Hiroshima Math. J.
33 (2003), 167–187

Behavior of the life span for solutions to the system of reaction-diffusion equations

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(Received August 29, 2001)
(Revised February 7, 2003)

Abstract. We consider the weakly coupled system of reaction-diffusion equations[1,2]

$$u_t = \Delta u + a(x)v^p, \qquad v_t = \Delta v + b(x)u^q,$$

$$u(x,0) = \lambda^{\mu}\varphi(x), \qquad v(x,0) = \lambda^{\nu}\psi(x),$$

where $0 \le a(x),\ b(x) \in C(\mathbf{R}^N),\ \varphi(x), \psi(x) \ge 0$ are bounded continuous functions in $\mathbf{R}^N,\ p, q > 1,\ \mu, \nu > 0$, and $\lambda > 0$ are parameters. The existense of solutions, blow-up conditions, and global solutions of the above equations with $a(x) \equiv |x|^{\sigma_1},\ b(x) \equiv |x|^{\sigma_2}$ $(0 \le \sigma_1 < N(p-1),\ 0 \le \sigma_2 < N(q-1))$ are studied by Mochizuki and Huang. In this paper, we consider an estimate of maximal existence time of blow-up solutions as λ goes to 0 or ∞, when $a(x), b(x)$ are more general functions.

1. Introduction and statement of results

We consider bounded, nonnegative solutions to the Cauchy problem for a weakly coupled system

$$\begin{cases} u_t = \Delta u + a(x)v^p & (x \in \mathbf{R}^N, t > 0), \\ v_t = \Delta v + b(x)u^q & (x \in \mathbf{R}^N, t > 0), \\ u(x,0) = \lambda^{\mu}\varphi(x) & (x \in \mathbf{R}^N), \\ v(x,0) = \lambda^{\nu}\psi(x) & (x \in \mathbf{R}^N), \end{cases} \tag{1}$$

where $0 \le a(x),\ b(x) \in C(\mathbf{R}^N),\ 0 \le \varphi(x),\ \psi(x) \in BC(\mathbf{R}^N)$; here $BC(\mathbf{R}^N)$ is the set of bounded continuous functions on $\mathbf{R}^N,\ p, q > 1,\ \mu, \nu > 0$, and $\lambda > 0$ are parameters. Since the nonlinearities, $a(x)v^p, b(x)u^q$, are locally continuous in x and locally Lipschitz in u, v, it follows from standard results that any solution $u(x,t), v(x,t) \ge 0$ of the equation (1) is in fact classical; that is, $u, v \in C^{2,1}(\mathbf{R}^N \times (0,T)) \cap C(\mathbf{R}^N \times [0,T))$ for some $T > 0$. Thus, the comparison theorem holds from Theorem 1 in [1]; i.e. if

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[1] 2000 *Mathematics Subject Classification*. 35K57, 58J35.

[2] *Key Words and Phrases*. reaction-diffusion equations, heat and other parabolic equation methods.

$$\underline{f_0} \le u(x,0) \le \overline{f_0}, \qquad \underline{g_0} \le v(x,0) \le \overline{g_0},$$

it follows that for $x \in \mathbf{R}^N$, $0 \le t \le T$,

$$\underline{f}(t) \le u(x,t) \le \bar{f}(t), \qquad \underline{g}(t) \le v(x,t) \le \bar{g}(t),$$

where $(\underline{f}(t), \underline{g}(t))$ and $(\bar{f}(t), \bar{g}(t))$ are subsolution and supersolution of (1) with initial value $(\underline{f_0}, \underline{g_0})$ and $(\overline{f_0}, \overline{g_0})$.

We let $T_\lambda^* > 0$ be the maximal existence time. From the general theory of evolution equation [9], it follows that there exists a unique bounded solution $u(x,t)$ to the equation

$$\begin{cases} u_t = \Delta u + a(x)u^p & (x \in \mathbf{R}^N, t > 0), \\ u(x,0) = \lambda\varphi(x) & (x \in \mathbf{R}^N), \end{cases} \tag{2}$$

which satisfies

$$\sup_{t \in [0,T)} \|u(t)\|_\infty < \infty \qquad \text{for } 0 < \exists T \le \infty,$$

where $a(x)$ is a continuous function which satisfies that $a(x)/|x|^\sigma$ $(\sigma > -2)$ is bounded when $|x|$ is sufficiently large, and $0 \le \varphi(x) \le \delta \exp(-\gamma|x|^2)$ holds. So we define T_λ^* as follows:

$$T_\lambda^* := \sup\left\{ T > 0;\ \sup_{t \in [0,T)} \{\|u(t)\|_\infty + \|v(t)\|_\infty\} < \infty \right\}.$$

If $T_\lambda^* = \infty$, the solutions are global. The global existence and nonexistence are studied by Escobedo-Herrero [2] and Mochizuki [7] in the case $a(x) \equiv b(x) \equiv 1$, and are extended in [8] to the case $a(x) = |x|^{\sigma_1}$, $b(x) = |x|^{\sigma_2}$, where $0 \le \sigma_1 < N(p-1)$, $0 \le \sigma_2 < N(q-1)$.

In this paper, we shall consider a precise estimate of T_λ^* as λ goes to 0 or ∞. This problem is studied in Huang-Mochizuki-Mukai [5] and Mochizuki [7] in the special case $a(x) \equiv b(x) \equiv 1$. On the other hand, Pinsky [11] studied the life span of the single equation (2) where $a(x)$ is some kind of function. We shall extend the results of [5] and [7] and prove by the same methods as [11]. We put

$$\alpha = \frac{2(p+1)}{pq-1}, \qquad \beta = \frac{2(q+1)}{pq-1}.$$

THEOREM 1. *Assume that a, b satisfy*

$$a(x) \sim |x|^{\sigma_1}, \qquad b(x) \sim |x|^{\sigma_2} \qquad \text{as } |x| \to \infty,$$

where $\sigma_1, \sigma_2 > -2$ if $N \ge 2$, $\sigma_1, \sigma_2 > -1$ if $N = 1$, and that initial data φ, ψ satisfy

$$0 \le \varphi(x), \qquad \psi(x) \le \delta \exp(-\gamma|x|^2)$$

for some $\delta, \gamma > 0$.

