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Generalized solutions in the Egorov formulation of nonlinear diffusion equations and linear hyperbolic equations in the Colombeau algebra

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ABSTRACT. We investigate generalized solutions of nonlinear diffusion equations and linear hyperbolic equations with discontinuous coefficients in the framework of Colombeau's algebra of generalized functions. Under Egorov's formulation, we obtain results on existence and uniqueness of generalized solutions, which are shown to be consistent with classical solutions. The example of a linear hyperbolic equation given by Hurd and Sattinger [8] has no distributional solutions in Schwartz's sense, but has the unique generalized solution. We study what distribution is associated with it, namely, how it behaves on the level of information of distribution theory.

1. Introduction

In 1982, Colombeau introduced an algebra $\mathscr{G}$ of generalized functions to deal with the multiplication problem for distributions, see Colombeau [3, 4]. This algebra $\mathscr{G}$ is a differential algebra which contains the space $\mathscr{D}'$ of distributions. Furthermore, nonlinear operations more general than the multiplication make sense in the algebra $\mathscr{G}$ (cf. Section 2). Therefore the algebra $\mathscr{G}$ is a very convenient one to find and study solutions of nonlinear differential equations with singular data and coefficients.

In this paper we will study generalized solutions of the Cauchy problems

$$\begin{cases} u_t = \Delta_x u + a \cdot \nabla_x f(u) + g(u), & 0 < t < T, \quad x \in \mathbf{R}^d, \\ u|_{t=0} = u_0, & \qquad\qquad\quad x \in \mathbf{R}^d \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_t = a \cdot \nabla_x (b_1(t,x)u) + b_2(t,x)u, & -T < t < T, \quad x \in \mathbf{R}^d, \\ u|_{t=0} = u_0, & \qquad\qquad\quad x \in \mathbf{R}^d \end{cases} \tag{1.2}$$

in the framework of generalized functions introduced by Colombeau. We seek solutions in the algebras $\mathscr{G}_{s,g}([0,T] \times \mathbf{R}^d)$ and $\mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$ of generalized

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functions which will be defined in Section 2 below. We mention that $\mathscr{G}_{s,g}(\mathbf{R}^d)$ contains the space $\mathscr{D}'_{L^\infty}(\mathbf{R}^d)$ of bounded distributions.

Generalized solutions of various differential equations in Colombeau's algebra have been studied until now. Generalized solutions of problems like (1.1) have been studied in [2, 11]. It was proved there that there exists a unique generalized solution. Furthermore, the generalized solution was shown to be consistent with the classical solution. However, it is known that there exist linear partial differential equations whose generalized solutions fail to exist or are not unique. For instance, see Colombeau, Heibig and Oberguggenberger [6]. We mention that generalized solutions of problem (1.2) may not be unique in $\mathscr{G}_{s,g}([-T,T]\times\mathbf{R}^d)$ from [6]. Hence, in order to obtain a unique solution to such problems, we have to change the formulation of differential equations in Colombeau's algebra. Of course, problems which have a unique solution under the usual formulation should also have it under the new formulation. Furthermore, coherence with the classical solutions should be obtained if they exist. As such an attempt, Egorov [7] used the difference operators in place of the usual differentiations with respect to the space variable x for formulating evolution systems in an algebra of generalized functions of Colombeau's type defined by him. This means that if we consider a given differential equation as an equation on the algebra of generalized functions, then we have degrees of freedom of difference approximations of the differential equation. Although Egorov there proved results concerning existence and uniqueness of a generalized solution of the Cauchy problem for evolution systems, he did not discuss coherence with the classical solution in detail. We will also use the difference operators for formulating the above Cauchy problems in the algebra $\mathscr{G}_{s,g}$. Concerning problem (1.1) with the difference operators, we will study generalized solutions under weaker conditions than the ones he imposed for nonlinear terms to obtain the global existence in the case of nonlinear evolution systems. Furthermore, we will study how the generalized solutions in Egorov's formulation of problems (1.1) and (1.2) are related to the classical solutions, respectively. The formulation of the above Cauchy problems in $\mathscr{G}_{s,g}$ will be given by using this method in Sections 3 and 4. Problems other than the ones (1.1) and (1.2) remain to be unsolved.

It is well known (Hurd and Sattinger [8]) that problem (1.2) with $b_1 = H$, $b_2 = 0$ and $u_0 = 1$, namely, the problem

$$\begin{cases} u_t = (H(x)u)_x, & -T < t < T, \quad x \in \mathbf{R}, \\ u|_{t=0} = 1, & \qquad\qquad\quad\;\; x \in \mathbf{R}, \end{cases} \tag{1.3}$$

where H is the Heaviside function, fails to have a distributional solution on $(0,T)\times\mathbf{R}$. Oberguggenberger [9] used the algebra $\mathscr{G}$ introduced by Colombeau which is wider than $\mathscr{G}_{s,g}$, and proved that for each $T > 0$, there exists a unique generalized solution $V \in \mathscr{G}([-T, T] \times \mathbf{R})$ of problem (1.3) with the usual differentiation, which is associated with $1 + t\delta(x)$ on $(0, T) \times \mathbf{R}$. Here δ is the delta function. This means that to problem (1.3) having no distributional solution, we can give a unique solution in the Colombeau algebra $\mathscr{G}([-T, T] \times \mathbf{R})$, which behaves like $1 + t\delta(x)$ in $(0, T) \times \mathbf{R}$ on the level of information of distribution theory. In Section 4, we will study the distribution in Schwartz's sense with which the generalized solution $U \in \mathscr{G}_{s,g}([-T, T] \times \mathbf{R})$ in Egorov's formulation of problem (1.3) is associated, namely, how U behaves on the level of information of distribution theory.

The first purpose of this paper is to show the existence and uniqueness of generalized solutions in Egorov's formulation of the above Cauchy problems (1.1) and (1.2) in Colombeau's algebra $\mathscr{G}_{s,g}$, which are consistent with the classical solutions. The second is to show that the generalized solution $U \in \mathscr{G}_{s,g}([-T, T] \times \mathbf{R})$ in Egorov's formulation of the Cauchy problem (1.3) is associated with $1 + t\delta(x)$ on $(0, T) \times \mathbf{R}$.

This paper is organized as follows: First we recall the definition and properties of the Colombeau algebra $\mathscr{G}_{s,g}(\bar{\Omega})$ and define difference operators on $\mathscr{G}_{s,g}$ in Section 2. In Section 3, by using suitable difference operators, we give our formulation of problem (1.1) and prove results concerning existence and uniqueness of its generalized solution $U \in \mathscr{G}_{s,g}([0, T] \times \mathbf{R}^d)$ (Theorems 3.2 and 3.4). Next, we study how the generalized solutions are related to the classical solutions, if the initial data belong to $W^{2,1}(\mathbf{R}^d) \cap C_B^2(\mathbf{R}^d)$. In fact, we obtain that the generalized solution is associated with the unique classical solution (Theorem 3.7). Section 4 is devoted to the study of a generalized solution of problem (1.2). By a similar way to Section 3, we give our formulation of problem (1.2) and prove results concerning existence and uniqueness of its generalized solution $U \in \mathscr{G}_{s,g}([-T, T] \times \mathbf{R}^d)$ (Theorem 4.2). Furthermore, we prove that the generalized solution is associated with the unique weak solution if b_1 is globally Lipschitz continuous, if b_2 is equal to zero and if u_0 belongs to $L^2(\mathbf{R}^d)$, by using suitable difference operators (Theorem 4.4). In particular, in the case of problem (1.3), we obtain that its generalized solution U is associated with $1 + t\delta(x)$ on $(0, T) \times \mathbf{R}$, namely, U behaves like $1 + t\delta(x)$ in $(0, T) \times \mathbf{R}$ on the level of information of distribution theory (Theorem 4.6).

2. Colombeau's theory

2.1. Colombeau's generalized functions

We briefly recall the definition and properties of a modified version of the Colombeau algebra of generalized functions [3, 4].

Notation 2.1. Let Ω be a nonempty open subset of $\mathbf{R}^d$ and let $\bar{\Omega}$ be its closure. We denote by $\mathscr{D}_{L^\infty}(\bar{\Omega})$ the algebra of restrictions to $\bar{\Omega}$ of real valued and bounded smooth functions on $\mathbf{R}^d$, all whose derivatives are bounded. Let $\mathscr{E}_{s,g}[\bar{\Omega}]$ be the algebra of all maps from the interval $(0,1]$ into $\mathscr{D}_{L^\infty}(\bar{\Omega})$. We denote by $\mathscr{E}_{M,s,g}[\bar{\Omega}]$ the subset of $\mathscr{E}_{s,g}[\bar{\Omega}]$ composed of all $R(\varepsilon,x)$ with the property that for all $\alpha \in \mathbf{N}_0^d$, there exist $N \in \mathbf{N}$, $c > 0$ and $\eta > 0$ such that

$$\sup_{x \in \bar{\Omega}} |D_x^\alpha R(\varepsilon,x)| < c\varepsilon^{-N} \qquad \text{for all } 0 < \varepsilon < \eta.$$

We denote by $\mathscr{N}_{s,g}[\bar{\Omega}]$ the subset of $\mathscr{E}_{s,g}[\bar{\Omega}]$ composed of all $R(\varepsilon,x)$ with the property that for all $\alpha \in \mathbf{N}_0^d$ and $q \in \mathbf{N}$, there exist $c > 0$ and $\eta > 0$ such that

$$\sup_{x \in \bar{\Omega}} |D_x^\alpha R(\varepsilon,x)| < c\varepsilon^{q} \qquad \text{for all } 0 < \varepsilon < \eta.$$

The algebra $\mathscr{G}_{s,g}(\bar{\Omega})$ *of generalized functions* is defined by the quotient algebra

$$\mathscr{G}_{s,g}(\bar{\Omega}) = \mathscr{E}_{M,s,g}[\bar{\Omega}]/\mathscr{N}_{s,g}[\bar{\Omega}],$$

where the subscripts "s" and "g" stand for "simplified" and "global", respectively.

