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# A certain family of series associated with the Zeta and related functions

Junesang Choi and H. M. Srivastava

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**Abstract.** The history of problems of evaluation of series associated with the Riemann Zeta function can be traced back to Christian Goldbach (1690–1764) and Leonhard Euler (1707–1783). Many different techniques to evaluate various series involving the Zeta and related functions have since then been developed. The authors show how elegantly certain families of series involving the Zeta function can be evaluated by starting with a single known identity for the generalized (or Hurwitz) Zeta function. Some special cases and their connections with already developed series involving the Zeta and related functions are also considered.

## 1. Introduction, definitions, and the main result

The Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle (1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} & (\Re(s) > 0; \ \ s \neq 1) \end{cases} \tag{1.1}$$

satisfies the functional equation (see [24, p. 269]):

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \zeta(1-s) \sin\left(\frac{\pi s}{2}\right) \tag{1.2}$$

and takes on the following special or limit values (see [24, p. 271]):

$$\zeta(-1) = -\frac{1}{12}, \qquad \zeta(0) = -\frac{1}{2}, \qquad \text{and} \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi), \tag{1.3}$$

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and

$$\lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right)=\gamma \tag{1.4}$$

in terms of the Euler-Mascheroni constant $\gamma$ given by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right) \cong 0.577\,215\,664\,901\,532\,860\,606\,512\ldots.$$

The generalized (or Hurwitz) Zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) := \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad (\Re(s)>1;\ \ a\neq 0,-1,-2,\ldots), \tag{1.5}$$

which, just as $\zeta(s)$, can be continued meromorphically everywhere in the complex $s$-plane except for a simple pole at $s=1$ (with residue 1). It is not difficult to see from the definitions (1.1) and (1.5) that

$$\zeta(s,a+n)=\zeta(s,a)-\sum_{k=0}^{n-1}\frac{1}{(k+a)^s} \qquad (n\in\mathbf{N}:=\{1,2,3,\ldots\}) \tag{1.6}$$

and

$$\zeta(s,1)=\zeta(s)=(2^s-1)^{-1}\zeta\left(s,\frac{1}{2}\right). \tag{1.7}$$

The subject of evaluations of series involving the Zeta and related functions has a long history which can be traced back to Christian Goldbach (1690–1764) and Leonhard Euler (1707–1783) (see, for details, [19] and [20]). Many different techniques to evaluate various families of series involving the Zeta and related functions have since then been developed (*cf.*, *e.g.*, [2], [5], [8] to [12], [13], [19], and [20]). The main object of this paper is to show how nicely certain families of series involving the Zeta and related functions can be evaluated by starting with the following known identity for $\zeta(s,a)$ [19, p. 18, Eq. (6.13)]:

$$\sum_{k=0}^{\infty}\frac{(s)_k}{k!}\zeta(s+k,a)t^k=\zeta(s,a-t) \qquad (|t|<|a|). \tag{1.8}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of the familiar Gamma function, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0;\ \ \lambda\neq 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbf{N};\ \ \lambda\in\mathbf{C}). \end{cases} \tag{1.9}$$

The general series identity, which is to be proved in this paper, is contained in the following

THEOREM. *For every nonnegative integer n,*

$$\sum_{k=2}^{\infty} \frac{\zeta(k,a)}{(k)_{n+1}} t^{n+k} = \frac{(-1)^n}{n!}[\zeta'(-n,a-t) - \zeta'(-n,a)] + \sum_{k=1}^{n} \frac{(-1)^{n+k}}{n!}\binom{n}{k}[(H_n - H_{n-k})\zeta(k-n,a) - \zeta'(k-n,a)]t^k + [H_n + \psi(a)]\frac{t^{n+1}}{(n+1)!} \qquad (|t| < |a|;\ n \in \mathbf{N}_0), \tag{1.10}$$

*where $H_n$ denotes the harmonic numbers defined by*

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \tag{1.11}$$

*and it is understood* (*as elsewhere in this paper*) *that an empty sum is nil.*

For the sake of ready reference, we recall here the following identities and relationships which will be required in our proof of the above Theorem.

