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Oscillation and nonoscillation theorems for a class of fourth order quasilinear functional differential equations

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ABSTRACT. The oscillatory behavior of fourth order functional differential equations

(A) $$(|y''(t)|^{\alpha} \operatorname{sgn} y''(t))'' + q(t)|y(g(t))|^{\beta} \operatorname{sgn} y(g(t)) = 0$$

is investigated. First, criteria are given for the existence of nonoscillatory solutions with specific asymptotic behavior, and then criteria for all solutions to be oscillatory are derived by comparing (A) with the associated differential equation without functional argument.

1. Introduction

The objective of this paper is to study the oscillatory and nonoscillatory behavior of fourth order nonlinear functional differential equations

(A) $$(|y''(t)|^{\alpha} \operatorname{sgn} y''(t))'' + q(t)|y(g(t))|^{\beta} \operatorname{sgn} y(g(t)) = 0$$

where

(a) α and β are positive constants;
(b) $q : [0, \infty) \to (0, \infty)$ is a continuous function;
(c) $g : [0, \infty) \to (0, \infty)$ is a continuously differentiable function such that $g'(t) > 0$, $t \geq 0$, and $\lim_{t\to\infty} g(t) = \infty$.

By a solution of (A) we mean a function $y : [T_y, \infty) \to \mathbf{R}$ which is twice continuously differentiable together with $|y''|^{\alpha} \operatorname{sgn} y''$ and satisfies the equation (A) at all sufficiently large t. Those solutions which vanish in a neighborhood of infinity will be excluded from our consideration. A solution is said to be oscillatory if it has a sequence of zeros clustering around ∞, and nonoscillatory otherwise.

We first (in Section 1) study the existence of nonoscillatory solutions. The set of nonoscillatory solutions of (A) is decomposed into six disjoint classes according to their asymptotic behavior at ∞, and existence criteria are established for each of these classes. Some of the criteria are shown to be necessary as well.

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We next derive criteria for any solution of (A) to be oscillatory. Our derivations depends heavily on oscillaiton theory of fourth order nonlinear ordinary differential equations

$$\text{(B)} \qquad (|y''|^{\alpha} \operatorname{sgn} y'')'' + q(t)|y|^{\beta} \operatorname{sgn} y = 0$$

recently developed by Wu [6]. A comparison principle enables us to deduce oscillation of an equation of the form (A) from that of a similar equation with a different functional argument. As a result, we are able to demonstrate the existence of classes of equations of the form (A) for which sharp oscillation criteria can be established.

We note that oscillation properties of second order functional differential equations involving nonlinear Sturm-Liouville type differential operators have been investigated by Kusano and Lalli [1], Kusano and Wang [3] and Wang [5]. The present paper is a step toward generalizing the above results to higher order functional differential equations whose principal parts are genuinely nonlinear.

2. Nonoscillation theorems

Our purpose here is to make a detailed analysis of the structure of the set of all possible nonoscillatory solutions of the equation (A), which can also be expressed as

$$\text{(A)} \qquad ((y''(t))^{\alpha *})'' + q(t)(y(g(t)))^{\beta *} = 0$$

in terms of the asterisk notation

$$\text{(2.1)} \qquad \xi^{\gamma *} = |\xi|^{\gamma} \operatorname{sgn} \xi = |\xi|^{\gamma - 1}\xi, \qquad \xi \in \mathbf{R},\ \gamma > 0.$$

A) *Classification of nonoscillatory solutions*. It suffices to restrict our consideration to eventually positive solutions of (A), since if $y(t)$ is a solution of (A) then so is $-y(t)$. Let $y(t)$ be one such solution. Then, as is easily verified, $y(t)$ satisfies either

I: $y'(t) > 0$, $y''(t) > 0$, $((y''(t))^{\alpha *})' > 0$ for all large t

or

II: $y'(t) > 0$, $y''(t) < 0$, $((y''(t))^{\alpha *})' > 0$ for all large t.

(See Wu [6].) It follows that $y(t), y'(t), y''(t)$ and $((y''(t))^{\alpha *})'$ are eventually monotone, so that they tend to finite or infinite limits as $t \to \infty$. Let

$$\lim_{t \to \infty} y^{(i)}(t) = \omega_i, \quad i = 0, 1, 2, \qquad \text{and} \qquad \lim_{t \to \infty} ((y''(t))^{\alpha *})' = \omega_3.$$

It is clear that ω_3 is a finite nonnegative number. One can easily show that:

(i) if $y(t)$ satisfies I, then the set of its asymptotic values $\{\omega_i\}$ falls into one of the following three cases:

I_1: $\omega_0 = \omega_1 = \omega_2 = \infty,\ \omega_3 \in (0, \infty)$;
I_2: $\omega_0 = \omega_1 = \omega_2 = \infty,\ \omega_3 = 0$;
I_3: $\omega_0 = \omega_1 = \infty,\ \omega_2 \in (0, \infty),\ \omega_3 = 0$.