We denote by $\{u^\varepsilon(x)\}_{\varepsilon \in (0,1]}$ or simply $u^\varepsilon(x)$ a representative of a generalized function $U \in \mathscr{G}_{s,g}(\bar{\Omega})$. Then, for generalized functions $U, V \in \mathscr{G}_{s,g}(\bar{\Omega})$ and any $\alpha \in \mathbf{N}_0^d$, the product UV is defined by the class of $\{u^\varepsilon(x)v^\varepsilon(x)\}_{\varepsilon \in (0,1]}$ and the partial derivative $D_x^\alpha U$ by the class of $\{D_x^\alpha u^\varepsilon(x)\}_{\varepsilon \in (0,1]}$. Also, for a generalized function $U \in \mathscr{G}_{s,g}([0,T] \times \mathbf{R}^d)$, its restriction $U|_{t=0}$ to $\{t=0\}$ is defined by the class of $\{u^\varepsilon(0,x)\}_{\varepsilon \in (0,1]}$. For a generalized function $U \in \mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$, we can similarly define $U|_{t=0}$.

In the algebra $\mathscr{G}_{s,g}$, we can also define nonlinear operations more general than the multiplication. To see this, we introduce the following notion.

Definition 2.2. We say that a function $f \in C^\infty(\mathbf{R}^d)$ is *slowly increasing at infinity* if for every $\alpha \in \mathbf{N}_0^d$ there exist $c > 0$ and $r \in \mathbf{N}$ such that, for all $x \in \mathbf{R}^d$,

$$|D^\alpha f(x)| \le c(1+|x|)^r.$$

We denote by $\mathscr{O}_M(\mathbf{R}^d)$ the space of slowly increasing functions at infinity.

If $f \in \mathscr{O}_M(\mathbf{R}^p)$ and $U_i \in \mathscr{G}_{s,g}(\mathbf{R}^d)$ for $i = 1, \dots, p$, then we can define $f(U_1, \dots, U_p) \in \mathscr{G}_{s,g}(\mathbf{R}^d)$ to be the class of $\{f(u_1^\varepsilon(x), \dots, u_p^\varepsilon(x))\}_{\varepsilon \in (0,1]}$. For details see [1, 3, 4].

Definition 2.3. A generalized function $U \in \mathscr{G}_{s,g}(\bar{\Omega})$ is said to be *associated with a distribution* $w \in \mathscr{D}'(\Omega)$ if it has a representative $u^\varepsilon \in \mathscr{E}_{M,s,g}[\bar{\Omega}]$ such that

$$u^{\varepsilon} \to w \qquad \text{in } \mathscr{D}'(\Omega) \text{ as } \varepsilon \to 0.$$

We denote by $U \approx w$ if U is associated with w. We say that generalized functions $U, V \in \mathscr{G}_{s,g}(\bar{\Omega})$ are *associated with each other* if $U - V \approx 0$. We denote by $U \approx V$ if U and V are associated with each other.

In other words, a generalized function $U \in \mathscr{G}_{s,g}(\bar{\Omega})$ is associated with a distribution w if U behaves like w on the level of information of distribution theory.

Remark 2.4. The algebra $\mathscr{G}_{s,g}(\mathbf{R}^d)$ contains the space $\mathscr{D}'_{L^\infty}(\mathbf{R}^d)$ of bounded distributions as follows: Let T be an element of $\mathscr{D}'_{L^\infty}(\mathbf{R}^d)$. Then $R(\varepsilon, x) = T * \rho_\varepsilon(x)$ can be a representative of T and the class of $\{R(\varepsilon, x)\}_{\varepsilon \in (0,1]}$ is associated with T, where $\rho(x)$ is a fixed element of $\mathscr{D}(\mathbf{R}^d)$ such that $\int \rho(x) dx = 1$ and

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right).$$

In this sense, we obtain the inclusion $\mathscr{D}'_{L^\infty}(\mathbf{R}^d) \subset \mathscr{G}_{s,g}(\mathbf{R}^d)$.

Throughout this paper, we assume that $\rho(x)$ satisfies the above conditions.

Definition 2.5. We say that $U \in \mathscr{G}_{s,g}(\bar{\Omega})$ is of *bounded type* if it has a representative $u^\varepsilon \in \mathscr{E}_{M,s,g}[\bar{\Omega}]$ such that there exist $c > 0$ and $\eta > 0$ such that

$$\sup_{x \in \bar{\Omega}} |u^\varepsilon(x)| < c \qquad \text{for } 0 < \varepsilon < \eta.$$

It is called of *logarithmic type* if

$$\sup_{x \in \bar{\Omega}} |u^\varepsilon(x)| < c \log \frac{1}{\varepsilon} \qquad \text{for } 0 < \varepsilon < \eta.$$

We note that $u \in L^\infty(\mathbf{R}^d)$, viewed as an element of $\mathscr{G}_{s,g}(\mathbf{R}^d)$, is of bounded type. On the other hand, for any distribution $T \in \mathscr{D}'(\mathbf{R}^d)$ there exists a generalized function $U \in \mathscr{G}_{s,g}(\mathbf{R}^d)$ of logarithmic type which is associated with T. Therefore, we also note that any distribution on $\mathbf{R}^d$ can be interpreted as an element of logarithmic type of $\mathscr{G}_{s,g}(\mathbf{R}^d)$ in the sense of the association, for details see Colombeau and Heibig [5].

2.2. Difference operators

Egorov [7] used difference operators in place of the usual differentiations with respect to the space variable x for formulating evolution systems in a space of generalized functions. First we define the difference operators $\partial_{x_i}^{\chi,h}$ and $(\partial_{x_i}^2)^{\chi,h}$ on $\mathscr{G}_{s,g}(\mathbf{R}^d)$ for $i = 1, \ldots, d$. The difference operators with respect to the space variable $x \in \mathbf{R}^d$ on $\mathscr{G}_{s,g}([0,T] \times \mathbf{R}^d)$ and $\mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$ will be defined at the end of this subsection.

Let $u^\varepsilon(x)$ be a representative of a generalized function $U \in \mathscr{G}_{s,g}(\mathbf{R}^d)$ and let χ be a step function on $\mathbf{R}$ given by

$$\chi(y) = \begin{cases} \dfrac{1}{c_2 - c_1}, & c_1 \le y \le c_2, \\ 0, & \text{otherwise,} \end{cases}$$

where c_1, c_2 are constants, $c_1 < c_2$ and $0 \in [c_1, c_2]$. Furthermore, let h be a scaling function, i.e. $h : (0,1] \to (0,1]$ is increasing, and $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then we define

$$\partial_{x_i}^{\chi,h} U = \text{class of } \left\{(\partial_{x_i} u^\varepsilon \underset{i}{*} \chi_{h(\varepsilon)})(x)\right\}_{\varepsilon \in (0,1]},$$

$$(\partial_{x_i}^2)^{\chi,h} U = \text{class of } \left\{(\partial_{x_i}^2 u^\varepsilon \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)})(x)\right\}_{\varepsilon \in (0,1]}$$

for $i = 1, \dots, d$, where

$$(\partial_{x_i} u^\varepsilon \underset{i}{*} \chi_{h(\varepsilon)})(x) = \int_{\mathbf{R}} \partial_{x_i} u^\varepsilon(x_1, \dots, x_{i-1}, x_i - y, x_{i+1}, \dots, x_d) \chi_{h(\varepsilon)}(y) dy, \quad (2.1)$$

$\chi_{h(\varepsilon)}(y) = (1/h(\varepsilon))\chi(y/h(\varepsilon))$ and $\check{\chi}(y) = \chi(-y)$. Also, let $h_i : (0,1] \to \mathbf{R}^d$ be a function such that ith element of $h_i(\varepsilon)$ is $h(\varepsilon)$ and the other elements are equal to zero. Then expression (2.1) becomes

$$(\partial_{x_i} u^\varepsilon \underset{i}{*} \chi_{h(\varepsilon)})(x) = -\frac{u^\varepsilon(x - c_2 h_i(\varepsilon)) - u^\varepsilon(x - c_1 h_i(\varepsilon))}{(c_2 - c_1) h(\varepsilon)}$$

for $i = 1, \dots, d$. It follows immediately that $\partial_{x_i}^{\chi,h} U, (\partial_{x_i}^2)^{\chi,h} U \in \mathscr{G}_{s,g}(\mathbf{R}^d)$ for $i = 1, \dots, d$. Furthermore, we obtain that $\partial_{x_i}^{\chi,h} U \approx \partial_{x_i} U$ and $(\partial_{x_i}^2)^{\chi,h} U \approx \partial_{x_i}^2 U$ for $i = 1, \dots, d$ if $U \in \mathscr{D}'_{L^\infty}(\mathbf{R}^d)$.