First of all, there exists a relationship between the generalized Zeta function $\zeta(s,a)$ and the Bernoulli polynomials $B_n(a)$ in the form (see [3, p. 264, Theorem 12.13]):

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1} \qquad (n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}), \tag{1.12}$$

which can be applied in order to evaluate $\zeta(-n,a)$ for special values of $n \in \mathbf{N}_0$.

The following identity involving the Bernoulli polynomials:

$$\frac{B_{n+1}(a+t)}{n+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{B_{k+1}(a)}{k+1}t^{n-k} + \frac{t^{n+1}}{n+1} \qquad (n \in \mathbf{N}_0) \tag{1.13}$$

results from the known formula (see [3, p. 275]):

$$B_n(a+t) = \sum_{k=0}^{n}\binom{n}{k}B_k(a)t^{n-k} \qquad (n \in \mathbf{N}_0) \tag{1.14}$$

when we replace $n$ in (1.14) by $n+1$ and divide both sides of the resulting equation by $n+1$.

We also recall here the following Laurent series expansion of $\zeta(s,a)$ at $s=1$ (see [24, p. 271]):

$$\zeta(s,a) = \frac{1}{s-1} - \psi(a) + \sum_{n=1}^{\infty} c_n (s-1)^n, \tag{1.15}$$

where the coefficients $c_n$ are constants to be determined and the Psi (or Digamma) function $\psi(z)$ defined by

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(t)\,dt \tag{1.16}$$

is meromorphic in the complex $z$-plane with simple poles at $z = 0, -1, -2, \ldots$ (with residue $-1$). In fact, we have (see, *e.g.*, [21, pp. 24–25]):

$$\psi(z+n) = \psi(z) + \sum_{j=1}^{n} \frac{1}{z+j-1} \qquad (n \in \mathbf{N}) \tag{1.17}$$

and

$$\frac{d}{dz}\{(z)_n\} = (z)_n[\psi(z+n) - \psi(z)], \tag{1.18}$$

which follows easily from the definitions (1.9) and (1.16).

## 2. Proof of the Theorem

Upon transposing the first $n+2$ terms from $k=0$ to $k=n+1$ in (1.8) to the right-hand side, if we divide both sides of the resulting equation by $s+n$, we get

$$\sum_{k=n+2}^{\infty} (s)_n (s+n+1)_{k-n-1} \zeta(s+k,a) \frac{t^k}{k!} = \frac{g_n(s,t,a)}{s+n} \qquad (|t| < |a|;\ \ n \in \mathbf{N}_0), \tag{2.1}$$

where, for convenience,

$$g_n(s,t,a) := \zeta(s,a-t) - \sum_{k=0}^{n+1} \frac{(s)_k}{k!} \zeta(s+k,a) t^k. \tag{2.2}$$

Now we shall show that

$$\lim_{s \to -n} g_n(s,t,a) = 0. \tag{2.3}$$

Since $\zeta(s+n+1,a)$ has a simple pole at $s=-n$ with its residue 1, we find that

$$\lim_{s\to -n}(s+n)\zeta(s+n+1,a)=1. \tag{2.4}$$

By rewriting (2.2) in the form:

$$g_n(s,t,a)=\zeta(s,a-t)-\sum_{k=0}^{n}\frac{(s)_k}{k!}\zeta(s+k,a)t^k-(s)_n(s+n)\zeta(s+n+1,a)\frac{t^{n+1}}{(n+1)!},$$

if we take the limit as $s\to -n$ with the aid of (2.4) and make use of the elementary identity:

$$\binom{\lambda}{n}=\frac{(-1)^n(-\lambda)_n}{n!}\qquad (n\in\mathbf{N}_0;\ \ \lambda\in\mathbf{C}),$$