(ii) if $y(t)$ satisfies II, then the set of its asymptotic values $\{\omega_i\}$ falls into one of the following three cases:

II_1: $\omega_0 = \infty,\ \omega_1 \in (0, \infty),\ \omega_2 = \omega_3 = 0$;
II_2: $\omega_0 = \infty,\ \omega_1 = \omega_2 = \omega_3 = 0$;
II_3: $\omega_0 \in (0, \infty),\ \omega_1 = \omega_2 = \omega_3 = 0$.

Equivalent expressions for these six classes of positive solutions of (A) are as follows:

I_1: $\lim_{t\to\infty} \dfrac{y(t)}{t^{2+1/\alpha}} = \text{const} > 0$;

I_2: $\lim_{t\to\infty} \dfrac{y(t)}{t^{2+1/\alpha}} = 0,\ \lim_{t\to\infty} \dfrac{y(t)}{t^2} = \infty$;

I_3: $\lim_{t\to\infty} \dfrac{y(t)}{t^2} = \text{const} > 0$;

II_1: $\lim_{t\to\infty} \dfrac{y(t)}{t} = \text{const} > 0$;

II_2: $\lim_{t\to\infty} \dfrac{y(t)}{t} = 0,\ \lim_{t\to\infty} y(t) = \infty$;

II_3: $\lim_{t\to\infty} y(t) = \text{const} > 0$.

B) *Integral representations for nonoscillatory solutions.* We shall establish the existence of positive solutions for each of the above six classes. For this purpose a crucial role will be played by integral representations for those six types of solutions of (A) as derived below.

Let $y(t)$ be a positive solution of (A) such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq T > 0$. Integrating (A) from t to ∞ gives

$$((y''(t))^{\alpha*})' = \omega_3 + \int_t^\infty q(s)(y(g(s)))^\beta ds, \qquad t \geq T. \tag{2.2}$$

If $y(t)$ is a solution of type I_i $(i = 1, 2, 3)$, then we integrate (2.2) three times over $[T, t]$ to obtain

(2.3)
$$y(t) = k_0 + k_1(t - T) + \int_T^t (t - s)\left[k_2^\alpha + \int_T^s \left(\omega_3 + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma\right) dr\right]^{1/\alpha} ds,$$

for $t \geq T$, where $k_0 = y(T)$, $k_1 = y'(T)$ and $k_2 = y''(T)$ are nonnegative constants. The equality (2.3) gives an integral representation for a solution $y(t)$ of type I_1. A type-I_2 solution $y(t)$ of (A) is expressed by (2.3) with $\omega_3 = 0$.

If $y(t)$ is a solution of type I_3, then, first integrating (2.2) from t to ∞ and then integrating the resulting equation twice from T to t, we have

(2.4)
$$y(t) = k_0 + k_1(t-T) + \int_T^t (t-s)\left[\omega_2^\alpha - \int_s^\infty (r-s)q(r)(y(g(r)))^\beta dr\right]^{1/\alpha} ds, \quad t \geq T,$$

where T is chosen sufficiently large.

An integral representation for a solution $y(t)$ of type II_1 is derived by integrating (2.2) with $\omega_3 = 0$ twice from t to ∞ and then once from T to t:

(2.5)
$$y(t) = k_0 + \int_T^t \left(\omega_1 + \int_s^\infty \left[\int_r^\infty (\sigma - r)q(\sigma)(y(g(\sigma)))^\beta d\sigma\right]^{1/\alpha} dr\right) ds, \qquad t \geq T.$$

An expression for a type-II_2 solution is given by (2.5) with $\omega_1 = 0$. If $y(t)$ is a solution of type II_3, then integrations of (2.2) with $\omega_3 = 0$ three times yield

$$(2.6) \quad y(t) = \omega_0 - \int_t^\infty (s-t)\left[\int_s^\infty (r-s)q(r)(y(g(r)))^\beta dr\right]^{1/\alpha} ds, \qquad t \geq T.$$

C) *Nonoscillation criteria (necessary and sufficient conditions)*. It will be shown that necessary and sufficient conditions can be established for the existence of positive solutions of the four types $\mathrm{I}_1, \mathrm{I}_3, \mathrm{II}_1$ and II_3.

Theorem 2.1. *The equation* (A) *has a positive solution of type* I_1 *if and only if*

$$(2.7) \qquad \int_0^\infty (g(t))^{(2+1/\alpha)\beta} q(t)dt < \infty.$$

Proof. Suppose that (A) has a solution $y(t)$ of type I_1. Then, it satisfies (2.3) for $t \geq T$, when $T > 0$ is sufficiently large, which implies that

$$\int_T^\infty q(t)(y(g(t)))^\beta dt < \infty.$$

This together with the asymptotic relation $\lim_{t\to\infty} y(t)/t^{2+1/\alpha} = \mathrm{const} > 0$, shows that the condition (2.7) is satisfied.