(i) *Suppose that* $\alpha + \delta_1 > N$ (*or* $\beta + \delta_2 > N$), *where*

$$\delta_1 = \frac{\sigma_2 p + \sigma_1}{pq - 1}, \qquad \delta_2 = \frac{\sigma_1 q + \sigma_2}{pq - 1}.$$

Then there exist $\lambda_1 > 0$ *and* $C > 0$ *such that*

$$T_\lambda^* \le C\lambda^{-2\mu/(\alpha+\delta_1-N)} \ (or \ \le C\lambda^{-2\nu/(\beta+\delta_2-N)}) \qquad for \ \lambda < \lambda_1.$$

(ii) *Suppose that*

$$p < p^* = 1 + \frac{2+\sigma_1}{N}, \qquad q < q^* = 1 + \frac{2+\sigma_2}{N}.$$

Let μ, ν *be chosen to satisfy*

$$\frac{\mu}{\nu} = \frac{\alpha + \delta_1 - N}{\beta + \delta_2 - N}.$$

Then we have

$$T_\lambda^* \sim \lambda^{-2\mu/(\alpha+\delta_1-N)} = \lambda^{-2\nu/(\beta+\delta_2-N)} \qquad as \ \lambda \to 0.$$

Theorem 2. *Assume that* $0 \le a, b, \varphi, \psi \in BC(\mathbf{R}^N)$ *and that there is a smooth bounded domain* $D \subset \mathbf{R}^N$ *such that*

$$\inf_{x \in D} a(x), \inf_{x \in D} b(x), \inf_{x \in D} \varphi(x), \inf_{x \in D} \psi(x) > 0.$$

(i) *Suppose that* $p\nu > \mu$, $q\mu > \nu$. *Then there exist* $\lambda_1 > 0$ *and* $C > 0$ *such that*

$$T_\lambda^* \le C\lambda^{-2\mu/\alpha} \ (or \ \le C\lambda^{-2\nu/\beta}) \qquad for \ \lambda > \lambda_1.$$

(ii) *Let* μ, ν *be chosen to satisfy* $\mu/\nu = \alpha/\beta$. *Then we have*

$$T_\lambda^* \sim \lambda^{-2\mu/\alpha} = \lambda^{-2\nu/\beta} \qquad as \ \lambda \to \infty.$$

Remark 1. *Theorems* 1 *and* 2 *are the extension of results of* [11]. *If we put* $u = v$, $\varphi = \psi$, $a = b$, $p = q$, $\sigma_1 = \sigma_2$, $\mu = \nu = 1$ *in these theorems, the same results as Theorem* 1 (i) *and Theorem* 3 (i) *in* [11] *are obtained respectively.*

We shall prove Theorems 1 and 2 in Sections 2 and 3, respectively. In the sequel, we will use the notation

$$P(x,t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

We conclude this section by noting the following well-known integral representation which holds for bounded solutions $u(x,t), v(x,t)$ to (1):

$$u(x,t) = \lambda^{\mu} \int_{\mathbf{R}^N} P(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbf{R}^N} P(x-y,t-s)a(y)v(y,s)^p dyds,$$
$$v(x,t) = \lambda^{\nu} \int_{\mathbf{R}^N} P(x-y,t)\psi(y)dy + \int_0^t \int_{\mathbf{R}^N} P(x-y,t-s)b(y)u(y,s)^q dyds. \tag{3}$$

2. Proof of Theorem 1

We begin with the proof of the upper bounds.

Lemma 2.1. *Let $u(x,t), v(x,t)$ satisfy (1). Then for any $t_0 \in (0, T_\lambda^*)$, there exists a $c > 0$ such that*

$$u(x,t) \geq \lambda^{\mu} c t^{-N/2} \exp\left(-\frac{|x|^2}{2t}\right),$$
$$v(x,t) \geq \lambda^{\nu} c t^{-N/2} \exp\left(-\frac{|x|^2}{2t}\right), \qquad \text{for } t \in [t_0, T_\lambda^*), x \in \mathbf{R}^N.$$

Proof. We prove only the first inequality. Since $\varphi(x) \not\equiv 0$, there exists $D_1 \subset \mathbf{R}^N$ such that

$$c_1 = \inf_{x \in D_1} \varphi(x) > 0.$$

From the inequality $|x-y|^2 \leq 2|x|^2 + 2|y|^2$ and (3), it follows that

$$\begin{aligned} u(x,t) &\geq \lambda^{\mu} \int_{\mathbf{R}^N} P(x-y,t)\varphi(y)dy \geq \lambda^{\mu} c_1 \int_{D_1} P(x-y,t)dy \\ &\geq \lambda^{\mu} (4\pi t)^{-N/2} c_1 \int_{D_1} \exp\left(-\frac{|x|^2}{2t} - \frac{|y|^2}{2t}\right) dy \\ &\geq \lambda^{\mu} (4\pi)^{-N/2} c_1 t^{-N/2} \exp\left(-\frac{|x|^2}{2t}\right) \int_{D_1} \exp\left(-\frac{|y|^2}{2t_0}\right) dy, \end{aligned}$$

for $t \geq t_0$. □

Let $D_n = \{x \in \mathbf{R}^N; n < |x| < 2n\}$ if $N \geq 2$, and $D_n = \{x \in \mathbf{R}^N; n < x < 2n\}$ if $N = 1$. Let $\theta_n > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in D_n, and let $\omega_n(x)$ denote the corresponding positive eigenfunction, normalized by $\int_{D_n} \omega_n(x)dx = 1$. Note that since D_n contains an N-dimensional cube of length kn for an appropriate constant $k \in (0,1)$, it follows that there exists a constant $c > 0$ such that

$$\theta_n \le cn^{-2}. \tag{4}$$

By assumption, there exist n_0 and $c_1 > 0$ such that

$$a(x) \ge c_1|x|^{\sigma_1}, \quad b(x) \ge c_1|x|^{\sigma_2}, \qquad \text{for } |x| \ge n_0. \tag{5}$$

From now on, we will always assume that $n \ge n_0$. Define

$$F_n(t) = \int_{D_n} u(x,t)\omega_n(x)dx,$$

$$G_n(t) = \int_{D_n} v(x,t)\omega_n(x)dx, \qquad \text{for } 0 \le t < T_\lambda^*.$$

Then it follows that $F_n(t) \le \|u(t)\|_\infty$, $G_n(t) \le \|v(t)\|_\infty$ for all $n > 0$. Thus, T_λ^* is no more than the blow up time of $(F_n(t), G_n(t))$. Let $\partial/\partial n$ be the outward normal derivative to D_n at $x \in \partial D_n$. From Green's formula and the fact that $\omega_n(x) = 0$ and $\partial\omega_n/\partial n \le 0$ on ∂D_n, we obtain

$$\int_{D_n} (\Delta u(x,t)\omega_n(x) - u(x,t)\Delta\omega_n(x))dx = \int_{\partial D_n} \left(\frac{\partial u}{\partial n}\omega_n - u\frac{\partial \omega_n}{\partial n}\right)dS \ge 0.$$

From Hölder's inequality, the inequality

$$\int_{D_n} v(x,t)\omega_n(x)dx \le \left(\int_{D_n} v(x,t)^p\omega_n(x)dx\right)^{1/p}$$

holds. Using (4), (5), we obtain from (1)

$$\begin{aligned}
F_n'(t) &= \int_{D_n} u_t(x,t)\omega_n(x)dx \\
&= \int_{D_n} (\Delta u(x,t) + a(x)v(x,t)^p)\omega_n(x)dx \\
&\ge \int_{D_n} u(x,t)\Delta\omega_n(x)dx + c_1\int_{D_n} |x|^{\sigma_1}v(x,t)^p\omega_n(x)dx \\
&\ge -\theta_n\int_{D_n} u(x,t)\omega_n(x)dx + c_0n^{\sigma_1}\int_{D_n} v(x,t)^p\omega_n(x)dx \\
&\ge -cn^{-2}F_n(t) + c_0n^{\sigma_1}G_n(t)^p.
\end{aligned}$$

