.
$$

Hence

$$
\left(\frac{\partial^2 F}{\partial x_1^2}, \frac{\partial^2 F}{\partial x_1 \partial x_2}, \frac{\partial^2 F}{\partial x_1 \partial x_3}\right)(0,0,0) = (-(\sigma' \cdot \delta)\delta' - (\sigma' \cdot \varepsilon)\varepsilon', \delta', \varepsilon')(t_0).
$$

This means that the condition (ii) of Lemma 4.2 is always satisfied for F. Furthermore, the condition (i) is equivalent to

$$\frac{\partial^2 F}{\partial x_1^2}(0,0,0) = -(\sigma_0' \cdot \delta_0)\delta_0' - (\sigma_0' \cdot \varepsilon_0)\varepsilon_0' \neq 0,$$

that is, either $\sigma_0' \cdot \delta_0 \neq 0$ or $\sigma_0' \cdot \varepsilon_0 \neq 0$. Since $\sigma' \in \langle \delta, \varepsilon \rangle$, this condition is equivalent to $\sigma' \neq 0$ at $t = t_0$. This completes the proof. □

5. Singularities of generic 2-ruled hypersurfaces

In this section, we will define *almost non-degenerate* 2-ruled hypersurfaces which are generic in the usual sense. They have *exceptional rulings* where the striction curve cannot be defined and there are no singular points. So, we get Theorem 5.3 which characterizes the singularities of generic 2-ruled hypersurfaces. We will also discuss the behavior of the striction curve near the exceptional rulings.

First, we define an almost non-degenerate 2-ruled hypersurface.

DEFINITION 5.1. A 2-ruled hypersurface $F_{(\gamma,\delta,\varepsilon)}(t,u,v) = \gamma(t) + u\delta(t) + v\varepsilon(t)$, $t \in I$, is said to be *almost non-degenerate on* I, if there exists a discrete subset $D \subset I$ such that the following four conditions hold.

(1) $F_{(\gamma,\delta,\varepsilon)}$ is non-degenerate at any $t \notin D$.

(2) $\dim\langle \delta(t_i), \delta'(t_i), \varepsilon(t_i), \varepsilon'(t_i) \rangle = 3$ for any $t_i \in D$.

(3) Let A_t denote $\det(\delta(t), \delta'(t), \varepsilon(t), \varepsilon'(t))$. Then $dA_t/dt|_{t=t_i} \neq 0$ for any $t_i \in D$.

(4) $\gamma'(t_i) \notin \langle \delta(t_i), \delta'(t_i), \varepsilon(t_i), \varepsilon'(t_i) \rangle$ for any $t_i \in D$.

It is easy to check that the condition (4) does not depend on the choice of the base curve γ. For an almost non-degenerate 2-ruled hypersurface the rulings $(u,v) \mapsto \gamma(t_i) + u\delta(t_i) + v\varepsilon(t_i)$ for $t_i \in D$ are called *exceptional rulings*. Note that the condition (4) implies that $F_{(\gamma,\delta,\varepsilon)}$ is non-singular at any point in the exceptional rulings.

The following lemma shows that there are plenty of almost non-degenerate 2-ruled hypersurfaces, that is, the condition is generic in the usual sense.

LEMMA 5.2. *The set*

$$\{(\gamma,\delta,\varepsilon) \mid F_{(\gamma,\delta,\varepsilon)} \text{ is an almost non-degenerate 2-ruled hypersurface}\}$$

is open and dense in $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$ *with respect to the Whitney* C^∞-*topology.*

PROOF. First, we put

$$Q_1 = \{j^1(\gamma,\delta,\varepsilon)(t) \mid \dim\langle \delta(t), \varepsilon(t) \rangle = 1, t \in I\} \subset J^1(I, \mathbf{R}^4 \times S^3 \times S^3).$$

Then Q_1 is a closed submanifold of codimension 3. Second, we put

$$Q_2 = \{j^1(\gamma,\delta,\varepsilon)(t) \in J^1(I,\mathbf{R}^4 \times S^3 \times S^3)\backslash Q_1 \mid \\ \dim\langle\delta(t),\delta'(t),\varepsilon(t),\varepsilon'(t)\rangle = 2, t \in I\}.$$