For a generalized function $U \in \mathscr{G}_{s,g}([0,T] \times \mathbf{R}^d)$, $\partial_{x_i}^{\chi,h} U$ and $(\partial_{x_i}^2)^{\chi,h} U$ are also defined by the classes of

$$\left\{\partial_{x_i} u^\varepsilon(t,\cdot) \underset{i}{*} \chi_{h(\varepsilon)}(x)\right\}_{\varepsilon \in (0,1]} \qquad \text{and} \qquad \left\{\partial_{x_i}^2 u^\varepsilon(t,\cdot) \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)}(x)\right\}_{\varepsilon \in (0,1]}$$

for $i = 1, \dots, d$, respectively. The difference operators with respect to the space variable $x \in \mathbf{R}^d$ on $\mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$ are defined in the same way.

For simplicity, throughout this paper, we put

$$a \cdot \nabla_x^{\chi,h} = \sum_{i=1}^d a_i \partial_{x_i}^{\chi,h} \qquad \text{and} \qquad \Delta_x^{\chi,h} = \sum_{i=1}^d (\partial_{x_i}^2)^{\chi,h}$$

for $a = (a_1, \dots, a_d) \in \mathbf{R}^d$.

3. Nonlinear diffusion equations

In this section, we will consider problem (1.1). By the difference operators defined in Section 2, problem (1.1) can be rewritten as the problem

$$\begin{cases} U_t = \Delta_x^{\chi,h} U + a \cdot \nabla_x^{\chi,h} f(U) + g(U) & \text{in } \mathscr{G}_{s,g}([0,T] \times \mathbf{R}^d), \\ U|_{t=0} = U_0 & \text{in } \mathscr{G}_{s,g}(\mathbf{R}^d) \end{cases} \tag{3.1}$$

in the space of generalized functions.

Definition 3.1. We say that $U \in \mathscr{G}_{s,g}([0,T] \times \mathbf{R}^d)$ is a (*generalized*) *solution* of problem (3.1) if it has a representative $u^\varepsilon \in \mathscr{E}_{M,s,g}[[0,T] \times \mathbf{R}^d]$ such that

$$\begin{cases} u_t^\varepsilon = \displaystyle\sum_{i=1}^d \partial_{x_i}^2 u^\varepsilon(t,\cdot) \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)} \\ \qquad + \displaystyle\sum_{i=1}^d a_i \partial_{x_i} f(u^\varepsilon(t,\cdot)) \underset{i}{*} \chi_{h(\varepsilon)} + g(u^\varepsilon) + N^\varepsilon, \quad 0 < t < T, \ x \in \mathbf{R}^d, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon + n^\varepsilon, & x \in \mathbf{R}^d \end{cases}$$

for some $N^\varepsilon \in \mathscr{N}_{s,g}[[0,T] \times \mathbf{R}^d]$ and some $n^\varepsilon \in \mathscr{N}_{s,g}[\mathbf{R}^d]$, where u_0^ε is a representative of U_0.

Throughout this section, we assume that χ is a step function given by

$$\chi(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

We first prove the existence of a generalized solution of problem (3.1).

Theorem 3.2. *Assume that $h(\varepsilon)$ satisfies the inequality $1/h(\varepsilon)^2 \le c \log(1/\varepsilon)$ for $0 < \varepsilon < 1/2$ with some $c > 0$ and that $f, g \in \mathscr{O}_M(\mathbf{R})$ satisfy the properties $f' \le 0$, $g' \le 0$ and $g(0) = 0$. Furthermore, let all components of $a \in \mathbf{R}^d$ be nonnegative. Finally, let U_0 be an element of $\mathscr{G}_{s,g}(\mathbf{R}^d)$ given by the class of $\{(u_0 * \rho_\varepsilon)(x)\}_{\varepsilon \in (0,1]}$ with $u_0 \in L^1(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$. Then for each $T > 0$ there exists a solution $U \in \mathscr{G}_{s,g}([0,T] \times \mathbf{R}^d)$ of problem* (3.1).

Proof. We use an argument similar to those in the proofs of Propositions 13.1, 13.5 in Oberguggenberger [10] and Proposition 1 in Colombeau and Heibig [5]. We put $u_0^\varepsilon(x) = (u_0 * \rho_\varepsilon)(x)$. In order to prove the existence of a generalized solution, it suffices to consider the equation

$$u^{\varepsilon}(t,x) = u_0^{\varepsilon}(x) + \sum_{i=1}^{d}\int_0^t \frac{u^{\varepsilon}(s,x-h_i(\varepsilon)) - 2u^{\varepsilon}(s,x) + u^{\varepsilon}(s,x+h_i(\varepsilon))}{h(\varepsilon)^2}\,ds$$
$$- \sum_{i=1}^{d}\int_0^t \frac{a_i\{f(u^{\varepsilon}(s,x-h_i(\varepsilon))) - f(u^{\varepsilon}(s,x))\}}{h(\varepsilon)}\,ds$$
$$+ \int_0^t g(u^{\varepsilon}(s,x))ds. \tag{3.2}$$

Define a transformation $F(u^{\varepsilon})$ by the right-hand side of equation (3.2) and fix $\varepsilon \in (0,1]$ arbitrarily. Let $B_k(T) = \{u \in C([0,T] : W^{k,1}(\mathbf{R}^d)) \cap C_B^k([0,T] \times \mathbf{R}^d) : \|u - u_0^{\varepsilon}\| \le 1\}$ for each $k \in \mathbf{N}_0$, where $C_B^k([0,T] \times \mathbf{R}^d)$ denotes the space of all k-times continuously differentiable functions whose derivatives are bounded, and $\|\cdot\|$ is the norm in the space $C([0,T] : W^{k,1}(\mathbf{R}^d)) \cap C_B^k([0,T] \times \mathbf{R}^d)$ defined by

$$\|u\| = \sup_{0 \le t \le T} \sum_{\alpha \in \mathbf{N}_0^d, |\alpha| \le k} \|D^{\alpha}u\|_{L^1(\mathbf{R}^d)}$$
$$+ \|u\|_{L^{\infty}([0,T]\times\mathbf{R}^d)} + T \sum_{\alpha \in \mathbf{N}_0^{d+1}, 1 \le |\alpha| \le k} \|D^{\alpha}u\|_{L^{\infty}([0,T]\times\mathbf{R}^d)}.$$

Obviously, for each $k \in \mathbf{N}_0$, there exists $T_k^{\varepsilon} > 0$ such that F is a contraction in $B_k(T_k^{\varepsilon})$. Hence the equation $u^{\varepsilon} = F(u^{\varepsilon})$ has a unique solution $u^{\varepsilon} \in B_k(T_k^{\varepsilon})$. Furthermore, we can show that for any $T > 0$, the solution $u^{\varepsilon} \in C_B([0,T] \times \mathbf{R}^d)$ of equation (3.2) is unique. Hence, for $k = 1$, we can extend the solution $u^{\varepsilon}(t,x)$ to an element of $C([0,T_0^{\varepsilon}] : W^{1,1}(\mathbf{R}^d)) \cap C_B^1([0,T_0^{\varepsilon}] \times \mathbf{R}^d)$ by repeating the above method finite times and by using the fact that u^{ε} exists in $B_0(T_0^{\varepsilon})$. Similarly, we can obtain a solution $u^{\varepsilon} \in C([0,T_0^{\varepsilon}] : W^{k,1}(\mathbf{R}^d)) \cap C_B^k([0,T_0^{\varepsilon}] \times \mathbf{R}^d)$ for any $k \in \mathbf{N}_0$.