Thus, we obtain the following inequalities:

$$\begin{cases} F_n'(t) \ge -cn^{-2}F_n(t) + c_0n^{\sigma_1}G_n(t)^p & (t > 0), \\ G_n'(t) \ge -cn^{-2}G_n(t) + c_0n^{\sigma_2}F_n(t)^q & (t > 0). \end{cases} \tag{6}$$

By Lemma 2.1, there exists a $C > 0$ such that $u(x, n^2) \geq C\lambda^{\mu} n^{-N}$, $v(x, n^2) \geq C\lambda^{\nu} n^{-N}$ for $n < |x| < 2n$, thus

$$F_n(n^2) \geq C\lambda^{\mu} n^{-N}, \qquad G_n(n^2) \geq C\lambda^{\nu} n^{-N}.$$

Let $f_n, g_n \in C^0([0, T_\lambda^*)) \cap C^1((0, T_\lambda^*))$ be the solution to the system of ordinary differential equations

$$\begin{cases} f_n'(t) = -cn^{-2} f_n(t) + c_0 n^{\sigma_1} g_n(t)^p & (t > 0), \\ g_n'(t) = -cn^{-2} g_n(t) + c_0 n^{\sigma_2} f_n(t)^q & (t > 0), \\ f_n(n^2) = C\lambda^{\mu} n^{-N}, \\ g_n(n^2) = C\lambda^{\nu} n^{-N}. \end{cases} \tag{7}$$

Then $(F_n(t), G_n(t))$ is a supersolution of (7). By the scaling

$$\begin{aligned} f(t) &= c^{-\alpha/2} c_0^{\alpha/2} n^{\alpha+\delta_1} f_n(c^{-1} n^2 (t + c)), \\ g(t) &= c^{-\beta/2} c_0^{\beta/2} n^{\beta+\delta_2} g_n(c^{-1} n^2 (t + c)), \end{aligned} \tag{8}$$

we obtain the simpler system of equations

$$\begin{cases} f'(t) = -f(t) + g(t)^p & (t > 0), \\ g'(t) = -g(t) + f(t)^q & (t > 0), \end{cases} \tag{9}$$

with the initial data

$$f(0) = C_p \lambda^{\mu} n^{\alpha+\delta_1-N}, \qquad g(0) = C_q \lambda^{\nu} n^{\beta+\delta_2-N},$$

where $C_p = Cc^{-\alpha/2} c_0^{\alpha/2}$, $C_q = Cc^{-\beta/2} c_0^{\beta/2}$.

Lemma 2.2. *Let $(f(t), g(t))$ be the solution to (9) with the initial data*

$$f(0) > 1, \qquad g(0) = 0.$$

If $f(0)$ is sufficiently large, then $(f(t), g(t))$ blows up in finite time. Moreover, the life span T_0 of $(f(t), g(t))$ is estimated from above by

$$T_0 \leq t_0 + \int_{f(t_0)g(t_0)}^{\infty} \{C(p,q)\xi^{(p+1)(q+1)/(p+q+2)} - 2\xi\}^{-1} d\xi, \tag{10}$$

where

$$C(p,q) = \left(\frac{p+q+2}{p+1}\right)^{(p+1)/(p+q+2)} \left(\frac{p+q+2}{q+1}\right)^{(q+1)/(p+q+2)}$$

and $0 < t_0 < T_0$ is chosen to satisfy $\{f(t_0)g(t_0)\}^{(pq-1)/(p+q+2)} > 2$.

Proof. See e.g., K. Mochizuki [7]. □

Proof of Theorem 1 (i). As is shown in the above lemma, there exist $A_1 > 0$ and $B_1 > 0$ such that if

$$f(0) > A_1 \qquad \text{or} \qquad g(0) > B_1, \tag{11}$$

then $(f(t), g(t))$ blows up in finite time. We see that (11) will be satisfied if $n = n(\lambda)$ is chosen so that

$$\lambda^\nu = \gamma n^{-\alpha-\delta_1+N},$$

where $\gamma > 0$ is a constant which satisfies $\gamma > C_p^{-1} A_1$. If λ is sufficiently small, $n > n_0$, so we can apply this argument. From (8) and Lemma 2.2, there exists a $\lambda_0 > 0$ such that

$$T_\lambda^* \le c^{-1} n^2 (T_0 + c) = C\lambda^{-2\mu/(\alpha+\delta_1-N)}$$

for $\lambda < \lambda_0$. □

Note that there is only one equilibrium of system (9) in $\mathbf{R}_+^2$, say $P = (1,1)$. As is easily seen, P is a saddle point. One of the separatrix starts from 0 and runs to ∞. Another one intersects f-axis and g-axis at A_1 and B_1, respectively. Moreover, every solution $(f(t), g(t))$ of (9) with the initial value $(f(0), g(0))$ lying above this separatrix runs into

$$Q = \{(f, g) \in \mathbf{R}_+^2; f^{1/p} < g < f^q\},$$

and then blows up in finite time. As for these arguments, see e.g., Galaktionov-Kurdyumov-Samarskii [3], [4] or Qi-Levine [12].

We now turn to the proof of the lower bound. For the proof, we will need the following two lemmas from advanced calculus which appear as Lemmas 5 and 6 in [10].

Lemma 2.3. *For each $\sigma > 0$, there exists a constant $c > 0$ such that*

$$\int_{\mathbf{R}^N} P(x-y,t)(1+|y|)^\sigma dy \le c(1 + t^{\sigma/2} + |x|^\sigma), \qquad \textit{for } x \in \mathbf{R}^N, t > 0.$$

Proof. Using the inequality $|a+b|^\sigma \le 2^\sigma(|a|^\sigma + |b|^\sigma)$ for $\sigma > 0$, we obtain

$$\begin{aligned}\int_{\mathbf{R}^N} P(x-y,t)(1+|y|)^\sigma dy &= \int_{\mathbf{R}^N} P(z,t)(1+|x+z|)^\sigma dz \\ &\le 2^\sigma \int_{\mathbf{R}^N} P(z,t)(1+|x+z|^\sigma) dz \\ &\le 2^\sigma + 2^{2\sigma} \int_{\mathbf{R}^N} P(z,t)(|x|^\sigma + |z|^\sigma) dz \\ &= 2^\sigma + 2^{2\sigma}|x|^\sigma + 2^{2\sigma} c_\sigma t^{\sigma/2},\end{aligned}$$

where

$$c_\sigma = (4\pi)^{-N/2} \int_{\mathbf{R}^N} |\xi|^\sigma \exp\left(-\frac{|\xi|^2}{4}\right) d\xi. \qquad \square$$

LEMMA 2.4. *For $\sigma \le 0$ and $t > 0$, the function*

$$H(x) \equiv \int_{\mathbf{R}^N} P(x-y,t)(1+|y|)^\sigma dy$$

attains its maximum at $x = 0$.