Then Q_2 is a closed submanifold of $J^1(I,\mathbf{R}^4 \times S^3 \times S^3)\backslash Q_1$ of codimension 4. Note that $X = J^1(I,\mathbf{R}^4 \times S^3 \times S^3)\backslash(Q_1 \cup Q_2)$ is an open submanifold of $J^1(I,\mathbf{R}^4 \times S^3 \times S^3)$. Third, we put

$$Q_3 = \{j^1(\gamma,\delta,\varepsilon)(t) \in X \mid \dim\langle\delta(t),\delta'(t),\varepsilon(t),\varepsilon'(t)\rangle = 3, t \in I\}.$$

We define a C^∞-map ζ by

$$\zeta : X \ni j^1(\gamma,\delta,\varepsilon)(t) \mapsto \det(\delta(t),\delta'(t),\varepsilon(t),\varepsilon'(t)) \in \mathbf{R}.$$

Then $Q_3 = \zeta^{-1}(0)$ and we see that $0 \in \mathbf{R}$ is a regular value for ζ. So, Q_3 is a closed submanifold of X of codimension 1. Moreover, the set

$$S = \{j^1(\gamma,\delta,\varepsilon)(t) \in Q_3 \mid \gamma'(t) \in \langle\delta(t),\delta'(t),\varepsilon(t),\varepsilon'(t)\rangle, t \in I\}$$

is a closed submanifold of Q_3 of codimension 1.

By Thom's jet transversality theorem, the set

$$\mathscr{R} = \{(\gamma,\delta,\varepsilon) \in C^\infty(I,\mathbf{R}^4 \times S^3 \times S^3) \mid \\ j^1(\gamma,\delta,\varepsilon) \text{ is transverse to } Q_1, Q_2, Q_3 \text{ and } S\}$$

is a residual subset of $C^\infty(I,\mathbf{R}^4 \times S^3 \times S^3)$ with respect to the Whitney C^∞-topology. So $\mathscr{R}$ is dense in $C^\infty(I,\mathbf{R}^4 \times S^3 \times S^3)$.

We can easily check that $j^1(\gamma,\delta,\varepsilon)$ is transverse to Q_1, Q_2, Q_3 and S if and only if $F_{(\gamma,\delta,\varepsilon)}$ is an almost non-degenerate 2-ruled hypersurface. So, $\mathscr{R}$ coincides with the set $\{(\gamma,\delta,\varepsilon) \mid F_{(\gamma,\delta,\varepsilon)}$ is an almost non-degenerate 2-ruled hypersurface$\}$.

Now we prove that $\mathscr{R}$ is an open set. Since X is open in $J^1(I,\mathbf{R}^4 \times S^3 \times S^3)$, $C^\infty(I,X)$ is open in $C^\infty(I,J^1(I,\mathbf{R}^4 \times S^3 \times S^3))$. On the other hand, since Q_3 and S are closed submanifolds of X, the set

$$\{g \in C^\infty(I,X) \mid g \text{ is transverse to } Q_3 \text{ and } S\}$$

is open in $C^\infty(I,X)$. Hence the set

$$\begin{aligned}\mathscr{R}' &= \{g \in C^\infty(I,J^1(I,\mathbf{R}^4 \times S^3 \times S^3)) \mid g \text{ is transverse to } Q_1, Q_2, Q_3 \text{ and } S\} \\ &= \{g \in C^\infty(I,X) \mid g \text{ is transverse to } Q_3 \text{ and } S\}\end{aligned}$$

is open in $C^\infty(I,J^1(I,\mathbf{R}^4 \times S^3 \times S^3))$. Since the map

$$j^1 : C^\infty(I,\mathbf{R}^4 \times S^3 \times S^3) \to C^\infty(I,J^1(I,\mathbf{R}^4 \times S^3 \times S^3))$$

is continuous (see [2, p. 46], for example), $\mathscr{R} = (j^1)^{-1}(\mathscr{R}')$ is an open subset of $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$.

Therefore $\mathscr{R} = \{(\gamma, \delta, \varepsilon) \mid F_{(\gamma,\delta,\varepsilon)}$ is an almost non-degenerate 2-ruled hypersurface$\}$ is an open and dense subset of $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$. $\square$

Now, we prove the following theorem which shows that the generic singularities of 2-ruled hypersurfaces are the cross cap $\times$ interval. Since any singularity of a generic smooth map germ of a 3-manifold into $\mathbf{R}^4$ is the cross cap $\times$ interval, the following theorem asserts that the generic singularities of 2-ruled hypersurfaces are the same as those of generic C^∞-maps of 3-manifolds into $\mathbf{R}^4$, although the set of 2-ruled hypersurfaces is a thin subset in the space of all C^∞-maps.