Next, we prove that $\|u^{\varepsilon}\|_{L^{\infty}([0,T_0^{\varepsilon}]\times\mathbf{R}^d)} \le \|u_0^{\varepsilon}\|_{L^{\infty}(\mathbf{R}^d)}$. We obtain that $u^{\varepsilon}(t,x)$ takes the maximum in some bounded domain of $[0,T_0^{\varepsilon}] \times \mathbf{R}^d$, since u^{ε} belongs to $C([0,T_0^{\varepsilon}] : W^{k,1}(\mathbf{R}^d)) \cap C_B^k([0,T_0^{\varepsilon}] \times \mathbf{R}^d)$ for any $k \in \mathbf{N}_0$. We assume that (t_0,x_0) is a point in $(0,T_0^{\varepsilon}] \times \mathbf{R}^d$ satisfying $u^{\varepsilon}(t_0,x_0) = \sup_{(t,x)\in[0,T_0^{\varepsilon}]\times\mathbf{R}^d} u^{\varepsilon}(t,x)$. Then we see $u^{\varepsilon}(t_0,x_0) \ge 0$. If $u^{\varepsilon}(t_0,x_0) = 0$, then we can easily see that $u^{\varepsilon}(t,x) \le \sup_{x\in\mathbf{R}^d} u_0^{\varepsilon}(x)$. We assume that $u^{\varepsilon}(t_0,x_0) > 0$. Then, obviously, $u_t^{\varepsilon}(t_0,x_0) \ge 0$. It follows from equation (3.2) that $u^{\varepsilon}(t,x)$ satisfies the equation

$$u_t^{\varepsilon}(t,x) = \sum_{i=1}^{d} \frac{u^{\varepsilon}(t,x-h_i(\varepsilon)) - 2u^{\varepsilon}(t,x) + u^{\varepsilon}(t,x+h_i(\varepsilon))}{h(\varepsilon)^2}$$
$$- \sum_{i=1}^{d} \frac{a_i\{f(u^{\varepsilon}(t,x-h_i(\varepsilon))) - f(u^{\varepsilon}(t,x))\}}{h(\varepsilon)} + g(u^{\varepsilon}(t,x)). \tag{3.3}$$

Since f, g are decreasing and $g(0) = 0$, we have $u_t^\varepsilon(t_0, x_0) \le 0$. Hence $u_t^\varepsilon(t_0, x_0) = 0$. Furthermore, we obtain that $u^\varepsilon(t_0, x_0) = u^\varepsilon(t_0, x_0 - h_i(\varepsilon))$ for $i = 1, \dots, d$ from the first term of the right-hand side of equation (3.3). Repeating this method, we have $u^\varepsilon(t_0, x_0) = u^\varepsilon(t_0, x_0 - h_i(\varepsilon)n)$ for any $n \in \mathbf{N}$. But it is impossible, because u^ε belongs to $C([0, T_0^\varepsilon] : W^{k,1}(\mathbf{R}^d)) \cap C_B^k([0, T_0^\varepsilon] \times \mathbf{R}^d)$ for any $k \in \mathbf{N}_0$. This contradiction shows that $u^\varepsilon(t, x) \le \sup_{x \in \mathbf{R}^d} u_0^\varepsilon(x)$. For the minimum of $u^\varepsilon(t, x)$, we can similarly discuss. Thus we have

$$\inf_{x \in \mathbf{R}^d} u_0^\varepsilon(x) \le u^\varepsilon(t, x) \le \sup_{x \in \mathbf{R}^d} u_0^\varepsilon(x)$$

for every $(t, x) \in [0, T_0^\varepsilon] \times \mathbf{R}^d$. T_0^ε depends only on $\|u_0^\varepsilon\|_{L^1(\mathbf{R}^d)}$, $\|u_0^\varepsilon\|_{L^\infty(\mathbf{R}^d)}$, a, f and g. Note that $1/T_0^\varepsilon$ grows linearly when $\|u_0^\varepsilon\|_{L^1(\mathbf{R}^d)}$ goes to ∞ and that $\|u^\varepsilon(T_0^\varepsilon, \cdot)\|_{L^1(\mathbf{R}^d)} \le \|u_0^\varepsilon\|_{L^1(\mathbf{R}^d)} + 1$ and $\|u^\varepsilon(T_0^\varepsilon, \cdot)\|_{L^\infty(\mathbf{R}^d)} \le \|u_0^\varepsilon\|_{L^\infty(\mathbf{R}^d)}$. Hence it turns out that the time interval of existence of a solution $u^\varepsilon(t, x)$ obtained by repeating the above argument can be extended to $[0, \infty)$. Thus we can obtain the existence of a solution $u^\varepsilon(t, x)$ for any $T > 0$.

Finally, we prove that the solution u^ε given above belongs to $\mathscr{E}_{M,s,g}[[0, T] \times \mathbf{R}^d]$. Differentiating equation (3.2) with respect to x_i for $i = 1, \dots, d$ and applying Gronwall's inequality, we obtain that there exist $c > 0$ and $r \in \mathbf{N}$ such that

$$\begin{aligned}
&\|\partial_{x_i} u^\varepsilon(t, \cdot)\|_{L^\infty(\mathbf{R}^d)} \\
&\quad \le \|\partial_{x_i} u_0^\varepsilon\|_{L^\infty(\mathbf{R}^d)} \\
&\qquad + \left(\frac{4d}{h(\varepsilon)^2} + \frac{2d \max_i a_i}{h(\varepsilon)} \|f'(u^\varepsilon)\|_{L^\infty([0,T] \times \mathbf{R}^d)} + \|g'(u^\varepsilon)\|_{L^\infty([0,T] \times \mathbf{R}^d)} \right) \\
&\qquad\qquad \cdot \int_0^t \|\partial_{x_i} u^\varepsilon(s, \cdot)\|_{L^\infty(\mathbf{R}^d)} ds \\
&\quad \le \|\partial_{x_i} u_0^\varepsilon\|_{L^\infty(\mathbf{R}^d)} \exp\left[T \left\{ \frac{4d}{h(\varepsilon)^2} + \left(\frac{2d \max_i a_i}{h(\varepsilon)} + 1 \right) c (1 + \|u_0^\varepsilon\|_{L^\infty(\mathbf{R}^d)})^r \right\} \right],
\end{aligned}$$

since f and g belong to $\mathscr{O}_M(\mathbf{R})$. Therefore we obtain that there exist $N \in \mathbf{N}$, $c > 0$ and $\eta > 0$ such that

$$\|\partial_{x_i} u^\varepsilon\|_{L^\infty([0,T] \times \mathbf{R}^d)} \le c\varepsilon^{-N}$$

for $i = 1, \dots, d$, for all $0 < \varepsilon < \eta$, from the boundedness of u_0 and the assumption on h. Similarly we can prove that all derivatives of u^ε with respect to t and x are dominated by $c\varepsilon^{-N}$ with suitable c and N. Consequently, by defining U as the class of $\{u^\varepsilon(t, x)\}_{\varepsilon \in (0,1]}$, the assertion is obtained. □

Remark 3.3. In Theorem 3.2, if f and g are globally Lipschitz continuous, then the boundedness assumption on u_0 can be dropped.

Next, we prove the uniqueness of a generalized solution of problem (3.1) under weaker conditions than the ones which are imposed to obtain the existence.

Theorem 3.4. *Let $h(\varepsilon)$ be as in Theorem 3.2. Furthermore, let f and g belong to $\mathcal{O}_M(\mathbf{R})$ and let $a \in \mathbf{R}^d$. Then for each $T > 0$ the solution $U \in \mathcal{G}_{s,g}([0,T] \times \mathbf{R}^d)$ of bounded type of problem (3.1) is unique.*

Proof. Let $U_1, U_2 \in \mathcal{G}_{s,g}([0,T] \times \mathbf{R}^d)$ be two solutions of problem (3.1). For their representatives $u_1^\varepsilon, u_2^\varepsilon$, there exist $N^\varepsilon \in \mathcal{N}_{s,g}[[0,T] \times \mathbf{R}^d]$ and $n^\varepsilon \in \mathcal{N}_{s,g}[\mathbf{R}^d]$ such that

$$\begin{aligned} &u_1^\varepsilon(t,x) - u_2^\varepsilon(t,x) \\ &\quad = n^\varepsilon(x) + \frac{1}{h(\varepsilon)^2} \sum_{i=1}^d \int_0^t [\{u_1^\varepsilon(s, x - h_i(\varepsilon)) - u_2^\varepsilon(s, x - h_i(\varepsilon))\} \\ &\qquad\qquad - 2\{u_1^\varepsilon(s,x) - u_2^\varepsilon(s,x)\} + \{u_1^\varepsilon(s, x + h_i(\varepsilon)) - u_2^\varepsilon(s, x + h_i(\varepsilon))\}]ds \\ &\qquad - \frac{1}{h(\varepsilon)} \sum_{i=1}^d \int_0^t a_i[\{f(u_1^\varepsilon(s, x - h_i(\varepsilon))) - f(u_2^\varepsilon(s, x - h_i(\varepsilon)))\} \\ &\qquad\qquad - \{f(u_1^\varepsilon(s,x)) - f(u_2^\varepsilon(s,x))\}]ds \\ &\qquad + \int_0^t [\{g(u_1^\varepsilon(s,x)) - g(u_2^\varepsilon(s,x))\} + N^\varepsilon(s,x)]ds. \end{aligned}$$