PROOF. $H(x)$ depends only on $|x|$, thus it is enough to show that $(x, \nabla H(x)) \le 0$ for all $x \in \mathbf{R}^N$. We have

$$\begin{aligned} \nabla H(x) &= \int_{\mathbf{R}^N} \nabla_x P(x-y,t)(1+|y|)^\sigma dy \\ &= -\int_{\mathbf{R}^N} \nabla_y P(x-y,t)(1+|y|)^\sigma dy \\ &= \int_{\mathbf{R}^N} P(x-y,t)\nabla(1+|y|)^\sigma dy. \end{aligned}$$

Thus,

$$\begin{aligned} (x, \nabla H(x)) = \sigma(4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right) \int_{\mathbf{R}^N} \exp\left(\frac{(x,y)}{2t}\right)(x,y) \\ \times \exp\left(-\frac{|y|^2}{4t}\right)(1+|y|)^{\sigma-1}|y|^{-1} dy. \end{aligned} \tag{12}$$

Since $(x, \nabla H(x))$ depends only on $|x|$, it is enough to show that $\int_{|x|=r} (x, \nabla H(x)) dx \le 0$ for all $r > 0$. Considering symmetry of functions, we see

$$\begin{aligned} &\int_{|x|=r} \exp\left(\frac{(x,y)}{2t}\right)(x,y) dx \\ &\quad = \left\{ \int_{|x|=r,\,(x,y)\ge 0} + \int_{|x|=r,\,(x,y)\le 0} \right\} \exp\left(\frac{(x,y)}{2t}\right)(x,y) dx \\ &\quad = \int_{|x|=r,\,(x,y)\ge 0} \left\{ \exp\left(\frac{(x,y)}{2t}\right) - \exp\left(-\frac{(x,y)}{2t}\right) \right\}(x,y) dx \\ &\quad \ge 0, \end{aligned} \tag{13}$$

for all $y \in \mathbf{R}^N$. From (12) and (13), we obtain $\int_{|x|=r} (x, \nabla H(x)) dx \le 0$. $\square$

To prove that a given number $T > 0$ provides a lower bound for T_λ^*, we will make the following argument. Define

$$u_0(x,t) = \lambda^\mu \int_{\mathbf{R}^N} P(x-y,t)\varphi(y)dy,$$

$$v_0(x,t) = \lambda^\nu \int_{\mathbf{R}^N} P(x-y,t)\psi(y)dy,$$

where φ, ψ satisfy

$$0 \le \varphi(x), \qquad \psi(x) \le \delta P(x,k) \tag{14}$$

for some $\delta, k > 0$, and

$$\begin{aligned} u_{n+1}(x,t) &= u_0(x,t) + \int_0^t \int_{\mathbf{R}^N} P(x-y,t-s)a(y)v_n(y,s)^p dyds, \\ v_{n+1}(x,t) &= v_0(x,t) + \int_0^t \int_{\mathbf{R}^N} P(x-y,t-s)b(y)u_n(y,s)^q dyds, \end{aligned} \tag{15}$$

for $n \ge 0$. By induction, $u_{n+1}(x,t) \ge u_n(x,t)$, $v_{n+1}(x,t) \ge v_n(x,t)$. If there exists a $T > 0$ such that

$$\sup_{n\ge 0} u_n(x,t), \sup_{n \ge 0} v_n(x,t) < \infty, \qquad \text{for } x \in \mathbf{R}^N, t \in [0,T),$$

then

$$\tilde{u}(x,t) \equiv \lim_{n\to\infty} u_n(x,t), \qquad \tilde{v}(x,t) \equiv \lim_{n\to\infty} v_n(x,t)$$

converge uniformly in $x \in \mathbf{R}^N$, $t \in [0,T)$, and it follows from the monotone convergence theorem and (15) that $\tilde{u}, \tilde{v}$ satisfy (3) for $x \in \mathbf{R}^N$, $t \in (0,T)$; hence $T_\lambda^* \ge T$. Thus, to obtain an estimate of the form $T_\lambda^* \ge T$, it is enough to show the following lemma:

Lemma 2.5. *If* (14) *holds,*

$$u_n(x,t) \le 2\lambda^\mu \delta P(x,t+k), \qquad v_n(x,t) \le 2\lambda^\nu \delta P(x,t+k) \tag{16}$$

holds for all $n \ge 0$ *in* $x \in \mathbf{R}^N$, $t \in [0, T(\lambda))$, *where*

$$T(\lambda) = C \min\{\lambda^{2(-p\nu+\mu)/N(p^*-p)}, \lambda^{2(-q\mu+\nu)/N(q^*-q)}\} - k.$$

Proof. From (14) and the relation

$$\begin{aligned}\int_{\mathbf{R}^N} & P(x-y,t)P(y,k)dy \\ & = (4\pi t)^{-N/2}(4\pi k)^{-N/2}\exp\left(-\frac{|x|^2}{4(t+k)}\right) \\ & \quad \times \int_{\mathbf{R}^N}\exp\left(-\frac{t+k}{4tk}\left|y-\frac{kx}{t+k}\right|^2\right)dy \\ & = (4\pi(t+k))^{-N/2}\exp\left(-\frac{|x|^2}{4(t+k)}\right)\int_{\mathbf{R}^N}P(z,k)dz \\ & = P(x,t+k),\end{aligned}$$

it follows that

$$\begin{aligned} u_0(x,t) &\le \lambda^\mu \delta P(x,t+k) \le 2\lambda^\mu \delta P(x,t+k), \\ v_0(x,t) &\le \lambda^\nu \delta P(x,t+k) \le 2\lambda^\nu \delta P(x,t+k), \end{aligned} \tag{17}$$

for all $t \ge 0$. Hence (16) holds for $n = 0$ when $0 \le t < \infty$.

Next, we shall assume that (16) holds for some $n \ge 0$. In the sequel C will denote a positive constant whose value will change from term to term. Since $a(x) \le C(1+|x|)^{\sigma_1}$ for some $C > 0$ by assumption, using (15), (16), and (17), we obtain

$$\begin{aligned} u_{n+1}(x,t) &\le \lambda^\mu \delta P(x,t+k) \\ & \quad + (2\lambda^\nu\delta)^p \int_0^t\int_{\mathbf{R}^N} a(y)P(x-y,t-s)P(y,s+k)^p dyds \\ & \le \lambda^\mu \delta P(x,t+k) \\ & \quad + (2\lambda^\nu\delta)^p C\int_0^t\int_{\mathbf{R}^N}(t-s)^{-N/2}(s+k)^{-Np/2} \\ & \quad \times (1+|y|)^{\sigma_1}\exp\left(-\frac{|x-y|^2}{4(t-s)}-\frac{p|y|^2}{4(s+k)}\right)dyds. \end{aligned} \tag{18}$$