Theorem 5.3. *There exists an open and dense subset* $\mathscr{O} \subset C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$ *such that for any* $(\gamma, \delta, \varepsilon) \in \mathscr{O}$ *the* 2*-ruled hypersurface map germ* $F_{(\gamma,\delta,\varepsilon)}$ *is an immersion germ or is right-left equivalent to the cross cap* $\times$ *interval at any point* (t, u, v).

Proof. First, by Lemma 5.2 the set $\mathscr{R} = \{(\gamma, \delta, \varepsilon) \in C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3) \mid (\gamma, \delta, \varepsilon)$ gives an almost non-degenerate 2-ruled hypersurface$\}$ is an open and dense subset. By the condition (4) in Definition 5.1, $F_{(\gamma,\delta,\varepsilon)}$ for $(\gamma, \delta, \varepsilon) \in \mathscr{R}$ is non-singular at any point in the exceptional rulings.

We take $(\gamma, \delta, \varepsilon) \in C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$ such that

$$j^2(\gamma, \delta, \varepsilon)(t_0) \in J^2(I, \mathbf{R}^4 \times S^3 \times S^3) \backslash (\tilde{Q}_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3),$$

where $\pi_1^2 : J^2(I, \mathbf{R}^4 \times S^3 \times S^3) \to J^1(I, \mathbf{R}^4 \times S^3 \times S^3)$ is the natural projection, $\tilde{Q}_i = (\pi_1^2)^{-1}(Q_i)$ $(i = 1, 2, 3)$, and Q_i are the submanifolds defined in the proof of Lemma 5.2. Then, since $F_{(\gamma,\delta,\varepsilon)}$ is non-degenerate at t_0, there exists a striction curve $\sigma(t)$ near t_0. Now we rewrite the condition $\sigma'(t_0) = 0$ by using the formula in Remark 3.7. By replacing ε with

$$\varepsilon_1 = \frac{\varepsilon - (\delta \cdot \varepsilon)\delta}{\|\varepsilon - (\delta \cdot \varepsilon)\delta\|}$$

so that $\delta \cdot \varepsilon_1 = 0$, we get

$$\begin{aligned} \sigma'(t_0) = 0 &\Leftrightarrow \sigma'(t_0) \cdot \delta(t_0) = 0 \qquad \text{and} \qquad \sigma'(t_0) \cdot \varepsilon_1(t_0) = 0 \\ &\Leftrightarrow G = 0 \qquad \text{and} \qquad H = 0, \end{aligned}$$

where

$$G = (\gamma'(t_0) \cdot \delta(t_0)) + \frac{d}{dt}\left(\frac{A(t)}{C(t)}\right)\bigg|_{t=t_0} + \left(\frac{B(t_0)}{C(t_0)}\right)(\varepsilon_1'(t_0) \cdot \delta(t_0))$$

and

$$H = (\gamma'(t_0)\cdot\varepsilon_1(t_0)) + \left(\frac{A(t_0)}{C(t_0)}\right)(\delta'(t_0)\cdot\varepsilon_1(t_0)) + \frac{d}{dt}\left(\frac{B(t)}{C(t)}\right)\bigg|_{t=t_0}.$$

Here, $A(t)$, $B(t)$ and $C(t)$ are obtained by replacing ε with ε_1 in the formulas in Remark 3.7. Note that G and H are C^∞-functions of the partial derivatives at $t=t_0$ of the components of γ, δ and ε of order at most two. Then we define a C^∞-map Φ by:

$$\Phi : J^2(I,\mathbf{R}^4\times S^3\times S^3)\backslash(\tilde{Q}_1\cup\tilde{Q}_2\cup\tilde{Q}_3)\ni j^2(\gamma,\delta,\varepsilon)(t_0)\mapsto(G,H)\in\mathbf{R}^2.$$