Hence $u_1^\varepsilon(t,x) - u_2^\varepsilon(t,x)$ satisfies the inequality

$$\begin{aligned} &|u_1^\varepsilon(t,x) - u_2^\varepsilon(t,x)| \\ &\quad \le \|n^\varepsilon\|_{L^\infty(\mathbf{R}^d)} + T\|N^\varepsilon\|_{L^\infty([0,T]\times\mathbf{R}^d)} \\ &\qquad + \left(\frac{4d}{h(\varepsilon)^2} + \frac{2d \max_i |a_i|}{h(\varepsilon)} \|f'(w_1^\varepsilon)\|_{L^\infty([0,T]\times\mathbf{R}^d)} + \|g'(w_2^\varepsilon)\|_{L^\infty([0,T]\times\mathbf{R}^d)} \right) \\ &\qquad\qquad \cdot \int_0^t \|u_1^\varepsilon(s,\cdot) - u_2^\varepsilon(s,\cdot)\|_{L^\infty(\mathbf{R}^d)} ds, \end{aligned}$$

where w_i^ε is a value between u_1^ε and u_2^ε for $i = 1, 2$. Applying Gronwall's inequality, we obtain that there exist $c > 0$ and $r \in \mathbf{N}$ such that

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^\infty([0,T]\times\mathbf{R}^d)}$$
$$\le (\|n^\varepsilon\|_{L^\infty(\mathbf{R}^d)} + T\|N^\varepsilon\|_{L^\infty([0,T]\times\mathbf{R}^d)})$$
$$\cdot \exp\left[T\left\{\frac{4d}{h(\varepsilon)^2} + \left(\frac{2d\,\max_i|a_i|}{h(\varepsilon)} + 1\right)c\left(1 + \max_{i=1,2}\|w_i^\varepsilon\|_{L^\infty([0,T]\times\mathbf{R}^d)}\right)^r\right\}\right],$$

since f and g belong to $\mathcal{O}_M(\mathbf{R})$. Therefore, for all $q \in \mathbf{N}$, there exist $c > 0$ and $\eta > 0$ such that

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^\infty([0,T]\times\mathbf{R}^d)} \le c\varepsilon^q$$

for all $0 < \varepsilon < \eta$, by the boundedness of u_i^ε for $i = 1, 2$ and the assumption on h. Similarly we can prove the same type of estimate for all derivatives of $u_1^\varepsilon - u_2^\varepsilon$ with respect to t and x. Hence $u_1^\varepsilon - u_2^\varepsilon \in \mathcal{N}_{s,g}[[0,T]\times\mathbf{R}^d]$, that is, $U_1 - U_2 = 0$ in $\mathcal{G}_{s,g}([0,T]\times\mathbf{R}^d)$. Thus the assertion follows. □

Remark 3.5. We note that the assertion of Theorem 3.4 does not depend on the choices of the step function χ.

Remark 3.6. In Theorem 3.4, if f and g are globally Lipschitz continuous, then the boundedness assumption on U can be dropped.

Now we turn to a comparison between classical solutions of problem (1.1) and generalized solutions of problem (3.1). We assume that f, g and a are as in Theorem 3.2 and that u_0 belongs to $W^{2,1}(\mathbf{R}^d) \cap C_B^2(\mathbf{R}^d)$. Then, we can see that for each $T > 0$, there exists a classical solution $u \in C([0,T] : W^{2,1}(\mathbf{R}^d) \cap C_B^2(\mathbf{R}^d))$ of problem (1.1), by using a fixed point argument and the maximum principle. Furthermore, we can show that it is unique by a similar way to the proof for Theorem 7 in Oleĭnik [12]. On the other hand, U_0, given by the class of $\{(u_0 * \rho_\varepsilon)(x)\}_{\varepsilon\in(0,1]}$, defines an element of $\mathcal{G}_{s,g}(\mathbf{R}^d)$. According to Theorems 3.2 and 3.4, problem (3.1) has a unique solution $U \in \mathcal{G}_{s,g}([0,T]\times\mathbf{R}^d)$ of bounded type for each $T > 0$, under the assumption on $h(\varepsilon)$ stated in Theorem 3.2. As seen from the following theorem, the generalized solution U is equal to the classical solution u in the sense of the association.

Theorem 3.7. *Assume that f, g, a and $h(\varepsilon)$ are as in Theorem* 3.2 *and that $u_0 \in W^{2,1}(\mathbf{R}^d) \cap C_B^2(\mathbf{R}^d)$. Let U_0 be an element of $\mathcal{G}_{s,g}(\mathbf{R}^d)$ given by the class of $\{(u_0 * \rho_\varepsilon)(x)\}_{\varepsilon\in(0,1]}$. Furthermore, let $U \in \mathcal{G}_{s,g}([0,T]\times\mathbf{R}^d)$ be the generalized solution of bounded type of problem* (3.1). *Then U is associated with the unique classical solution $u \in C([0,T] : W^{2,1}(\mathbf{R}^d) \cap C_B^2(\mathbf{R}^d))$ of problem* (1.1).

Proof. Let $u^\varepsilon(t,x)$ be a representative of U satisfying equation (3.3). Then it follows that

$$
\begin{aligned}
&u_t^\varepsilon(t,x) - u_t(t,x)\\
&\quad = \sum_{i=1}^d \partial_{x_i}^2 (u^\varepsilon - u) \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)}\\
&\qquad + \sum_{i=1}^d a_i \partial_{x_i} \{f(u^\varepsilon) - f(u)\} \underset{i}{*} \chi_{h(\varepsilon)} + \{g(u^\varepsilon(t,x)) - g(u(t,x))\}\\
&\qquad + \sum_{i=1}^d \{\partial_{x_i}^2 u \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)} - \partial_{x_i}^2 u\} + \sum_{i=1}^d a_i \{\partial_{x_i} f(u) \underset{i}{*} \chi_{h(\varepsilon)} - \partial_{x_i} f(u)\}\\
&\quad = \sum_{i=1}^d \frac{1}{h(\varepsilon)^2} [\{u^\varepsilon(t, x - h_i(\varepsilon)) - u(t, x - h_i(\varepsilon))\} - 2\{u^\varepsilon(t,x) - u(t,x)\}\\
&\qquad\qquad + \{u^\varepsilon(t, x + h_i(\varepsilon)) - u(t, x + h_i(\varepsilon))\}]\\
&\qquad - \sum_{i=1}^d \frac{a_i}{h(\varepsilon)} [\{f(u^\varepsilon(t, x - h_i(\varepsilon))) - f(u(t, x - h_i(\varepsilon)))\}\\
&\qquad\qquad - \{f(u^\varepsilon(t,x)) - f(u(t,x))\}]\\
&\qquad + \{g(u^\varepsilon(t,x)) - g(u(t,x))\}\\
&\qquad + \sum_{i=1}^d \{\partial_{x_i}^2 u \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)} - \partial_{x_i}^2 u\} + \sum_{i=1}^d a_i \{\partial_{x_i} f(u) \underset{i}{*} \chi_{h(\varepsilon)} - \partial_{x_i} f(u)\}. \qquad (3.4)
\end{aligned}
$$

Multiplying equation (3.4) by $\operatorname{sgn}(u^\varepsilon(t,x) - u(t,x))$ and integrating it in t and x, we have

$$
\begin{aligned}
&\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^1(\mathbf{R}^d)}\\
&\quad \le \|u_0^\varepsilon - u_0\|_{L^1(\mathbf{R}^d)} + \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} \frac{1}{h(\varepsilon)^2} (u^\varepsilon(s,x) - u(s,x))\\
&\qquad \cdot \{\operatorname{sgn}(u^\varepsilon(s, x + h_i(\varepsilon)) - u(s, x + h_i(\varepsilon))) - 2 \operatorname{sgn}(u^\varepsilon(s,x) - u(s,x))\\
&\qquad\quad + \operatorname{sgn}(u^\varepsilon(s, x - h_i(\varepsilon)) - u(s, x - h_i(\varepsilon)))\} dxds\\
&\qquad - \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} \frac{a_i}{h(\varepsilon)} \{f(u^\varepsilon(s,x)) - f(u(s,x))\}\\
&\qquad \cdot \{\operatorname{sgn}(u^\varepsilon(s, x + h_i(\varepsilon)) - u(s, x + h_i(\varepsilon))) - \operatorname{sgn}(u^\varepsilon(s,x) - u(s,x))\} dxds\\
&\qquad + \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} |\partial_{x_i}^2 u \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)} - \partial_{x_i}^2 u| dxds
\end{aligned}
$$

$$+\sum_{i=1}^{d}\int_0^t\int_{\mathbf{R}^d} a_i \cdot \left|\partial_{x_i} f(u) \underset{i}{*} \chi_{h(\varepsilon)} - \partial_{x_i} f(u)\right| dxds$$

$$\leq \|u_0^\varepsilon - u_0\|_{L^1(\mathbf{R}^d)} + \sum_{i=1}^{d}\int_0^t \left\|\partial_{x_i}^2 u \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)} - \partial_{x_i}^2 u\right\|_{L^1(\mathbf{R}^d)} ds$$

$$+\sum_{i=1}^{d} a_i \int_0^t \left\|\partial_{x_i} f(u) \underset{i}{*} \chi_{h(\varepsilon)} - \partial_{x_i} f(u)\right\|_{L^1(\mathbf{R}^d)} ds,$$

since f and g are decreasing, where $u_0^\varepsilon(x) = (u_0 * \rho_\varepsilon)(x)$. Therefore we have the inequality

$$\sup_{0\leq t\leq T} \|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^1(\mathbf{R}^d)}$$

$$\leq \|u_0^\varepsilon - u_0\|_{L^1(\mathbf{R}^d)} + \sum_{i=1}^{d}\int_0^T \left\|\partial_{x_i}^2 u \underset{i}{*} \chi_{h(\varepsilon)} \underset{i}{*} \check{\chi}_{h(\varepsilon)} - \partial_{x_i}^2 u\right\|_{L^1(\mathbf{R}^d)} ds$$

$$+\sum_{i=1}^{d} a_i \int_0^T \left\|\partial_{x_i} f(u) \underset{i}{*} \chi_{h(\varepsilon)} - \partial_{x_i} f(u)\right\|_{L^1(\mathbf{R}^d)} ds.$$