Suppose now that (2.7) holds. Let $k > 0$ be any given constant. Choose $T > 0$ large enough so that

$$(2.8) \qquad \left(\frac{\alpha^2}{(\alpha+1)(2\alpha+1)}\right)^\beta \int_T^\infty (g(t))^{(2+1/\alpha)\beta} q(t)dt \leq \frac{(2k)^\alpha - k^\alpha}{(2k)^\beta}.$$

Put $T_* = \min\left\{T, \inf_{t \geq T} g(t)\right\}$, and define

$$G(t,T) = \int_T^t (t-s)(s-T)^{1/\alpha} ds = \frac{\alpha^2}{(\alpha+1)(2\alpha+1)} (t-T)^{2+1/\alpha}, \qquad t \geq T, \tag{2.9}$$

$$G(t,T) = 0, \qquad t \leq T.$$

Let $Y \subset C[T_*, \infty)$ and $\mathscr{F} : Y \to C[T_*, \infty)$ be defined as follows:

$$Y = \{ y \in C[T_*, \infty) : kG(t,T) \leq y(t) \leq 2kG(t,T), t \geq T_* \}, \tag{2.10}$$

$$\mathscr{F}y(t) = \int_T^t (t-s) \left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma \right) dr \right]^{1/\alpha} ds, \qquad t \geq T, \tag{2.11}$$

$$\mathscr{F}y(t) = 0, \qquad T_* \leq t \leq T.$$

Clearly, Y is a closed convex subset of the Frechét space $C[T_*, \infty)$ with the topology of uniform convergence on compact subintervals of $[T_*, \infty)$.

If $y \in Y$, then for $t \geq T$

$$\mathscr{F}y(t) \geq k \int_T^t (t-s)(s-T)^{1/\alpha} ds = kG(t,T)$$

and

$$\begin{aligned}
\mathscr{F}y(t) &\leq \int_T^t (t-s) \left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(2kG(g(\sigma),T))^\beta d\sigma \right) dr \right]^{1/\alpha} ds \\
&\leq \int_T^t (t-s) \left[\int_T^s \left(k^\alpha + \left(\frac{\alpha^2 \cdot 2k}{(\alpha+1)(2\alpha+1)} \right)^\beta \int_r^\infty q(\sigma)(g(\sigma))^{(2+1/\alpha)\beta} d\sigma \right) dr \right]^{1/\alpha} ds \\
&\leq 2k \int_T^t (t-s)(s-T)^{1/\alpha} ds = 2kG(t,T),
\end{aligned}$$

and hence $\mathscr{F}y \in Y$. Thus, $\mathscr{F}$ maps Y into itself. Let $\{y_n\}$ be a sequence of functions in Y converging to $y \in Y$ in the metric topology of $C[T_*, \infty)$. Then, by using Lebesgue's dominated convergence theorem, we can prove that the sequence $\{\mathscr{F}y_n(t)\}$ converges to $\mathscr{F}y(t)$ as $n \to \infty$ uniformly on every compact subinterval of $[T_*, \infty)$, that is, $\mathscr{F}y_n \to \mathscr{F}y$ as $n \to \infty$ in $C[T_*, \infty)$. Hence $\mathscr{F}$ is a continuous mapping.

For any $y \in Y$ we have

$$(\mathscr{F}y(t))' = \int_T^t \left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma \right) dr \right]^{1/\alpha} ds, \qquad t \geq T,$$

which implies that

$$0 \leq (\mathscr{F}y(t))' \leq 2k \int_T^t (s-T)^{1/\alpha} ds = \frac{2k\alpha}{\alpha+1} (t-T)^{1+1/\alpha}, \qquad t \geq T.$$

From this inequality, together with the fact that $\mathscr{F}y \in Y$, we conclude that the set $\mathscr{F}(Y)$ is relatively compact in the topology of $C[T_*, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists a fixed element $y \in Y$ of $\mathscr{F}$, i.e., $y = \mathscr{F}y$, which satisfies the integral equation

$$y(t) = \int_T^t (t-s)\left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma\right) dr\right]^{1/\alpha} ds, \qquad t \geq T. \tag{2.12}$$

This is a special case of (2.3) with $k_0 = k_1 = k_2 = 0$ and $\omega_3 = k^\alpha$. Differentiation of (2.12) shows that $y(t)$ is a positive solution of (A) for all large t. Since $\lim_{t\to\infty}((y''(t))^\alpha)' = k^\alpha > 0$, $y(t)$ is a desired solution of type I_1. This completes the proof.

Theorem 2.2. *The equation* (A) *has a positive solution of type* I_3 *if and only if*

$$\int_0^\infty t(g(t))^{2\beta} q(t)dt < \infty. \tag{2.13}$$

Proof. A positive solution $y(t)$ of type I_3, if exists, has an integral representation (2.4) for some $T > 0$, which implies that

$$\int_T^\infty (t-T)q(t)(y(g(t)))^\beta dt < \infty.$$

Since $\lim_{t\to\infty} y(t)/t^2 = \text{const} > 0$, we see that the condition (2.13) is satisfied.