To determine the rank of the Jacobian matrix of Φ at $j^2(\gamma,\delta,\varepsilon)(t_0)$, we calculate the derivative of Φ with respect to the coordinates of $J^2(I,\mathbf{R}^4\times S^3\times S^3)$ corresponding to the second order derivatives of the four components of γ. Then the derivatives of G coincide with the four components of

$$\begin{aligned}&(\delta'(t_0)\cdot\varepsilon_1(t_0))(\delta'(t_0)\cdot\varepsilon_1'(t_0))\delta(t_0)\\ &\quad+((\delta'(t_0)\cdot\varepsilon_1(t_0))(\varepsilon_1'(t_0)\cdot\varepsilon_1'(t_0))-(\delta'(t_0)\cdot\varepsilon_1(t_0))^3)\varepsilon_1(t_0)\\ &\quad+(-(\varepsilon_1'(t_0)\cdot\varepsilon_1'(t_0))+(\delta'(t_0)\cdot\varepsilon_1(t_0))^2)\delta'(t_0)+(\delta'(t_0)\cdot\varepsilon_1'(t_0))\varepsilon_1'(t_0),\end{aligned}$$

and the derivatives of H coincide with those of

$$\begin{aligned}&(-(\delta'(t_0)\cdot\delta'(t_0))(\delta'(t_0)\cdot\varepsilon_1(t_0))+(\delta'(t_0)\cdot\varepsilon_1(t_0))^3)\delta(t_0)\\ &\quad-(\delta'(t_0)\cdot\varepsilon_1(t_0))(\delta'(t_0)\cdot\varepsilon_1'(t_0))\varepsilon_1(t_0)\\ &\quad+(\delta'(t_0)\cdot\varepsilon_1'(t_0))\delta'(t_0)+(-(\delta'(t_0)\cdot\delta'(t_0))+(\delta'(t_0)\cdot\varepsilon_1(t_0))^2)\varepsilon_1'(t_0).\end{aligned}$$

Now we calculate the determinant of the matrix formed by the coefficients of $\delta'(t_0)$ and $\varepsilon_1'(t_0)$ of the above two formulas:

$$\begin{aligned}&(-(\varepsilon_1'(t_0)\cdot\varepsilon_1'(t_0))+(\delta'(t_0)\cdot\varepsilon_1(t_0))^2)(-(\delta'(t_0)\cdot\delta'(t_0))\\ &\qquad+(\delta'(t_0)\cdot\varepsilon_1(t_0))^2)-(\delta'(t_0)\cdot\varepsilon_1'(t_0))(\delta'(t_0)\cdot\varepsilon_1'(t_0))\\ &=C(t_0)\neq 0.\end{aligned}$$

So the rank of the Jacobian matrix is always equal to 2. Hence $(0,0)\in\mathbf{R}^2$ is a regular value of Φ and $T=\Phi^{-1}((0,0))$ is a closed submanifold of $J^2(I,\mathbf{R}^4\times S^3\times S^3)\backslash(\tilde{Q}_1\cup\tilde{Q}_2\cup\tilde{Q}_3)$ of codimension 2.

Therefore, the set $\mathcal{O}=\{(\gamma,\delta,\varepsilon)\in C^\infty(I,\mathbf{R}^4\times S^3\times S^3)\,|\,F_{(\gamma,\delta,\varepsilon)}$ is an almost non-degenerate 2-ruled hypersurface and the striction curve is an immersion} coincides with the set

$$\mathcal{O}' = \{(\gamma,\delta,\varepsilon) \in C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3) \mid \\ j^2(\gamma,\delta,\varepsilon) \text{ is transverse to } \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{S} \text{ and } T\},$$

where S is defined in the proof of Lemma 5.2 and $\tilde{S} = (\pi_1^2)^{-1}(S)$. By Thom's jet transversality theorem, the set $\mathcal{O}'$ is dense in $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$. Hence $\mathcal{O}$ is dense in $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$.

On the other hand, we define a map $F_\sharp : C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3) \to C^\infty(I \times J_1 \times J_2, \mathbf{R}^4)$ by $F_\sharp(\gamma,\delta,\varepsilon) = F_{(\gamma,\delta,\varepsilon)}$. Then, $F_\sharp$ is continuous. Furthermore, it is easy to check that the set $\mathscr{S} = \{f \in C^\infty(I \times J_1 \times J_2, \mathbf{R}^4) \mid f$ is an immersion or is the right-left equivalent to the cross cap $\times$ interval at any point of $I \times J_1 \times J_2\}$ is an open set.