Using the relation

$$\begin{aligned}\exp & \left(-\frac{|x-y|^2}{4(t-s)}-\frac{p|y|^2}{4(s+k)}\right) \\ & = \exp\left(-\frac{|y-R(s,t)x|^2}{4(t-s)R(s,t)}\right)\exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)}\right),\end{aligned}$$

where $R(s,t) = (s+k)/(s+k+p(t-s))$, (18) can be rewritten as

$$u_{n+1}(x,t) \le \lambda^{\mu}\delta P(x,t+k) + (2\lambda^{\nu}\delta)^{p} C \int_0^t \int_{\mathbf{R}^N} P(R(s,t)x - y, R(s,t)(t-s)) \times (1+|y|)^{\sigma_1} (s+k)^{-Np/2} R(s,t)^{N/2} \exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)}\right) dyds. \qquad (19)$$

At this stage in the proof, we must consider two cases separately. The first case is when $\sigma_1 > 0$, and the second case is when $\sigma_1 \le 0$. We treat the case $\sigma_1 > 0$ first. Carrying out the integration over $\mathbf{R}^N$ in (19), and using Lemma 2.3 with t, x and σ being replaced by $R(s,t)(t-s)$, $R(s,t)x$ and σ_1 respectively, the final term on the right hand side of (19) reduces to

$$(2\lambda^{\nu}\delta)^{p} C \int_0^t [1 + R(s,t)^{\sigma_1/2}(t-s)^{\sigma_1/2} + R(s,t)^{\sigma_1}|x|^{\sigma_1}] \times (s+k)^{-Np/2} R(s,t)^{N/2} \exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)}\right) ds. \qquad (20)$$

Multiplying outside the integral in (20) by the factor $\exp(-|x|^2/4(t+k))$, multiplying inside the integral by its reciprocal, and simplifying the argument in the exponential term, (20) may be rewritten as

$$(2\lambda^{\nu}\delta)^{p} C \exp\left(-\frac{|x|^2}{4(t+k)}\right) \int_0^t (s+k)^{-Np/2} R(s,t)^{N/2} \times [1 + R(s,t)^{\sigma_1/2}(t-s)^{\sigma_1/2} + R(s,t)^{\sigma_1}|x|^{\sigma_1}] \times \exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right) ds. \qquad (21)$$

We now write

$$R(s,t)^{\sigma_1}|x|^{\sigma_1} \exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right) = R(s,t)^{\sigma_1/2} z^{\sigma_1/2} \exp\left(-\frac{(p-1)z}{4(t+k)}\right), \qquad (22)$$

where $z = R(s,t)|x|^2$. Differentiating this as a function of $z > 0$, we have

$$R(s,t)^{\sigma_1/2}\left(\frac{\sigma_1}{2}z^{\sigma_1/2-1}-\frac{p-1}{4(t+k)}z^{\sigma_1/2}\right)\exp\left(-\frac{(p-1)z}{4(t+k)}\right).$$

By the inequality $p>1$, the function (22) of z attains its maximum at $z=2\sigma_1(t+k)/(p-1)$. The maximum value then is

$$R(s,t)^{\sigma_1/2}\left(\frac{2\sigma_1(t+k)}{p-1}\right)^{\sigma_1/2}e^{-\sigma_1/2}.$$

From this it follows that

$$\begin{aligned}R(s,t)^{\sigma_1}|x|^{\sigma_1}\exp\left(-\frac{(p-1)R(s,t)|x|^2}{4(t+k)}\right)\\ \le CR(s,t)^{\sigma_1/2}(t+k)^{\sigma_1/2},\end{aligned}\tag{23}$$

for all $x\in\mathbf{R}^N$, $t>0$ and $0<s<t$. From (23) and the fact that $p>1$, it follows that the quantity in (21) is smaller than

$$\begin{aligned}(2\lambda^\nu\delta)^pC\exp\left(-\frac{|x|^2}{4(t+k)}\right)\bigg\{\int_0^t(s+k)^{-Np/2}R(s,t)^{N/2}ds\\ +\int_0^t(s+k)^{-Np/2}R(s,t)^{(N+\sigma_1)/2}[(t-s)^{\sigma_1/2}+(t+k)^{\sigma_1/2}]ds\bigg\}.\end{aligned}\tag{24}$$

We now carry out the integration in (24). Recalling that $p<p^*=1+(2+\sigma_1)/N$, recalling that $R(s,t)=(s+k)/(s+k+p(t-s))$, and noting that $t+k\le s+k+p(t-s)<p(t+k)$ for $s\in[0,t]$, we have

$$\begin{aligned}&\int_0^t(s+k)^{-Np/2}R(s,t)^{N/2}ds\\ &\quad=(t+k)^{-N/2}\int_0^t(s+k)^{N(1-p)/2}\left(\frac{t+k}{s+k+p(t-s)}\right)^{N/2}ds\\ &\quad\le(t+k)^{-N/2}\int_0^t(s+k)^{N(1-p)/2}ds\\ &\quad\le\begin{cases}C(t+k)^{1-Np/2}, & \text{if } p<1+2/N,\\ C(t+k)^{-N/2}\log(t/k+1), & \text{if } p=1+2/N,\\ C(t+k)^{-N/2}, & \text{if } p>1+2/N,\end{cases}\end{aligned}\tag{25}$$

and

$$\int_0^t (s+k)^{-Np/2} R(s,t)^{(N+\sigma_1)/2} [(t-s)^{\sigma_1/2} + (t+k)^{\sigma_1/2}] ds$$

$$\leq C(t+k)^{-N/2} \int_0^t (s+k)^{(N(1-p)+\sigma_1)/2} \left(\frac{t+k}{s+k+p(t-s)} \right)^{(N+\sigma_1)/2} ds$$

$$\leq C(t+k)^{-N/2} \int_0^t (s+k)^{(N(1-p)+\sigma_1)/2} ds$$

$$\leq C(t+k)^{-N/2+N(p^*-p)/2}. \tag{26}$$

From (20), (21), (24), (25) and (26), we conclude now that the final term on the right hand side of (19) is smaller than

$$(2\lambda^\nu \delta)^p C(t+k)^{-N/2+N(p^*-p)/2} \exp\left(-\frac{|x|^2}{4(t+k)} \right).$$

Substituting this in (19), we obtain

$$u_{n+1}(x,t)$$

$$\leq \lambda^\mu \delta P(x, t+k) + (2\lambda^\nu \delta)^p C(t+k)^{-N/2+N(p^*-p)/2} \exp\left(-\frac{|x|^2}{4(t+k)} \right)$$

$$= (\lambda^\mu \delta + (2\lambda^\nu \delta)^p C(t+k)^{N(p^*-p)/2}) P(x, t+k), \tag{27}$$

for $x \in \mathbf{R}^N$, $t \geq 0$.