Suppose now that (2.13) holds. Let $k > 0$ be an arbitrarily fixed constant and choose $T > 0$ so large that

$$\int_T^\infty t(g(t))^{2\beta} q(t)dt \leq \frac{(2k)^\alpha - k^\alpha}{k^\beta}. \tag{2.14}$$

Let $T_* = \min\left\{T, \inf_{t\geq T} g(t)\right\}$ and consider the set $Y \subset C[T_*, \infty)$ defined by

$$Y = \left\{y \in C[T_*, \infty) : \frac{k}{2}(t-T)_+^2 \leq y(t) \leq k(t-T)_+^2, t \geq T_*\right\}, \tag{2.15}$$

where $(t-T)_+ = t-T$ if $t \geq T$, and $(t-T)_+ = 0$ if $t \leq T$. Define the mapping $\mathscr{G} : Y \to C[T_*, \infty)$ by

$$\mathscr{G}y(t) = \int_T^t (t-s)\left[(2k)^\alpha - \int_s^\infty (r-s)q(r)(y(g(r)))^\beta dr\right]^{1/\alpha} ds, \qquad t \geq T, \tag{2.16}$$

$$\mathscr{G}y(t) = 0, \qquad T_* \leq t \leq T.$$

That $\mathscr{G}(Y) \subset Y$ is an immediate consequence of (2.14). Since the continuity of $\mathscr{G}$ and the relative compactness of $\mathscr{G}(Y)$ can be proved routinely, there exists an element $y \in Y$ such that $y = \mathscr{G}y$, which satisfies

$$(2.17) \quad y(t) = \int_T^t (t-s)\left[(2k)^\alpha - \int_s^\infty (r-s)q(r)(y(g(r)))^\beta dr\right]^{1/\alpha} ds, \qquad t \geq T.$$

This is a special case of (2.4) with $k_0 = k_1 = 0$ and $\omega_2 = 2k$. Differentiating (2.17), we see that $y(t)$ is a positive solution of (A) for all large t with the property that $\lim_{t\to\infty} y''(t) = 2k > 0$. Thus, $y(t)$ is a type-I_3 solution of (A). This completes the proof.

Theorem 2.3. *The equation* (A) *has a positive solution of type* II_1 *if and only if*

$$(2.18) \qquad \int_0^\infty \left[\int_t^\infty (s-t)(g(s))^\beta q(s)ds\right]^{1/\alpha} dt < \infty.$$

Proof. To prove the "only if" part of the theorem it suffices to observe that a positive solution $y(t)$ of type II_1 satisfies $\lim_{t\to\infty} y(t)/t = \text{const} > 0$ and

$$\int_T^\infty \left[\int_t^\infty (s-t)q(s)(y(g(s)))^\beta ds\right]^{1/\alpha} dt < \infty.$$

To prove the "if" part, assume that (2.18) holds, and for any fixed constant $k > 0$ choose $T > 0$ so that

$$(2.19) \qquad \int_T^\infty \left[\int_t^\infty (s-t)(g(s))^\beta q(s)ds\right]^{1/\alpha} dt \leq 2^{-\beta/\alpha}k^{1-\beta/\alpha}.$$

Let $T_* = \min\left\{T, \inf_{t\geq T} g(t)\right\}$ and consider the set $Y \subset C[T_*, \infty)$ and the mapping $\mathscr{H} : Y \to C[T_*, \infty)$ defined by

$$(2.20) \qquad Y = \{y \in C[T_*, \infty) : kt \leq y(t) \leq 2kt, t \geq T_*\}$$

and

$$(2.21) \quad \mathscr{H}y(t) = kt + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)q(\sigma)(y(g(\sigma)))^\beta d\sigma\right]^{1/\alpha} dr ds, \qquad t \geq T,$$

$$\mathscr{H}y(t) = kt, \qquad T_* \leq t \leq T.$$

It can be verified as in the preceding theorems that (i) $\mathscr{H}(Y) \subset Y$, (ii) $\mathscr{H}$ is continuous, and (iii) $\mathscr{H}(Y)$ is relatively compact. Therefore, $\mathscr{H}$ has a fixed point $y \in Y$, which gives rise to a positive type-II$_1$ solution of (A), since it satisfies

$$y(t) = kt + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)q(\sigma)(y(g(\sigma)))^\beta d\sigma\right]^{1/\alpha} drds, \qquad t \geq T. \tag{2.22}$$

Note that (2.22) is a special case of (2.5), and $\lim_{t\to\infty} y'(t) = k$. The proof is thus complete.