From the assumptions on u_0, u and f, it follows that u^ε converges to the unique classical solution u in $\mathscr{D}'((0,T)\times\mathbf{R}^d)$ as ε tends to 0. □

Remark 3.8. For the case $f' \geq 0$, we can obtain the same results as Theorems 3.2 and 3.7 if we choose the step function χ to be

$$\chi(y) = \begin{cases} 1, & -1 \leq y \leq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Remark 3.9. Theorems 3.2, 3.4 and 3.7 guarantee that fundamental properties, which we usually expect for generalized solutions of differential equations, also hold in Egorov's formulation.

4. Linear hyperbolic equations with discontinuous coefficients

In this section we will consider problem (1.2). As in Section 3, problem (1.2) can be rewritten as the problem

$$\begin{cases} U_t = a \cdot \nabla_x^{\chi,h}(B_1 U) + B_2 U & \text{in } \mathscr{G}_{s,g}([-T,T]\times\mathbf{R}^d), \\ U|_{t=0} = U_0 & \text{in } \mathscr{G}_{s,g}(\mathbf{R}^d) \end{cases} \tag{4.1}$$

in the space of generalized functions, where B_1 and B_2 are elements of $\mathscr{G}_{s,g}(\mathbf{R}^{d+1})$.

Definition 4.1. We say that $U \in \mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$ is a (*generalized*) *solution* of problem (4.1) if it has a representative $u^\varepsilon \in \mathscr{E}_{M,s,g}[[-T,T] \times \mathbf{R}^d]$ such that

$$\begin{cases} u_t^\varepsilon = \displaystyle\sum_{i=1}^d a_i \partial_{x_i}(b_1^\varepsilon u^\varepsilon)(t,\cdot) \underset{i}{*} \chi_{h(\varepsilon)} + b_2^\varepsilon u^\varepsilon + N^\varepsilon, & -T < t < T, \ x \in \mathbf{R}^d, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon + n^\varepsilon, & x \in \mathbf{R}^d \end{cases}$$

for some $N^\varepsilon \in \mathscr{N}_{s,g}[[-T,T] \times \mathbf{R}^d]$ and some $n^\varepsilon \in \mathscr{N}_{s,g}[\mathbf{R}^d]$, where $u_0^\varepsilon, b_1^\varepsilon$ and b_2^ε are representatives of U_0, B_1 and B_2, respectively.

The existence and uniqueness results of a generalized solution of problem (4.1) are shown easily by similar arguments to Section 3. In fact, we can obtain the following theorem:

Theorem 4.2. *Assume that $h(\varepsilon)$ satisfies the inequality $1/h(\varepsilon) \le c \log(1/\varepsilon)$ for $0 < \varepsilon < 1/2$ with some $c > 0$. Let $a \in \mathbf{R}^d$ and let $U_0 \in \mathscr{G}_{s,g}(\mathbf{R}^d)$. Furthermore, let $B_1 \in \mathscr{G}_{s,g}(\mathbf{R}^{d+1})$ be of bounded type and let $B_2 \in \mathscr{G}_{s,g}(\mathbf{R}^{d+1})$ be of logarithmic type. Then for each $T > 0$ there exists a solution $U \in \mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$ of problem (4.1) and in addition it is unique.*

Remark 4.3. As stated in Section 1, there exist coefficients B_1, B_2 and an initial data U_0 such that problem (4.1) has no unique solution in the algebra $\mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$, if the difference operator $a \cdot \nabla_x^{\chi,h}$ is replaced by the usual differentiation $a \cdot \nabla_x$, see [6].

Next, we investigate the relationship between classical solutions of problem (1.2) and generalized solutions of problem (4.1). We assume that b_1 is Lipschitz continuous on $\mathbf{R}^{d+1}$ and that there exist $M_1, M_2 > 0$ such that $M_1 \le |b_1(t,x)| \le M_2$. Furthermore, we assume that $b_2 = 0$, $a \in \mathbf{R}^d$ and $u_0 \in L^2(\mathbf{R}^d)$. Then we see that problem (1.2) has a unique weak solution $u \in L^2([-T,T] \times \mathbf{R}^d)$ for each $T > 0$, from Theorems 1 and 2 of Hurd and Sattinger [8]. On the other hand, U_0, B_1 and B_2, given by the classes of $\{(u_0 * \rho_\varepsilon)(x)\}_{\varepsilon \in (0,1]}$, $\{(b_1 * \rho_\varepsilon)(t,x)\}_{\varepsilon \in (0,1]}$ and $\{(b_2 * \rho_\varepsilon)(t,x)\}_{\varepsilon \in (0,1]}$, respectively, define elements of $\mathscr{G}_{s,g}$. According to Theorem 4.2, problem (4.1) has a unique solution $U \in \mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$ for each $T > 0$, under the assumption on $h(\varepsilon)$ stated in Theorem 4.2. Then we obtain the following theorem:

Theorem 4.4. *Let $b_2 = 0$, $a \in \mathbf{R}^d$ and $u_0 \in L^2(\mathbf{R}^d)$. We assume that b_1 is Lipschitz continuous on $\mathbf{R}^{d+1}$ and that there exist $M_1, M_2 > 0$ such that $M_1 \le |b_1(t,x)| \le M_2$. Let U_0, B_1 and B_2 be elements of $\mathscr{G}_{s,g}$ which are given by the classes of $\{(u_0 * \rho_\varepsilon)(x)\}_{\varepsilon \in (0,1]}$, $\{(b_1 * \rho_\varepsilon)(t,x)\}_{\varepsilon \in (0,1]}$ and $\{(b_2 * \rho_\varepsilon)(t,x)\}_{\varepsilon \in (0,1]}$, respectively. Furthermore, we assume that a step function χ is even and that $h(\varepsilon)$*

is as in Theorem 4.2. *Finally, let* $U \in \mathscr{G}_{s,g}([-T,T] \times \mathbf{R}^d)$ *be the generalized solution of problem* (4.1). *Then U is associated with the unique weak solution* $u \in L^2([-T,T] \times \mathbf{R}^d)$ *of problem* (1.2).

Proof. Put $u_0^\varepsilon(x) = (u_0 * \rho_\varepsilon)(x)$ and $b_1^\varepsilon(t,x) = (b_1 * \rho_\varepsilon)(t,x)$. Let $u^\varepsilon(t,x)$ be a representative of U satisfying the equation

$$u_t^\varepsilon(t,x) = \sum_{i=1}^d a_i \{\partial_{x_i}(b_1^\varepsilon u^\varepsilon)(t,\cdot) \underset{i}{*} \chi_{h(\varepsilon)}(x)\}. \tag{4.2}$$

Then, we can see that u^ε belongs to $C([-T,T] : W^{k,2}(\mathbf{R}^d))$ for any $k \in \mathbf{N}_0$, since u_0^ε belongs to $W^{k,2}(\mathbf{R}^d)$ for any $k \in \mathbf{N}_0$. Multiplying equation (4.2) by $b_1^\varepsilon(t,x)u^\varepsilon(t,x)$ and integrating it in t and x, we have

$$\int_{\mathbf{R}^d} b_1^\varepsilon(t,x)(u^\varepsilon(t,x))^2 dx - \int_{\mathbf{R}^d} b_1^\varepsilon(0,x)(u_0^\varepsilon(x))^2 dx$$
$$- \int_0^t \int_{\mathbf{R}^d} \partial_s b_1^\varepsilon(s,x)(u^\varepsilon(s,x))^2 dx ds = 0,$$

since χ is even. Therefore we have the inequality

$$\|u^\varepsilon(t,\cdot)\|_{L^2(\mathbf{R}^d)}^2 \le \frac{M_2}{M_1}\|u_0^\varepsilon\|_{L^2(\mathbf{R}^d)}^2 + \frac{1}{M_1}\|\partial_t b_1^\varepsilon\|_{L^\infty([-T,T]\times\mathbf{R}^d)} \left| \int_0^t \|u^\varepsilon(s,\cdot)\|_{L^2(\mathbf{R}^d)}^2 ds \right|.$$

By weak compactness, we can find a subsequence of $\{u^\varepsilon(t,x)\}_{\varepsilon \in (0,1]}$ converging weakly in $L^2([-T,T] \times \mathbf{R}^d)$. From the uniqueness of weak solutions of problem (1.2), it follows that u^ε converges to the unique weak solution u in $\mathscr{D}'((-T,T) \times \mathbf{R}^d)$ as ε tends to 0. □

Remark 4.5. According to Theorem 4.2, problem (4.1) has a unique generalized solution even if its coefficients are discontinuous. But, the same does not hold for problem (1.2). This can be illustrated by the following example: Let

$$b_1(t,x) = \begin{cases} 0, & x/t < 1, \\ -2, & x/t > 1 \end{cases}$$

and let $b_2 = 0$. Furthermore, let $a = 1$. Then, it is well known ([8]) that problem (1.2) with initial data 0 has many solutions in $(0,T) \times \mathbf{R}$. But, problem (4.1) has the unique solution $U = 0$ from Theorem 4.2, if U_0, B_1 and B_2 are elements of $\mathscr{G}_{s,g}$ which are defined by the classes of $\{(u_0 * \rho_\varepsilon)(x)\}_{\varepsilon \in (0,1]}$, $\{(b_1 * \rho_\varepsilon)(t,x)\}_{\varepsilon \in (0,1]}$ and $\{(b_2 * \rho_\varepsilon)(t,x)\}_{\varepsilon \in (0,1]}$, respectively.