We now turn to the case $\sigma_1 \leq 0$. It follows from Lemma 2.4 that the inside integral,

$$\int_{\mathbf{R}^N} P(R(s,t)x - y, R(s,t)(t-s))(1+|y|)^{\sigma_1} dy,$$

appearing on the right hand side of (19), attains its maximum as a function of x when $x = 0$. Thus, the final term on the right hand side of (19) is less than or equal to

$$(2\lambda^\nu \delta)^p C \int_0^t \int_{\mathbf{R}^N} P(y, R(s,t)(t-s))(1+|y|)^{\sigma_1}$$

$$\times (s+k)^{-Np/2} R(s,t)^{N/2} \exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)} \right) dyds. \tag{28}$$

By the facts that $\int_{\mathbf{R}^N} P(y,t)(1+|y|)^{\sigma_1} dy \leq 1$ for $t \in [0,1]$, and that

$$\int_{\mathbf{R}^N} P(y,t)(1+|y|)^{\sigma_1}dy \le \int_{\mathbf{R}^N} P(y,t)|y|^{\sigma_1}dy$$
$$= t^{\sigma_1/2}(4\pi)^{-N/2}\int_{\mathbf{R}^N}|z|^{\sigma_1}\exp\left(-\frac{|z|^2}{4}\right)dz \qquad \text{for } t \ge 1.$$

by the assumption that $\sigma_1 \in (-2,0]$ if $N \ge 2$ or that $\sigma_1 \in (-1,0]$ if $N = 1$, it follows that there exists a $C > 0$ such that

$$\int_{\mathbf{R}^N} P(y,t)(1+|y|)^{\sigma_1}dy \le C(1+t)^{\sigma_1/2}. \tag{29}$$

Applying this with t being replaced by $R(s,t)(t-s)$, it follows that the quantity in (28) is less than or equal to

$$(2\lambda^{\nu}\delta)^p C\int_0^t [1+R(s,t)(t-s)]^{\sigma_1/2}$$
$$\times (s+k)^{-Np/2}R(s,t)^{N/2}\exp\left(-\frac{pR(s,t)|x|^2}{4(s+k)}\right)ds. \tag{30}$$

Since $p > 1$, $R(s,t) \le 1$ and $pR(s,t)/(s+k) = p/(s+k+p(t-s)) \ge 1/(t+k)$ for $s \in [0,t]$, the quantity in (30) is less than or equal to

$$(2\lambda^{\nu}\delta)^p C\exp\left(-\frac{|x|^2}{4(t+k)}\right)\int_0^t R(s,t)^{(N+\sigma_1)/2}(1+t-s)^{\sigma_1/2}(s+k)^{-Np/2}ds. \tag{31}$$

We now carry out the integration in (31). Recalling that $p < p^* = 1 + (2+\sigma_1)/N$, that $\sigma_1 \in (-2,0]$ if $N \ge 2$ or that $\sigma_1 \in (-1,0]$ if $N = 1$, and that $R(s,t) = (s+k)/(s+k+p(t-s))$, and noting that $t+k \le s+k+p(t-s) < p(t+k)$ for $s \in [0,t]$, we have

$$\int_0^t R(s,t)^{(N+\sigma_1)/2}(1+t-s)^{\sigma_1/2}(s+k)^{-Np/2}ds$$
$$\le (t+k)^{-(N+\sigma_1)/2}\int_0^t (s+k)^{(N(1-p)+\sigma_1)/2}(1+t-s)^{\sigma_1/2}ds$$
$$\le C(t+k)^{-(N+\sigma_1)/2}\left\{(t+k)^{\sigma_1/2}\int_0^{t/2}(s+k)^{(N(1-p)+\sigma_1)/2}ds\right.$$
$$\left. + (t+k)^{(N(1-p)+\sigma_1)/2}\int_{t/2}^t (t-s)^{\sigma_1/2}ds\right\}$$
$$\le C(t+k)^{-N/2+N(p^*-p)/2}. \tag{32}$$

From (19), (28), (30), (31) and (32), we conclude that

$$u_{n+1}(x,t) \le (\lambda^{\mu}\delta + (2\lambda^{\nu}\delta)^{p} C(t+k)^{N(p^*-p)/2}) P(x, t+k), \tag{33}$$

for $x \in \mathbf{R}^N$, $t \ge 0$.

In the same way as (18) through (32), we conclude that

$$\begin{aligned} u_{n+1}(x,t) &\le (\lambda^{\mu}\delta + (2\lambda^{\nu}\delta)^{p} C(t+k)^{N(p^*-p)/2}) P(x, t+k), \\ v_{n+1}(x,t) &\le (\lambda^{\nu}\delta + (2\lambda^{\mu}\delta)^{q} C(t+k)^{N(q^*-q)/2}) P(x, t+k) \end{aligned} \tag{34}$$

for $x \in \mathbf{R}^N$, $t \ge 0$. From (34), we find that (16) with n being replaced by $n+1$ holds as long as

$$(2\lambda^{\nu}\delta)^{p} C(t+k)^{N(p^*-p)/2} \le \lambda^{\mu}\delta, (2\lambda^{\mu}\delta)^{q} C(t+k)^{N(q^*-q)/2} \le \lambda^{\nu}\delta.$$

Thus, (16) holds for all $n \ge 0$ when

$$\begin{aligned} t &\le \min\{((2\lambda^{\nu}\delta)^{-p} C\lambda^{\mu}\delta)^{2/N(p^*-p)}, ((2\lambda^{\mu}\delta)^{-q} C\lambda^{\nu}\delta)^{2/N(q^*-q)}\} - k \\ &= C \min\{\lambda^{2(-p\nu+\mu)/N(p^*-p)}, \lambda^{2(-q\mu+\nu)/N(q^*-q)}\} - k = T(\lambda). \end{aligned}$$

□

Proof of Theorem 1 (ii). Recall here that we have assumed

$$\frac{\mu}{\nu} = \frac{\alpha + \delta_1 - N}{\beta + \delta_2 - N}.$$

Then since $p\beta - \alpha = q\alpha - \beta = 2$, $p\delta_2 - \delta_1 = \sigma_1$, $q\delta_1 - \delta_2 = \sigma_2$, it follows that

$$\begin{aligned} \frac{-p\nu + \mu}{2 + \sigma_1 + N(1-p)} &= \frac{-\nu}{2 + \sigma_1 + N(1-p)} \cdot \left(p - \frac{\mu}{\nu}\right) \\ &= \frac{-\nu}{2 + \sigma_1 + N(1-p)} \cdot \left(p - \frac{\alpha + \delta_1 - N}{\beta + \delta_2 - N}\right) = \frac{-\nu}{\beta + \delta_2 - N}, \\ \frac{-q\mu + \nu}{2 + \sigma_2 + N(1-q)} &= \frac{-\mu}{\alpha + \delta_1 - N}. \end{aligned}$$

Thus, we obtain

$$T_{\lambda}^* \ge T(\lambda) \ge C\lambda^{-2\mu/(\alpha+\delta_1-N)} = C\lambda^{-2\nu/(\beta+\delta_2-N)}$$

when $\lambda > 0$ is sufficiently small. □

3. Proof of Theorem 2

We begin with the proof of the upper bounds. Let $D \subset \mathbf{R}^N$ be a smooth bounded domain such that

$$\inf_{x \in D} a(x), \inf_{x \in D} b(x), \inf_{x \in D} \varphi(x), \inf_{x \in D} \psi(x) \ge c > 0. \tag{35}$$