we obtain

$$\lim_{s\to -n} g_n(s,t,a)=\zeta(-n,a-t)-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\zeta(-k,a)t^{n-k}+\frac{(-1)^{n+1}}{n+1}t^{n+1},$$

which, in view of the relationship (1.12), can be put in its equivalent form:

$$\lim_{s\to -n} g_n(s,t,a)=-\frac{B_{n+1}(a-t)}{n+1}+\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{k+1}(a)}{k+1}t^{n-k}+\frac{(-1)^{n+1}}{n+1}t^{n+1},$$

from which our assertion (2.3) follows easily by applying (1.13). Thus, by l'Hôpital's rule, we have

$$\lim_{s\to -n}\frac{g_n(s,t,a)}{s+n}=\lim_{s\to -n}\frac{\partial}{\partial s}\{g_n(s,t,a)\}. \tag{2.5}$$

Next, by appealing to (2.2) and (1.18), we observe that

$$\begin{aligned}\frac{\partial}{\partial s}\{g_n(s,t,a)\}&=\zeta'(s,a-t)-\zeta'(s,a)\\&\quad-\sum_{k=1}^{n}h(s,a,k)\frac{t^k}{k!}-h(s,a,n+1)\frac{t^{n+1}}{(n+1)!},\end{aligned} \tag{2.6}$$

where, for convenience,

$$h(s,a,k):=(s)_k[\{\psi(s+k)-\psi(s)\}\zeta(s+k,a)+\zeta'(s+k,a)]. \tag{2.7}$$

In view of (1.17), (2.7) yields

$$\lim_{s\to -n}h(s,a,k)=-(-n)_k[(H_n-H_{n-k})\zeta(k-n,a)-\zeta'(k-n,a)] \tag{2.8}$$

$$(k=1,\ldots,n)$$

and

$$\lim_{s\to -n} h(s,a,n+1) \tag{2.9}$$

$$= \lim_{s\to -n}\left[\left(\sum_{j=0}^{n}\frac{(s)_n}{s+j}\right)(s+n)\zeta(s+n+1,a)+(s)_{n+1}\zeta'(s+n+1,a)\right],$$

which, upon writing

$$\sum_{j=0}^{n}\frac{(s)_n}{s+j}=\sum_{j=0}^{n-1}\frac{(s)_n}{s+j}+\frac{(s)_n}{s+n},$$

reduces to the form:

$$\lim_{s\to -n} h(s,a,n+1) = -\sum_{j=0}^{n-1}\frac{(-n)_n}{n-j}+\lim_{s\to -n}\phi(s,a,n), \tag{2.10}$$

where, for convenience,

$$\phi(s,a,n) := (s)_n\zeta(s+n+1,a)+(s)_{n+1}\zeta'(s+n+1,a). \tag{2.11}$$

Now, by virtue of (1.15), we readily have

$$\begin{aligned}\lim_{s\to -n}\phi(s,a,n) &= \lim_{s\to -n}(s)_n\left[-\psi(a)+\sum_{j=1}^{\infty}(j+1)c_j(s+n)^j\right]\\ &= -(-n)_n\psi(a)\\ &= (-1)^{n+1}n!\psi(a).\end{aligned} \tag{2.12}$$

It follows from (2.10) and (2.12) that

$$\lim_{s\to -n} h(s,a,n+1) = (-1)^{n+1}n![H_n+\psi(a)], \tag{2.13}$$

where $H_n$ denotes the harmonic numbers defined by (1.11).