Finally, we consider the case that the coefficients in problem (1.2) are discontinuous, more precisely, b_1 is the Heaviside function H and b_2 is equal to zero. Let $a = 1$. Then, problem (1.2) with initial data 1 has no solution in $(0, T) \times \mathbf{R}$ in the sense of distributions, see [8]. But, under the same assumptions with respect to coefficients and initial data, it is known (Oberguggenberger [9]) that for each $T > 0$, there exists a unique generalized solution $V \in \mathscr{G}([-T, T] \times \mathbf{R})$ of problem (4.1) with the usual differentiations, which is associated with $1 + t\delta(x)$ on $(0, T) \times \mathbf{R}$. We can show that a similar result to this one holds for a generalized solution $U \in \mathscr{G}_{s,g}([-T, T] \times \mathbf{R})$ of problem (4.1). Now we choose the step function χ to be

$$\chi(y) = \begin{cases} 1, & -1 \le y \le 0, \\ 0, & \text{otherwise} \end{cases} \tag{4.3}$$

and a function ρ to be nonnegative. As seen from the following theorem, the generalized solution $U \in \mathscr{G}_{s,g}([-T, T] \times \mathbf{R})$ of problem (4.1) is associated with $1 + t\delta(x)$ on $(0, T) \times \mathbf{R}$.

Theorem 4.6. *Let $u_0 = 1$ and let $b_2 = 0$. Let b_1 be the Heaviside function H and let $a = 1$. Furthermore, let U_0, B_1 and B_2 be elements of $\mathscr{G}_{s,g}$ which are given by the classes of $\{(u_0 * \rho_\varepsilon)(x)\}_{\varepsilon \in (0,1]}$, $\{(b_1 * \rho_\varepsilon)(x)\}_{\varepsilon \in (0,1]}$ and $\{(b_2 * \rho_\varepsilon)(t, x)\}_{\varepsilon \in (0,1]}$, respectively. We assume that χ is the step function given by* (4.3) *and that ρ is nonnegative. Finally, we assume that $h(\varepsilon)$ is as in Theorem* 4.2. *Then the generalized solution $U \in \mathscr{G}_{s,g}([-T, T] \times \mathbf{R})$ of problem* (4.1) *is associated with $1 + t\delta(x)$ on $(0, T) \times \mathbf{R}$.*

Proof. Put $u_0^\varepsilon(x) = (u_0 * \rho_\varepsilon)(x)$ and $b_1^\varepsilon(x) = (b_1 * \rho_\varepsilon)(x)$. Let $u^\varepsilon(t, x)$ be a representative of U satisfying the equation

$$u_t^\varepsilon(t, x) = \int_{\mathbf{R}} \partial_x \{b_1^\varepsilon(x - y) u^\varepsilon(t, x - y)\} \chi_{h(\varepsilon)}(y) dy. \tag{4.4}$$

Put $v^\varepsilon(t, x) = u^\varepsilon(t, x) - 1$. By equation (4.4), $v^\varepsilon(t, x)$ satisfies the equation

$$\begin{aligned} v_t^\varepsilon(t, x) = & \int_{\mathbf{R}} \partial_x b_1^\varepsilon(x - y) \chi_{h(\varepsilon)}(y) dy \\ & + \int_{\mathbf{R}} \partial_x \{b_1^\varepsilon(x - y) v^\varepsilon(t, x - y)\} \chi_{h(\varepsilon)}(y) dy. \end{aligned} \tag{4.5}$$

Then, we can see that v^ε belongs to $C([-T, T] : W^{k,1}(\mathbf{R}))$ for any $k \in \mathbf{N}_0$. Hence, integrating equation (4.5) in t and x, we have

$$\int_{\mathbf{R}} v^\varepsilon(t, x) dx = t$$

for each $\varepsilon \in (0, 1]$.

Next, we prove that $v^\varepsilon(t,x)=0$ in $\{(t,x)\in[0,T]\times\mathbf{R}; x-\varepsilon y\geq 0$ for all $y\in\operatorname{supp}\rho\}$ and $\{(t,x)\in[0,T]\times\mathbf{R}; x+h(\varepsilon)-\varepsilon y\leq 0$ for all $y\in\operatorname{supp}\rho\}$ for each $\varepsilon\in(0,1]$. Fix $\varepsilon\in(0,1]$ arbitrarily. Transforming equation (4.5), we have

$$v_t^\varepsilon(t,x)=\frac{1}{h(\varepsilon)}\{b_1^\varepsilon(x+h(\varepsilon))-b_1^\varepsilon(x)\}$$
$$+\frac{1}{h(\varepsilon)}\{b_1^\varepsilon(x+h(\varepsilon))v^\varepsilon(t,x+h(\varepsilon))-b_1^\varepsilon(x)v^\varepsilon(t,x)\}. \tag{4.6}$$

It follows from the definition of $b_1^\varepsilon(x)$ that

$$v_t^\varepsilon(t,x)=\frac{1}{h(\varepsilon)}\{v^\varepsilon(t,x+h(\varepsilon))-v^\varepsilon(t,x)\} \tag{4.7}$$

for $x\in\mathbf{R}$ such that $x-\varepsilon y\geq 0$ for all $y\in\operatorname{supp}\rho$. Since v^ε belongs to $C([-T,T]:W^{k,1}(\mathbf{R}))$ for any $k\in\mathbf{N}_0$, there exists a point (t_0,x_0) in $\{(t,x)\in[0,T]\times\mathbf{R}; x-\varepsilon y\geq 0$ for all $y\in\operatorname{supp}\rho\}$ such that $v^\varepsilon(t_0,x_0)$ is the maximum of $v^\varepsilon(t,x)$ on $\{(t,x)\in[0,T]\times\mathbf{R}; x-\varepsilon y\geq 0$ for all $y\in\operatorname{supp}\rho\}$. Obviously $v_t^\varepsilon(t_0,x_0)\geq 0$. Furthermore, by equation (4.7), we have $v_t^\varepsilon(t_0,x_0)\leq 0$. Hence $v_t^\varepsilon(t_0,x_0)=0$. Again by equation (4.7), $v^\varepsilon(t_0,x_0)=v^\varepsilon(t_0,x_0+h(\varepsilon))$. Repeating this method, we have $v^\varepsilon(t_0,x_0)=v^\varepsilon(t_0,x_0+h(\varepsilon)n)$ for all $n\in\mathbf{N}$. Since v^ε belongs to $C([-T,T]:W^{k,1}(\mathbf{R}))$ for any $k\in\mathbf{N}_0$, it follows that $v^\varepsilon(t_0,x_0)=0$. Hence we have $v^\varepsilon(t,x)\leq 0$ in $\{(t,x)\in[0,T]\times\mathbf{R}; x-\varepsilon y\geq 0$ for all $y\in\operatorname{supp}\rho\}$. As shown below, $v^\varepsilon(t,x)\geq 0$ in $[0,T]\times\mathbf{R}$. Therefore $v^\varepsilon(t,x)=0$ in $\{(t,x)\in[0,T]\times\mathbf{R}; x-\varepsilon y\geq 0$ for all $y\in\operatorname{supp}\rho\}$. Also, integrating equation (4.6) in t, we have

$$v^\varepsilon(t,x)=\frac{t}{h(\varepsilon)}\{b_1^\varepsilon(x+h(\varepsilon))-b_1^\varepsilon(x)\}$$
$$+\int_0^t\frac{1}{h(\varepsilon)}\{b_1^\varepsilon(x+h(\varepsilon))v^\varepsilon(s,x+h(\varepsilon))-b_1^\varepsilon(x)v^\varepsilon(s,x)\}ds.$$

It follows from the definition of $b_1^\varepsilon(x)$ that $v^\varepsilon(t,x)=0$ for $x\in\mathbf{R}$ such that $x+h(\varepsilon)-\varepsilon y\leq 0$ for all $y\in\operatorname{supp}\rho$.