Let $\theta > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in D, and let $\omega(x)$ denote the corresponding positive eigenfunction, normalized by $\int_D \omega(x)dx = 1$. Define

$$F(t) = \int_D u(x,t)\omega(x)dx,$$

$$G(t) = \int_D v(x,t)\omega(x)dx, \qquad \text{for } 0 \le t < T_\lambda^*.$$

Using (35), we obtain from (1) that

$$\begin{aligned} F'(t) &= \int_D u_t(x,t)\omega(x)dx \\ &= \int_D (\Delta u(x,t) + a(x)v(x,t)^p)\omega(x)dx \\ &\ge -\theta F(t) + cG(t)^p. \end{aligned}$$

Thus, we obtain the following inequalities:

$$\begin{cases} F'(t) \ge -\theta F(t) + cG(t)^p & (t > 0), \\ G'(t) \ge -\theta G(t) + cF(t)^q & (t > 0). \end{cases} \tag{36}$$

From (35), $F(0) \ge c\lambda^\mu$, $G(0) \ge c\lambda^\nu$.

Let $f, g \in C^0([0, T_\lambda^*)) \cap C^1((0, T_\lambda^*))$ be the solution of the system of ordinary differential equations

$$\begin{cases} f'(t) = -\theta f(t) + cg(t)^p & (t > 0), \\ g'(t) = -\theta g(t) + cf(t)^q & (t > 0), \\ f(0) = c\lambda^\mu, g(0) = c\lambda^\nu. \end{cases} \tag{37}$$

Then $(F(t), G(t))$ is a supersolution of (37).

Lemma 3.1. *Define*

$$Q = \{(f,g) \in \mathbf{R}_+^2; (2\theta c^{-1} f)^{1/p} < g < (2\theta)^{-1} c f^q\},$$

and let $(f(t), g(t))$ *be the solution to* (37). *If* $(f(0), g(0)) \in Q$, *then* $(f(t), g(t)) \in Q$ *for all* $t \in [0, T_\lambda^*)$.

Proof. We shall first show that

$$f(t) > f(0) > (2\theta c^{-1})^{\alpha/2} \qquad \text{and} \qquad g(t) > g(0) > (2\theta c^{-1})^{\beta/2} \tag{38}$$

hold for all $t \in (0, T_\lambda^*)$. Since $f(t), g(t)$ are continuous at $t = 0$ and

$$-\theta f(0) + cg(0)^p > \theta f(0) > 0, \qquad -\theta g(0) + cf(0)^q > \theta g(0) > 0, \tag{39}$$

there exists an $\varepsilon_1 > 0$ such that

$$f'(t) = -\theta f(t) + cg(t)^p > 0,$$
$$g'(t) = -\theta g(t) + cf(t)^q > 0, \qquad \text{for } 0 < t < \varepsilon_1.$$

So (38) holds for $0 < t < \varepsilon_1$. Assume contrary that there exists a $t_1 \in (0, T_\lambda^*)$ such that (38) holds for $0 < t < t_1$ and $f(t_1) = f(0)$. From (37), it follows that

$$(e^{\theta t} f(t))' = e^{\theta t} f'(t) + \theta e^{\theta t} f(t) = ce^{\theta t} g(t)^p.$$

Integrating the both sides of this equality from 0 to t_1, we obtain

$$e^{\theta t_1} f(0) - f(0) = c \int_0^{t_1} e^{\theta s} g(s)^p ds \geq cg(0)^p \theta^{-1}(e^{\theta t_1} - 1).$$

Since $e^{\theta t_1} > 1$, it follows that $\theta f(0) \geq cg(0)^p$. This leads to a contradiction to (39), so we obtain $f(t) > f(0)$ for all $t \in (0, T_\lambda^*)$. In the same way, we also obtain $g(t) > g(0)$ for all $t \in (0, T_\lambda^*)$.

Next, we shall show that $(f(t), g(t)) \in Q$ for all $t \in [0, T_\lambda^*)$. Since $f(t), g(t)$ are continuous at $t = 0$, there exists an $\varepsilon_2 > 0$ such that $(f(t), g(t)) \in Q$ for $0 \leq t < \varepsilon_2$. Assume contrarily that there exists a $t_2 \in (0, T_\lambda^*)$ such that $(f(t), g(t)) \in Q$ for $0 \leq t < t_2$ and $2\theta f(t_2) = cg(t_2)^p$. Since it follows from (38) that

$$(2\theta)^{-1} cf(t_2)^q - g(t_2) = \{((2\theta)^{-1} c)^{q+1} g(t_2)^{pq-1} - 1\} g(t_2) > 0,$$

we obtain

$$\begin{aligned} &cpg(t_2)^{p-1} g'(t_2) - 2\theta f'(t_2) \\ &\quad = cpg(t_2)^{p-1}(cf(t_2)^q - \theta g(t_2)) - 2\theta(cg(t_2)^p - \theta f(t_2)) \\ &\quad > \theta\{cpg(t_2)^p - 2\theta f(t_2)\} = c\theta(p-1) g(t_2)^p > 0. \end{aligned}$$

Considering the continuity of $f'(t), g'(t)$, there exists an $\varepsilon > 0$ such that

$$cpg(t)^{p-1} g'(t) - 2\theta f'(t) > 0, \qquad \text{for } t_2 - \varepsilon < t < t_2.$$

Integrating the left hand side of this inequality from t satisfying $t_2 - \varepsilon < t < t_2$ to t_2, it follows that

$$\begin{aligned} 0 &< c \int_t^{t_2} pg(s)^{p-1} g'(s) ds - 2\theta \int_t^{t_2} f'(s) ds \\ &= cg(t_2)^p - cg(t)^p - 2\theta f(t_2) + 2\theta f(t) \\ &= 2\theta f(t) - cg(t)^p. \end{aligned}$$

This leads to a contradiction, so we obtain $2\theta f(t) < cg(t)^p$ for all $t \in [0, T_\lambda^*)$. In the same way, we also obtain $2\theta g(t) < cf(t)^q$ for all $t \in [0, T_\lambda^*)$. □

Proof of Theorem 2 (i). Choosing $\lambda_0 > 0$ to satisfy $\lambda_0^{p\nu-\mu} \geq 2\theta c^{-p}$, $\lambda_0^{q\mu-\nu} \geq 2\theta c^{-q}$, we easily see from the inequalities $p\nu > \mu$, $q\mu > \nu$ that $(f(0), g(0)) \in Q$ holds if $\lambda > \lambda_0$. Then we can apply Lemma 3.1 to obtain $(f(t), g(t)) \in Q$ for all $t \in [0, T_\lambda^*)$. From now on, we will always assume that $\lambda > \lambda_0$. It follows from (37) that

$$\begin{aligned} f'(t) &= -\theta f(t) + c_1 g(t)^p \\ &> -\frac{1}{2}c_1 g(t)^p + c_1 g(t)^p = \frac{1}{2}c_1 g(t)^p \\ g'(t) &> \frac{1}{2}c_2 f(t)^q \qquad \text{for } t \in (0, T_\lambda^*). \end{aligned} \tag{40}$$