Making use of (2.8) and (2.13), we find from (2.6) that

$$\begin{aligned}\lim_{s\to -n}\frac{\partial}{\partial s}\{g_n(s,t,a)\} &= \zeta'(-n,a-t)-\zeta'(-n,a)\\ &\quad+\sum_{k=1}^{n}(-1)^k\binom{n}{k}[(H_n-H_{n-k})\zeta(k-n,a)\\ &\qquad-\zeta'(k-n,a)]t^k\\ &\quad+\frac{(-1)^n}{n+1}[H_n+\psi(a)]t^{n+1}.\end{aligned} \tag{2.14}$$

Finally, since

$$(2.15)\quad \lim_{s\to -n}\sum_{k=n+2}^{\infty}(s)_n(s+n+1)_{k-n-1}\zeta(s+k,a)\frac{t^k}{k!}=(-1)^n n!\sum_{k=2}^{\infty}\frac{\zeta(k,a)}{(k)_{n+1}}t^{k+n},$$

by equating the second members of (2.14) and (2.15), we are led immediately to the desired series identity (1.10). This evidently completes our proof of the Theorem.

Infinite sums of the type occurring in (1.10) can also be evaluated, in a *markedly different* manner, in terms of such higher transcendental functions as the *multiple* Gamma functions (see, for details, [16, p. 10, Theorem 1]).

## 3. Applications of the Theorem

Upon setting $n = 0, 1, 2, 3,$ and 4 in (1.10), if we make use of the appropriate identities which are readily available in the mathematical literature, we shall obtain the following known or new formulas for closed-form evaluations of several families of series involving the generalized (or Hurwitz) Zeta function $\zeta(s,a)$:

$$(3.1)\qquad \sum_{k=2}^{\infty}\frac{\zeta(k,a)}{k}t^k=\log\Gamma(a-t)-\log\Gamma(a)+t\psi(a)\qquad (|t|<|a|),$$

which is given (for example) in [24, p. 276], [15, p. 358, Entry (54.11.1)], and [8, p. 107, Eq. (2.11)];

$$(3.2)\qquad \sum_{k=2}^{\infty}\frac{\zeta(k,a)}{k(k+1)}t^{k+1}=\zeta'(-1,a)-\zeta'(-1,a-t)$$
$$+\left[\frac{1}{2}-a-\log\Gamma(a)+\frac{1}{2}\log(2\pi)\right]t$$
$$+[1+\psi(a)]\frac{t^2}{2}\qquad (|t|<|a|);$$

$$(3.3)\quad \sum_{k=2}^{\infty}\frac{\zeta(k,a)}{k(k+1)(k+2)}t^{k+2}$$
$$=\frac{1}{2}[\zeta'(-2,a-t)-\zeta'(-2,a)]+\left[\frac{1}{4}\left(a^2-a+\frac{1}{6}\right)+\zeta'(-1,a)\right]t$$
$$+\left[\frac{3}{2}-3a+\log(2\pi)-2\log\Gamma(a)\right]\frac{t^2}{2}+[3+2\psi(a)]\frac{t^3}{12}\qquad (|t|<|a|);$$

$$\text{(3.4)} \quad \sum_{k=2}^{\infty} \frac{\zeta(k,a)}{k(k+1)(k+2)(k+3)} t^{k+3}$$

$$= \frac{1}{6}[\zeta'(-3,a) - \zeta'(-3,a-t)] - \left[\frac{1}{3}\left(a^3 - \frac{3}{2}a^2 + \frac{1}{2}a\right) + 3\zeta'(-2,a)\right]\frac{t}{6}$$

$$+ \left[\frac{5}{2}\left(a^2 - a + \frac{1}{6}\right) + 6\zeta'(-1,a)\right]\frac{t^2}{12}$$

$$+ \left[\frac{11}{2} - 11a + 3\log(2\pi) - 6\log\Gamma(a)\right]\frac{t^3}{36}$$

$$+ [11 + 6\psi(a)]\frac{t^4}{144} \qquad (|t| < |a|);$$

$$\text{(3.5)} \quad \sum_{k=2}^{\infty} \frac{\zeta(k,a)}{k(k+1)(k+2)(k+3)(k+4)} t^{k+4}$$