Theorem 2.4. *The equation* (A) *has a positive solution of type* II$_3$ *if and only if*

$$\int_0^\infty t\left[\int_t^\infty (s - t)q(s)ds\right]^{1/\alpha} dt < \infty. \tag{2.23}$$

Proof. Let $y(t)$ be a type-II$_3$ solution of (A). Then $y(t)$ satisfies (2.6), which implies that

$$\int_T^\infty t\left[\int_t^\infty (s - t)q(s)(y(g(s)))^\beta ds\right]^{1/\alpha} dt < \infty.$$

Since $\lim_{t\to\infty} y(t) = \text{const} > 0$, (2.23) follows from the above inequality.

Suppose now that (2.23) holds. Let $k > 0$ be any fixed constant and take $T > 0$ so large that

$$\int_T^\infty t\left[\int_t^\infty (s - t)q(s)ds\right]^{1/\alpha} dt \leq \frac{1}{2}k^{1-\beta/\alpha}. \tag{2.24}$$

Let $T_* = \min\left\{T, \inf_{t\geq T} g(t)\right\}$, and define the mapping $\mathscr{I}$ by

$$\mathscr{I}y(t) = k - \int_t^\infty (s - t)\left[\int_s^\infty (r - s)q(r)(y(g(r)))^\beta dr\right]^{1/\alpha} ds, \qquad t \geq T, \tag{2.25}$$

$$\mathscr{I}y(t) = \mathscr{I}y(T), \qquad T_* \leq t \leq T.$$

Then, it can be verified without difficulty that $\mathscr{I}$ has a fixed element y in the set

$$Y = \left\{y \in C[T_*, \infty) : \frac{k}{2} \leq y(t) \leq k, t \geq T_*\right\}.$$

This fixed point gives rise to a required positive solution of (A), since it satisfies

$$y(t) = k - \int_t^\infty (s - t)\left[\int_s^\infty (r - s)q(r)(y(g(r)))^\beta dr\right]^{1/\alpha} ds, \qquad t \geq T, \tag{2.27}$$

which is nothing else but (2.6). Note that $\lim_{t\to\infty} y(t) = k$. This completes the proof.

D) *Nonoscillation criteria (sufficient conditions)*. Let us now turn our attention to positive solutions of types I_2 and II_2 of (A). We are content with sufficient conditions for the existence of positive solutions with "intermediate" growth. We observe that this kind of problem has not been dealt with even for ordinary differential equations without functional arguments of the form (B); see Wu [6].

THEOREM 2.5. *The equation* (A) *has a positive solution of type* I_2 *if*

$$\int_0^\infty (g(t))^{(2+1/\alpha)\beta} q(t)dt < \infty \tag{2.28}$$

and

$$\int_0^\infty t(g(t))^{2\beta} q(t)dt = \infty. \tag{2.29}$$

PROOF. Choose $T > 0$ large enough so that $T_* = \min\left\{T, \inf_{t \geq T} g(t)\right\} \geq 1$ and

$$\int_T^\infty (g(t))^{(2+1/\alpha)\beta} q(t)dt \leq \frac{1}{2^{\alpha+1}}\left(\frac{(\alpha+1)(2\alpha+1)}{\alpha^2}\right)^\alpha. \tag{2.30}$$

Define

$$Y = \left\{ y \in C[T_*, \infty) : \frac{1}{2^{1+1/\alpha}}(t-T)_+^2 \leq y(t) \leq t^{2+1/\alpha}, t \geq T_* \right\}, \tag{2.31}$$

$$\mathscr{J}y(t) = \int_T^t (t-s)\left[\frac{1}{2} + \int_T^s \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma dr\right]^{1/\alpha} ds, \qquad t \geq T, \tag{2.32}$$

$$\mathscr{J}y(t) = 0, \qquad T_* \leq t \leq T.$$

If $y \in Y$, then, using the inequality $(A+B)^{1/\alpha} \leq (2A)^{1/\alpha} + (2B)^{1/\alpha}$, $A \geq 0$, $B \geq 0$, (2.30) and (2.9), we have for $t \geq T$

$$\begin{aligned}
&\frac{1}{2^{1+1/\alpha}}(t-T)^2 \leq \mathscr{J}y(t) \\
&\quad \leq \int_T^t (t-s)\left\{1 + \left[2\int_T^\infty q(\sigma)(g(\sigma))^{(2+1/\alpha)\beta} d\sigma\right]^{1/\alpha} (s-T)^{1/\alpha}\right\} ds \\
&\quad \leq \int_T^t (t-s)ds + \frac{(\alpha+1)(2\alpha+1)}{2\alpha^2}\int_T^t (t-s)(s-T)^{1/\alpha} ds \\
&\quad = \frac{1}{2}(t-T)^2 + \frac{1}{2}(t-T)^{2+1/\alpha} \leq t^{2+1/\alpha},
\end{aligned}$$

which implies that $\mathscr{J}$ sends Y into itself. Since it is easy to verify that all the other conditions of the Schauder-Tychonoff fixed point theorem are fulfilled, there exists an element $y \in Y$ such that $y = \mathscr{J}y$, which satisfies the integral equation

$$y(t) = \int_T^t (t-s)\left[\frac{1}{2} + \int_T^s \int_r^\infty q(\sigma)(y(g(\sigma)))^\beta d\sigma dr\right]^{1/\alpha} ds, \qquad t \geq T. \tag{2.33}$$