Let us consider the system of ordinary differential equations

$$\begin{cases} x' = (1/2)cy^p, \; y' = (1/2)cx^q \quad (t > 0), \\ x(0) = c\lambda^\mu, \; y(0) = c\lambda^\nu. \end{cases} \tag{41}$$

Then $(f(t), g(t))$ is a supersolution of (41). From equation (41), it follows that $x^q x' = y^p y'$. Integrate the both sides from 0 to t. Then we have

$$\frac{x(t)^{q+1} - x(0)^{q+1}}{q+1} = \frac{y(t)^{p+1} - y(0)^{p+1}}{p+1}. \tag{42}$$

If $(q+1)^{-1}x(0)^{q+1} \geq (p+1)^{-1}y(0)^{p+1}$, it follows from (42) that

$$x(t) \geq \left(\frac{q+1}{p+1}\right)^{1/(q+1)} y(t)^{(p+1)(q+1)}.$$

Substitute this in the second equation of (41). Then we have

$$y'(t) \geq \frac{1}{2}C_1(p,q)y(t)^{q(p+1)/(q+1)},$$

where $C_1(p,q) = c((q+1)/(p+1))^{q/(q+1)}$. Multiplying $y(t)^{-q(p+1)/(q+1)}$ and integrating the both sides from 0 to t, we obtain

$$\begin{aligned} -\frac{\beta}{2}(y(t)^{-2/\beta} - (c\lambda^\nu)^{-2/\beta}) &\geq \frac{1}{2}C_1(p,q)t, \\ \beta y(t)^{-2/\beta} &\leq \beta(c\lambda^\nu)^{-2/\beta} - C_1(p,q)t. \end{aligned} \tag{43}$$

Since the right hand side of the second equation of (43) equals 0 when

$$t = \beta C_1(p,q)^{-1}(c\lambda^{\nu})^{-2/\beta},$$

it follows that $y(t)$ must blow up by the above t. This gives the upper bound

$$T_\lambda^* \le C\lambda^{-2\nu/\beta}, \qquad \text{for } \exists C > 0.$$

In the case when $(q+1)^{-1}x(0)^{q+1} \le (p+1)^{-1}y(0)^{p+1}$, we obtain by the same method

$$T_\lambda^* \le C\lambda^{-2\mu/\alpha}, \qquad \text{for } \exists C > 0. \qquad \square$$

We now turn to the proof the lower bound. We will use an idea of the same type as that used to prove the lower bound in Theorem 1. Define

$$u_0(x,t) = \lambda^{\mu}\int_{\mathbf{R}^N} P(x-y,t)\varphi(y)dy,$$

$$v_0(x,t) = \lambda^{\nu}\int_{\mathbf{R}^N} P(x-y,t)\psi(y)dy,$$

where φ, ψ satisfy

$$0 \le \varphi(x), \qquad \psi(x) \le \delta \tag{44}$$

for some $\delta > 0$, and

$$\begin{aligned} u_{n+1}(x,t) &= u_0(x,t) + \int_0^t\int_{\mathbf{R}^N} P(x-y,t-s)a(y)v_n(y,s)^p dyds,\\ v_{n+1}(x,t) &= v_0(x,t) + \int_0^t\int_{\mathbf{R}^N} P(x-y,t-s)b(y)u_n(y,s)^q dyds, \end{aligned} \tag{45}$$

for $n \ge 0$. By the same argument as in Section 2, it is enough to show the following lemma:

Lemma 3.2. *If* (44) *holds, the inequalities*

$$u_n(x,t) \le 2\lambda^{\mu}\delta, \qquad v_n(x,t) \le 2\lambda^{\nu}\delta \tag{46}$$

hold for all $n \ge 0$ *in* $x \in \mathbf{R}^N$, $t \in [0, T(\lambda))$, *where*

$$T(\lambda) = C\min\{\lambda^{-p\nu+\mu}, \lambda^{-q\mu+\nu}\}.$$

Proof. From (44), we easily see that

$$u_0(x,t) \le \lambda^{\mu}\delta \le 2\lambda^{\mu}\delta, \qquad v_0(x,t) \le \lambda^{\nu}\delta \le 2\lambda^{\nu}\delta, \tag{47}$$

for all $t \ge 0$. Hence (46) holds for $n = 0$ when $0 \le t < \infty$.

Next, we shall assume that (46) holds for some $n \geq 0$. In the sequel C will denote a positive constant whose value will change from term to term. Using (45), (46), and (47), we obtain

$$\begin{aligned} u_{n+1}(x,t) &\leq \lambda^{\mu}\delta + (2\lambda^{\nu}\delta)^{p} \int_0^t \int_{\mathbf{R}^N} a(y)P(x-y,t-s)dyds \\ &\leq \lambda^{\mu}\delta + (2\lambda^{\nu}\delta)^{p} Ct, \\ v_{n+1}(x,t) &\leq \lambda^{\nu}\delta + (2\lambda^{\mu}\delta)^{q} Ct, \end{aligned} \tag{48}$$

for $x \in \mathbf{R}^N$, $t \geq 0$. From (48), we find that (46) with n being replaced by $n+1$ holds as long as

$$(2\lambda^{\nu}\delta)^{p} Ct \leq \lambda^{\mu}\delta, \qquad (2\lambda^{\mu}\delta)^{q} Ct \leq \lambda^{\nu}\delta.$$

Thus, (46) holds for all $n \geq 0$ when

$$\begin{aligned} t &\leq \min\{(2\lambda^{\nu}\delta)^{-p} C\lambda^{\mu}\delta, (2\lambda^{\mu}\delta)^{-q} C\lambda^{\nu}\delta\} \\ &= C \min\{\lambda^{-p\nu+\mu}, \lambda^{-q\mu+\nu}\} = T(\lambda). \end{aligned}$$ □

Proof of Theorem 2 (ii). Recall here that we have assumed

$$\frac{\mu}{\nu} = \frac{\alpha}{\beta}.$$

Then since $p\beta - \alpha = q\alpha - \beta = 2$, it follows that

$$\begin{aligned} -p\nu + \mu &= -\nu \cdot \left(p - \frac{\mu}{\nu}\right) = -\nu \cdot \left(p - \frac{\alpha}{\beta}\right) = -\frac{2\nu}{\beta}, \\ -q\mu + \nu &= -\frac{2\mu}{\alpha}. \end{aligned}$$

Thus, we obtain

$$T_{\lambda}^{*} \geq T(\lambda) \geq C\lambda^{-2\mu/\alpha} = C\lambda^{-2\nu/\beta}$$

when $\lambda > 0$ is sufficiently large. □

Acknowledgements

This research was introduced to the author by Professor Kiyoshi Mochizuki in Tokyo Metropolitan University. The author would like to thank him for his proper guidanse. The author would also like to thank Kunio Hidano for useful discussions and friendly encouragement during the preparation of this paper.

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