$$= \frac{1}{24}[\zeta'(-4,a-t) - \zeta'(-4,a)]$$

$$+ \left[\frac{1}{4}\left(a^4 - 2a^3 + a^2 - \frac{1}{30}\right) + 4\zeta'(-3,a)\right]\frac{t}{24}$$

$$- \left[\frac{7}{3}\left(a^3 - \frac{3}{2}a^2 + \frac{1}{2}a\right) + 12\zeta'(-2,a)\right]\frac{t^2}{48}$$

$$+ \left[13\left(a^2 - a + \frac{1}{6}\right) + 24\zeta'(-1,a)\right]\frac{t^3}{144}$$

$$+ [25 - 50a + 12\log(2\pi) - 24\log\Gamma(a)]\frac{t^4}{576}$$

$$+ [25 + 12\psi(a)]\frac{t^5}{1440} \qquad (|t| < |a|).$$

Setting $a = 1$ in (3.5), we readily obtain

$$\text{(3.6)} \quad \sum_{k=2}^{\infty} \frac{\zeta(k)}{k(k+1)(k+2)(k+3)(k+4)} t^{k+4}$$

$$= \frac{1}{24}\left[\zeta'(-4,1-t) - \frac{3\zeta(5)}{4\pi^4}\right] - \left(\frac{5}{72} + 4\log C\right)\frac{t}{24} + \frac{\zeta(3)}{16\pi^2}t^2$$

$$+ \left(\frac{25}{6} - 24\log A\right)\frac{t^3}{144} + [12\log(2\pi) - 25]\frac{t^4}{576}$$

$$+ (25 - 12\gamma)\frac{t^5}{1440} \qquad (|t| < 1),$$

which, for $t = \frac{1}{2}$, yields

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k(k+1)(k+2)(k+3)(k+4)2^k} = \frac{1}{48}\log(2\pi) - \frac{\gamma}{240} - \frac{1}{3}\log A - \frac{4}{3}\log C + \frac{\zeta(3)}{4\pi^2} - \frac{31\zeta(5)}{32\pi^4}, \tag{3.7}$$

where we have made use of such results as (for example) the relationship (1.7) and the derivative formula (*cf.*, *e.g.*, [20, p. 387, Eq. (1.15)]):

$$\zeta'(-2n) = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}} \zeta(2n+1) \qquad (n \in \mathbf{N}),$$

which follows easily from

$$\zeta(-2n) = 0 \qquad (n \in \mathbf{N})$$

and Riemann's functional equation (1.2). Here, and elsewhere in this paper, $A$ denotes the Glaisher-Kinkelin constant defined by

$$\log A = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k \log k - \left(\frac{1}{2}n^2 + \frac{1}{2}n + \frac{1}{12}\right)\log n + \frac{1}{4}n^2\right\},$$

the numerical value of $A$ being given by

$$A \cong 1.282427130\ldots,$$

and $C$ is a mathematical constant defined by

$$\log C = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^3 \log k - \left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 - \frac{1}{120}\right)\log n + \frac{1}{16}n^4 - \frac{1}{12}n^2\right],$$

the numerical value of $C$ being given by

$$C \cong 301\,393\,392\,412\,467\,84 \times 10^{-2714341}.$$

Some of these and other mathematical constants have already occurred in the recent revival of the multiple Gamma functions [4] in the study of the determinants of the Laplacians on the $n$-dimensional unit sphere $\mathbf{S}^n$ (see [6], [18], [22], and [23]).