Differentiating (2.33) four times, we see that $y(t)$ is a solution of (A) on $[T, \infty)$. It is clear that $\lim_{t\to\infty}((y''(t))^{\alpha*})' = 0$. That $y(t)$ satisfies $\lim_{t\to\infty} y''(t) = \infty$ follows from the calculation below:

$$\begin{aligned} y''(t) &= \left[\frac{1}{2} + \int_T^t \int_s^\infty q(r)(y(g(r)))^\beta drds\right]^{1/\alpha} \\ &\geq \left[\frac{1}{2} + \frac{1}{2^{(1+1/\alpha)\beta}} \int_T^t \int_s^\infty q(r)(g(r)-T)_+^{2\beta} drds\right]^{1/\alpha} \\ &\geq \left[\frac{1}{2} + \frac{1}{2^{(1+1/\alpha)\beta}} \int_T^t (s-T)q(s)(g(s)-T)_+^{2\beta} ds\right]^{1/\alpha}, \qquad t \geq T. \end{aligned}$$

This completes the proof.

Theorem 2.6. *The equation* (A) *has a positive solution of type* II_2 *if*

$$\int_0^\infty \left[\int_t^\infty (s-t)(g(s))^\beta q(s) ds\right]^{1/\alpha} dt < \infty \tag{2.34}$$

and

$$\int_0^\infty t\left[\int_t^\infty (s-t)q(s)ds\right]^{1/\alpha} dt = \infty. \tag{2.35}$$

Proof. Let $k > 0$ be any fixed constant and choose $T > 0$ so large that $T_* = \min\left\{T, \inf_{t\geq T} g(t)\right\} \geq 1$ and

$$\int_T^\infty \left[\int_t^\infty (s-t)(g(s))^\beta q(s)ds\right]^{1/\alpha} dt \leq 2^{-\beta/\alpha} k^{1-\beta/\alpha}. \tag{2.36}$$

Consider the set $Y \subset C[T_*, \infty)$ and the mapping $\mathscr{K} : Y \to C[T_*, \infty)$ defined by

$$Y = \{y \in C[T_*, \infty) : k \leq y(t) \leq 2kt, t \geq T_*\}, \tag{2.37}$$

$$\mathscr{K}y(t) = k + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)q(\sigma)(y(g(\sigma)))^\beta d\sigma\right]^{1/\alpha} drds, \qquad t \geq T, \tag{2.38}$$

$$\mathscr{K}y(t) = k, \qquad T_* \leq t \leq T.$$

Then, the Schauder-Tychonoff theorem can be applied to the existence of a fixed element $y \in Y$ of $\mathscr{K}$. This $y = y(t)$ gives a solution of (A) on $[T, \infty)$, since

$$y(t) = k + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r) q(\sigma)(y(g(\sigma)))^\beta d\sigma\right]^{1/\alpha} dr ds, \qquad t \geq T. \tag{2.39}$$

That $\lim_{t\to\infty} y(t) = \infty$ is a consequence of the following observation:

$$\begin{aligned} y(t) &\geq k + \int_T^t (s - T)\left[\int_s^\infty (r - s) q(r)(y(g(r)))^\beta dr\right]^{1/\alpha} ds \\ &\geq k + k^{\beta/\alpha} \int_T^t (s - T)\left[\int_s^\infty (r - s) q(r) dr\right]^{1/\alpha} ds, \qquad t \geq T. \end{aligned}$$

Therefore, $y(t)$ is a positive solution of type II_2. This completes the proof.

3. Oscillation theorems

A) Our aim in this section is to establish criteria (preferably sharp) for all solutions of the equation (A) to be oscillatory. We are essentially based on some of the oscillation results of Wu [6], which are collected as Theorem W below, for the associated ordinary differential equation (B).