Hence the set $F_\sharp^{-1}(\mathscr{S}) \cap \mathscr{R}$ is an open subset of $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$. By Theorem 2.5, it is clear that $\mathcal{O} = F_\sharp^{-1}(\mathscr{S}) \cap \mathscr{R}$. So, $\mathcal{O}$ is an open set of $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$. Therefore, $\mathcal{O}$ is an open and dense subset of $C^\infty(I, \mathbf{R}^4 \times S^3 \times S^3)$. This completes the proof. $\square$

Before closing this section, we discuss the behavior of the striction curve near the exceptional rulings. Let $F_{(\gamma,\delta,\varepsilon)}$ be an almost non-degenerate 2-ruled hypersurface. Then $F_{(\gamma,\delta,\varepsilon)}$ has the striction curve except for $t_i \in D$ (see Definition 5.1). Moreover, recall that $F_{(\gamma,\delta,\varepsilon)}$ is non-singular at any point in the exceptional rulings. So, the singular points of $F_{(\gamma,\delta,\varepsilon)}$ are located only on the striction curve. To study the behavior of the striction curve near a given point $t_i \in D$, we take constrictively adapted director curves δ and ε. By interchanging δ and ε if necessary, we may assume that $\varepsilon'(t_i) = k\delta'(t_i)$ for some $k \in \mathbf{R}$. Since δ, δ', ε and γ' span $\mathbf{R}^4$ near $t = t_i$, we can write $\varepsilon'(t) = a(t)\gamma'(t) - a(t)(\gamma'(t)\cdot\delta(t))\delta(t) + b(t)\delta'(t) - a(t)(\gamma'(t)\cdot\varepsilon(t))\varepsilon(t)$ for t near t_i. The coefficients for the striction curve $\sigma(t) = \gamma(t) + (A(t)/C(t))\delta(t) + (B(t)/C(t))\varepsilon(t)$ are given by

$$A(t) = a(t)b(t)z(t), \qquad B(t) = -a(t)z(t) \qquad \text{and} \qquad C(t) = a(t)^2 z(t),$$

where $z = (\delta'\cdot\delta')((\gamma'\cdot\gamma') - (\gamma'\cdot\delta)^2 - (\gamma'\cdot\varepsilon)^2) - (\gamma'\cdot\delta')^2$. Since $\varepsilon'(t) \to k\delta'(t)$ as $t \to t_i$, we have $a(t) \to 0$, $b(t) \to k$ as $t \to t_i$ and $z(t) \neq 0$, $a(t) \neq 0$ for $t \neq t_i$. So, we have

$$\lim_{t\to t_i}\left|\frac{B(t)}{C(t)}\right| = \lim_{t\to t_i}\left|\frac{-1}{a(t)}\right| = \infty.$$

Furthermore, since $\sigma(t) = \gamma(t) + (B(t)/C(t))((A(t)/B(t))\delta(t) + \varepsilon(t))$ and $(A(t)/$

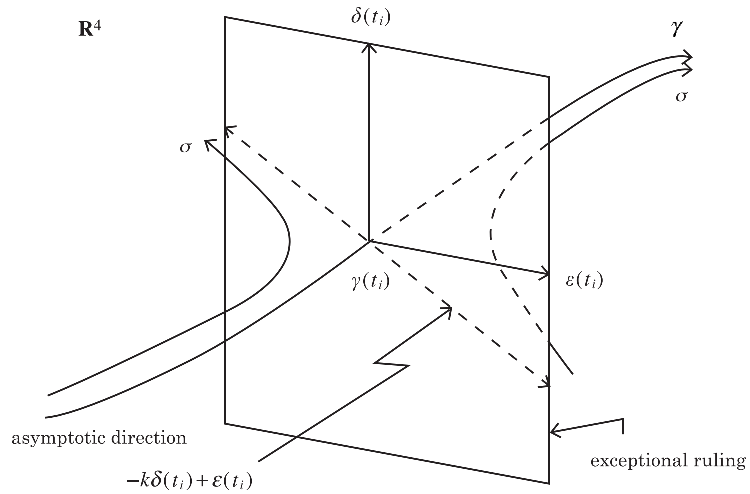

Fig. 1. The striction curve near an exceptional ruling

$B(t))\delta(t) + \varepsilon(t) \to -k\delta(t_i) + \varepsilon(t_i)$ as $t \to t_i$, the striction curve near t_i has an asymptotic direction

$$-k\delta(t_i) + \varepsilon(t_i)$$

in the exceptional ruling, that is, the two branches of the striction curve approaching to the exceptional ruling from the both sides have the same asymptotic direction.

Moreover, by the condition (3) of almost non-degeneracy (see Definition 5.1) we see that $a'(t_i) \neq 0$, so they diverge to opposite directions (see Figure 1).

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