Similarly, we obtain

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k(k+1)(k+2)(k+3)2^k} = \frac{59}{720}\log 2 + \frac{1}{12}\log \pi - \frac{\gamma}{48} - \log A - \frac{5}{2}\log C + \frac{\zeta(3)}{2\pi^2}; \tag{3.8}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k(k+1)(k+2)2^k} = -\frac{3}{8} + \frac{1}{2}\log(2\pi) - \frac{\gamma}{12} - 2\log A + \frac{7\zeta(3)}{8\pi^2}; \tag{3.9}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k(k+1)2^k} = -\frac{\gamma}{4} + \frac{7}{12}\log 2 + \frac{1}{2}\log \pi - 3\log A, \tag{3.10}$$

which is recorded in [8, p. 109, Eq. (2.23)];

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{k \cdot 2^k} = -\frac{\gamma}{2} + \frac{1}{2}\log \pi, \tag{3.11}$$

which is also recorded in [8, p. 109, Eq. (2.21)].

We remark in passing that the various series identities presented here are potentially useful in deriving further identities for series involving the Zeta and related functions. For example, if we replace $t$ in (3.2) by $-t$, we get

$$\begin{aligned}\sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k,a)}{k(k+1)} t^{k+1} &= \zeta'(-1, a-t) - \zeta'(-1,a) + \left[\frac{1}{2} - a - \log \Gamma(a) + \frac{1}{2}\log(2\pi)\right] t \\ &\quad - [1+\psi(a)]\frac{t^2}{2} \qquad (|t| < |a|).\end{aligned} \tag{3.12}$$

And since [7, p. 164, Eq. (2.8)]

$$\Gamma_2(a) = A \cdot (2\pi)^{(1/2)-(1/2)a} \cdot \exp\left[-\frac{1}{12} + \zeta'(-1,a) + (1-a)\zeta'(0,a)\right], \tag{3.13}$$

by making use of the following consequence of Hermite's formula for $\zeta(s,a)$ (see [24, pp. 270–271]):

$$\zeta'(0,a) = \frac{\partial}{\partial s}\{\zeta(s,a)\}|_{s=0} = \log \Gamma(a) - \frac{1}{2}\log(2\pi),$$

the series identity (3.12) can be written as follows in an equivalent form involving the double Gamma function $\Gamma_2$:

$$\begin{aligned}(3.14) \qquad \sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k,a)}{k(k+1)} t^{k+1} = {} & (1-a)\log\Gamma(a) - \log\Gamma_2(a) \\ & + (t+a-1)\log\Gamma(a+t) + \log\Gamma_2(a+t) \\ & + \left[\frac{1}{2} - a + \frac{1}{2}\log(2\pi) - \log\Gamma(a)\right]t \\ & - [\psi(a)+1]\frac{t^2}{2} \qquad (|t|<|a|),\end{aligned}$$

which was proven, in a markedly different way, by Choi and Srivastava [8, p. 108, Eq. (2.14)].

Finally, we deduce yet another interesting identity by suitably combining the special cases of (1.10) when $a=1$ and $a=2$. By applying (1.7), (1.8), and the familiar relationship:

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} \qquad (n \in \mathbf{N}),$$

we thus find that

$$\begin{aligned}(3.15) \qquad \sum_{k=1}^{\infty}\frac{t^{n+k}}{(k)_{n+1}} = {} & \frac{(-1)^{n+1}}{n!}(1-t)^n\log(1-t) \\ & + \sum_{k=1}^{n}\frac{(-1)^{n+k}}{n!}\binom{n}{k}(H_n - H_{n-k})t^k \qquad (|t|<1;\ n \in \mathbf{N}_0),\end{aligned}$$

which, in the special case when $n=2$, is a known result recorded (for example) by Hansen [15, p. 37, Entry (5.7.40); p. 74, Entry (5.16.26)].

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*Junesang Choi*
*Department of Mathematics*
*College of Natural Sciences*
*Dongguk University*
*Kyongju* 780-714
*Korea*
*E-Mail: junesang@mail.dongguk.ac.kr*

*H. M. Srivastava*
*Department of Mathematics and Statistics*
*University of Victoria*
*Victoria, British Columbia V*8*W* 3*P*4
*Canada*
*E-Mail: harimsri@math.uvic.ca*