Finally, we prove that $v^\varepsilon(t,x)\geq 0$ in $[0,T]\times\mathbf{R}$ for each $\varepsilon\in(0,1]$. Let $\mathscr{D}_{L^1}([0,T]\times\mathbf{R})$ denote the space of smooth functions whose derivatives belong to $L^1([0,T]\times\mathbf{R})$. We consider the problem

$$\begin{cases}\psi_t^\varepsilon-b_1^\varepsilon\partial_x\psi^\varepsilon(t,\cdot)*\check{\chi}_{h(\varepsilon)}=\varphi, & 0\leq t\leq T,\quad x\in\mathbf{R},\\ \psi^\varepsilon|_{t=T}=0, & x\in\mathbf{R}\end{cases} \tag{4.8}$$

for each nonnegative function $\varphi\in\mathscr{D}((0,T)\times\mathbf{R})$. Then, we can see that there exists a solution $\psi^\varepsilon\in\mathscr{D}_{L^1}([0,T]\times\mathbf{R})$ of problem (4.8) by solving the following problem (4.10). Furthermore, we can show that it is unique by Gronwall's inequality. Multiplying equation (4.4) by this solution $\psi^{\varepsilon}(t,x)$ and integrating it in t and x, we have

$$0 = \int_0^T \int_{\mathbf{R}} u_t^{\varepsilon}(t,x)\psi^{\varepsilon}(t,x)dxdt - \int_0^T \int_{\mathbf{R}} \int_{\mathbf{R}} \partial_x \{b_1^{\varepsilon}(x-y)u^{\varepsilon}(t,x-y)\}\chi_{h(\varepsilon)}(y)dy\psi^{\varepsilon}(t,x)dxdt.$$

Integration by parts yields

$$\begin{aligned} 0 &= -\int_{\mathbf{R}} \psi^{\varepsilon}(0,x)dx - \int_0^T \int_{\mathbf{R}} u^{\varepsilon}(t,x)\psi_t^{\varepsilon}(t,x)dxdt \\ &\quad + \int_0^T \int_{\mathbf{R}} b_1^{\varepsilon}(x)u^{\varepsilon}(t,x) \int_{\mathbf{R}} \psi_x^{\varepsilon}(t,x-y)\check{\chi}_{h(\varepsilon)}(y)dydxdt \\ &= -\int_{\mathbf{R}} \psi^{\varepsilon}(0,x)dx - \int_0^T \int_{\mathbf{R}} u^{\varepsilon}(t,x)\varphi(t,x)dxdt. \end{aligned} \tag{4.9}$$

Putting $\Psi^{\varepsilon}(t,x) = \psi^{\varepsilon}(T-t,x)$ and $\Phi(t,x) = \varphi(T-t,x)$, we have

$$\begin{cases} \Psi_t^{\varepsilon} + b_1^{\varepsilon}\partial_x \Psi^{\varepsilon}(t,\cdot) * \check{\chi}_{h(\varepsilon)} = -\Phi, & 0 \le t \le T, \ x \in \mathbf{R}, \\ \Psi^{\varepsilon}|_{t=0} = 0, & x \in \mathbf{R} \end{cases} \tag{4.10}$$

from problem (4.8). If $b_1^{\varepsilon}(x_1) = 0$ for some $x_1 \in \mathbf{R}$, then $\Psi_t^{\varepsilon}(t,x_1) = -\Phi(t,x_1) \le 0$ on $[0,T]$. Since $\Psi^{\varepsilon}|_{t=0} = 0$, $\Psi^{\varepsilon}(t,x_1) \le 0$ on $[0,T]$. Furthermore, we obtain that $\Psi^{\varepsilon}(t,x)$ takes the maximum in some bounded domain of $[0,T] \times \mathbf{R}$, since Ψ^{ε} belongs to $\mathscr{D}_{L^1}([0,T] \times \mathbf{R})$. Assume that (t_2,x_2) is a point in $(0,T] \times \mathbf{R}$ satisfying $\Psi^{\varepsilon}(t_2,x_2) = \sup_{(t,x) \in [0,T] \times \mathbf{R}} \Psi^{\varepsilon}(t,x)$ and that $\Psi^{\varepsilon}(t_2,x_2) > 0$. Transforming the first equation of problem (4.10) and substituting (t_2,x_2) into (t,x), we have the equation

$$\Psi_t^{\varepsilon}(t_2,x_2) + \frac{b_1^{\varepsilon}(x_2)}{h(\varepsilon)}\{\Psi^{\varepsilon}(t_2,x_2) - \Psi^{\varepsilon}(t_2,x_2-h(\varepsilon))\} = -\Phi(t_2,x_2). \tag{4.11}$$

Obviously, $\Psi_t^{\varepsilon}(t_2,x_2) \ge 0$, and $b_1^{\varepsilon}(x_2) > 0$ from the above argument and the assumption $\rho \ge 0$. Hence, the left-hand side of equation (4.11) is nonnegative, which means that $\Phi(t_2,x_2) = 0$. Again, from equation (4.11), we obtain that $\Psi^{\varepsilon}(t_2,x_2) = \Psi^{\varepsilon}(t_2,x_2-h(\varepsilon))$. Repeating this method, we have $\Psi^{\varepsilon}(t_2,x_2) = \Psi^{\varepsilon}(t_2,x_2-h(\varepsilon)n)$ for any $n \in \mathbf{N}$, which is impossible. This contradiction shows that $\Psi^{\varepsilon}(t,x) \le \sup_{x \in \mathbf{R}} \Psi^{\varepsilon}(0,x) = 0$, so that the inequality $\psi^{\varepsilon}(t,x) \le 0$ holds. Hence, we have

$$
\begin{aligned}
&-\int_{\mathbf{R}} \psi^{\varepsilon}(0,x)dx \\
&\quad = \int_0^T \int_{\mathbf{R}} b_1^{\varepsilon}(x)\partial_x \psi^{\varepsilon}(t,\cdot) * \check{\chi}_{h(\varepsilon)}\, dxdt + \int_0^T \int_{\mathbf{R}} \varphi(t,x)dxdt \\
&\quad = -\int_0^T \int_{\mathbf{R}} (\partial_x b_1^{\varepsilon} * \chi_{h(\varepsilon)})(x)\psi^{\varepsilon}(t,x)dxdt + \int_0^T \int_{\mathbf{R}} \varphi(t,x)dxdt \\
&\quad \geq \int_0^T \int_{\mathbf{R}} \varphi(t,x)dxdt
\end{aligned}
$$

by the first equation of problem (4.8) and the assumption $\rho \geq 0$. Therefore, by equation (4.9), we obtain that for any nonnegative function $\varphi \in \mathscr{D}((0,T) \times \mathbf{R})$

$$
\begin{aligned}
0 &= -\int_{\mathbf{R}} \psi^{\varepsilon}(0,x)dx - \int_0^T \int_{\mathbf{R}} u^{\varepsilon}(t,x)\varphi(t,x)dxdt \\
&\geq \int_0^T \int_{\mathbf{R}} \varphi(t,x)dxdt - \int_0^T \int_{\mathbf{R}} u^{\varepsilon}(t,x)\varphi(t,x)dxdt \\
&= \int_0^T \int_{\mathbf{R}} (1 - u^{\varepsilon}(t,x))\varphi(t,x)dxdt,
\end{aligned}
$$

which means that $u^{\varepsilon}(t,x) \geq 1$, that is, $v^{\varepsilon}(t,x) \geq 0$ in $[0,T] \times \mathbf{R}$ for each $\varepsilon \in (0,1]$. Consequently, we obtain that for all $\varphi \in \mathscr{D}((0,T) \times \mathbf{R})$

$$
\begin{aligned}
&\int_0^T \int_{\mathbf{R}} (v^{\varepsilon}(t,x) - t\delta(x))\varphi(t,x)dxdt \\
&\quad = \int_0^T \int_{\mathbf{R}} (v^{\varepsilon}(t,x)\varphi(t,x) - v^{\varepsilon}(t,x)\varphi(t,0))dxdt \\
&\qquad + \int_0^T \int_{\mathbf{R}} (v^{\varepsilon}(t,x)\varphi(t,0) - t\delta(x)\varphi(t,x))dxdt \\
&\quad = \int_0^T \int_{\mathbf{R}} v^{\varepsilon}(t,x)(\varphi(t,x) - \varphi(t,0))dxdt \\
&\quad \leq \int_0^T \int_{\mathbf{R}} v^{\varepsilon}(t,x)\|\varphi_x\|_{L^{\infty}((0,T)\times\mathbf{R})}|x|dxdt \\
&\quad \leq \frac{1}{2}\|\varphi_x\|_{L^{\infty}((0,T)\times\mathbf{R})}T^2 \max\left\{\varepsilon\left|\sup_{y\in\operatorname{supp}\rho} y\right|, h(\varepsilon) + \varepsilon\left|\inf_{y\in\operatorname{supp}\rho} y\right|\right\} \\
&\quad \to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Thus the assertion follows. □

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