Theorem W. (i) *Let $\alpha \geq 1 > \beta$. All solutions of* (B) *are oscillatory if and only if*

$$\int_0^\infty t^{(2+1/\alpha)\beta} q(t) dt = \infty. \tag{3.1}$$

(ii) *Let $\alpha \leq 1 < \beta$. All solutions of* (B) *are oscillatory if and only if*

$$\int_0^\infty t q(t) dt = \infty \tag{3.2}$$

or

$$\int_0^\infty t q(t) dt < \infty \quad \textit{and} \quad \int_0^\infty s\left[\int_s^\infty (r - s) q(r) dr\right]^{1/\alpha} ds = \infty. \tag{3.3}$$

B) *Comparison theorems.* Our idea is to deduce oscillation criteria for (A) from Theorem W by means of the following two lemmas (comparison theorems) which relate the oscillation (and nonoscillation) of the equation

$$(|u''(t)|^\alpha \operatorname{sgn} u''(t))'' + F(t, u(h(t))) = 0 \tag{3.4}$$

to that of the equations

$$(|v''(t)|^{\alpha} \operatorname{sgn} v''(t))'' + G(t, v(k(t))) = 0 \tag{3.5}$$

and

$$(|w''(t)|^{\alpha} \operatorname{sgn} w''(t))'' + \frac{l'(t)}{h'(h^{-1}(l(t)))} F(h^{-1}(l(t)), w(l(t))) = 0. \tag{3.6}$$

With regard to (3.4)–(3.6) it is assumed that $\alpha > 0$ is a constant, that h, k, l are continuously differentiable functions on $[0, \infty)$ such that

$$h'(t) > 0, \quad k'(t) > 0, \quad l'(t) > 0, \qquad \lim_{t\to\infty} h(t) = \lim_{t\to\infty} k(t) = \lim_{t\to\infty} l(t) = \infty,$$

and that F, G are continuous functions on $[0, \infty) \times \mathbf{R}$ such that $uF(t, u) \geq 0$, $uG(t, u) \geq 0$ and $F(t, u), G(t, u)$ are nondecreasing in u for any fixed $t \geq 0$. Naturally, h^{-1} denotes the inverse function of h.

Lemma 3.1. *Suppose that*

$$h(t) \geq k(t), \qquad t \geq 0 \tag{3.7}$$

$$F(t, x) \operatorname{sgn} x \geq G(t, x) \operatorname{sgn} x, \qquad (t, x) \in [0, \infty) \times \mathbf{R}. \tag{3.8}$$

If all the solutions of (3.5) *are oscillatory, then so are all the solutions of* (3.4).

Lemma 3.2. *Suppose that* $l(t) \geq h(t)$ *for* $t \geq 0$. *If all the solutions of* (3.6) *are oscillatory, then so are all the solutions of* (3.4).

These lemmas can be regarded as generalizations of the main comparison principles developed in the papers [2, 4] to differential equations involving higher order nonlinear differential operators. To prove these lemmas we need a result which describes the equivalence of nonoscillation situation between (3.4) and the differential inequality

$$(|z''(t)|^{\alpha} \operatorname{sgn} z''(t))'' + F(t, z(h(t))) \leq 0. \tag{3.9}$$

Lemma 3.3. *If there exists an eventually positive function satisfying* (3.9), *then* (3.4) *has an eventually positive solution.*

Proof of Lemma 3.3. Let $z(t)$ be an eventually positive solution of (3.9). It is easy to see that $z(t)$ satisfies either

I: $z'(t) > 0,\ z''(t) > 0,\ ((z''(t))^{\alpha*})' > 0,\ t \geq T,$

or

II: $z'(t) > 0,\ z''(t) < 0,\ ((z''(t))^{\alpha*})' > 0,\ t \geq T,$

provided $T > 0$ is sufficiently large.

If $z(t)$ satisfies I, integrating (3.9) from t to ∞, we have

$$((z''(t))^{\alpha*})' \geq \omega + \int_t^{\infty} F(s, z(h(s)))ds, \qquad t \geq T, \tag{3.10}$$

where $\omega = \lim_{t\to\infty}((z''(t))^{\alpha*})' \geq 0$. Further integrations of (3.10) three times from T to t yield the inequality

(3.11)
$$z(t) \geq z(T) + \int_T^t (t-s)\left[\int_T^s \left(\omega + \int_r^\infty F(\sigma, z(h(\sigma)))d\sigma\right)dr\right]^{1/\alpha} ds, \qquad t \geq T.$$

Let $T_* = \min\left\{T, \inf_{t\geq T} h(t)\right\}$. Put

$$U = \{u \in C[T_*, \infty) : 0 \leq u(t) \leq z(t), t \geq T_*\} \tag{3.12}$$

and define

$$\Phi u(t) = z(T) + \int_T^t (t-s)\left[\int_T^s \left(\omega + \int_r^\infty F(\sigma, u(h(\sigma)))d\sigma\right)dr\right]^{1/\alpha} ds, \quad t \geq T, \tag{3.13}$$
$$\Phi u(t) = z(t), \qquad T_* \leq t \leq T.$$

Then, it is easily verified that Φ maps continuously U into a relatively compact set of U, and so there exists a function $u \in U$ such that $u = \Phi u$, which implies that

(3.14)
$$u(t) = z(T) + \int_T^t (t-s)\left[\int_T^s \left(\omega + \int_r^\infty F(\sigma, u(h(\sigma)))d\sigma\right)dr\right]^{1/\alpha} ds, \qquad t \geq T.$$

This shows that $u(t)$ is a positive solution of the equation (3.4).

If $z(t)$ satisfies II, then (3.10) holds with $\omega = 0$, and integrating (3.10) from t to ∞, we find

$$-z''(t) \geq \left[\int_t^\infty (s-t)F(s, z(h(s)))ds\right]^{1/\alpha}, \qquad t \geq T, \tag{3.15}$$

from which, integrating twice, first from t to ∞ and then from T to t, we obtain

$$z(t) \geq z(T) + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)F(\sigma, z(h(\sigma)))d\sigma\right]^{1/\alpha} drds, \qquad t \geq T. \tag{3.16}$$

Let $T_* = \min\left\{T, \inf_{t\geq T} h(t)\right\}$ and let U and Ψ be defined, respectively, by (3.12) and

$$\Psi u(t) = z(T) + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r)F(\sigma, u(h(\sigma)))d\sigma\right]^{1/\alpha} drds, \qquad t \geq T, \tag{3.17}$$
$$\Psi u(t) = z(t), \qquad T_* \leq t \leq T.$$

The Schauder-Tychonoff fixed point theorem also applies to this case, and there exists a function $u \in U$ such that $u = \Psi u$, that is,

$$u(t) = z(T) + \int_T^t \int_s^\infty \left[\int_r^\infty (\sigma - r) F(\sigma, u(h(\sigma))) d\sigma\right]^{1/\alpha} dr ds, \qquad t \geq T. \tag{3.18}$$

It follows that $u(t)$ is a positive solution of (3.4). This completes the proof of Lemma 3.3.

Proof of Lemma 3.1. It suffices to prove that if (3.4) has an eventually positive solution, then so does (3.5).

Let $u(t)$ be an eventually positive solution of (3.4). Note that $u(t)$ is monotone increasing for all sufficiently large t. In view of (3.7) and (3.8), we see that there exists $T > 0$ such that $u(h(t)) \geq u(k(t))$, $t \geq T$, and

$$F(t, u(h(t))) \geq G(t, u(k(t))), \qquad t \geq T.$$

This together yields

$$(|u''(t)|^\alpha \operatorname{sgn} u''(t))'' + G(t, u(k(t))) \leq 0, \qquad t \geq T,$$

and application of Lemma 3.3 then shows that the equation (3.5) has an eventually positive solution $v(t)$. This completes the proof.

Proof of Lemma 3.2. The conclusion of the lemma is equivalent to the statement that if there exists an eventually positive solution of (3.4) then the same is true of (3.6).

Let $u(t)$ be an eventually positive solution of (3.4). The following two cases are possible:

I: $u'(t) > 0$, $u''(t) > 0$, $((u''(t))^{\alpha *})' > 0$ for all large t;

II: $u'(t) > 0$, $u''(t) < 0$, $((u''(t))^{\alpha *})' > 0$ for all large t.

Suppose that I holds. Then we have

(3.19)

$$u(t) \geq u(T) + \int_T^t (t - s) \left[\int_T^s \left(\omega + \int_r^\infty F(\sigma, u(h(\sigma))) d\sigma\right) dr\right]^{1/\alpha} ds, \qquad t \geq T,$$

where $\omega = \lim_{t\to\infty} ((u''(t))^{\alpha *})' \geq 0$. Combining (3.19) with the inequality

$$\begin{aligned}\int_r^\infty F(\sigma, u(h(\sigma))) d\sigma &= \int_{l^{-1}(h(r))}^\infty F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho \\ &\geq \int_r^\infty F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho,\end{aligned}$$

we get

$$u(t) \geq u(T) + \int_T^t (t-s) \left[\int_T^s \left(\omega + \int_r^\infty F(h^{-1}(l(\rho)), u(l(\rho))) \cdot \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho \right) dr \right]^{1/\alpha} ds, \qquad t \geq T. \tag{3.20}$$

We now observe that an essential part of the proof of Lemma 3.3 is the generation of a solution of the integral equation (3.14) [or (3.18)] on the basis of the existence of a function satisfying the corresponding integral inequality (3.11) [or (3.16)]. Here proceeding in a similar fashion, from the fact that $u(t)$ satisfies (3.20) we conclude that there exists a positive solution of the equation

$$w(t) = u(T) + \int_T^t (t-s) \left[\int_T^s \left(\omega + \int_r^\infty F(h^{-1}(l(\rho)), w(l(\rho))) \cdot \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho \right) dr \right]^{1/\alpha} ds, \qquad t \geq T. \tag{3.21}$$

It can be checked by differentiation that $w(t)$ provides a positive solution of the differential equation (3.6).

Suppose next that II holds. Then, $u(t)$ is shown to satisfy the inequality

$$u(t) \geq u(T) + \int_T^t \int_s^\infty \left[\int_r^\infty \int_\sigma^\infty F(\rho, u(h(\rho))) d\rho d\sigma \right]^{1/\alpha} dr ds, \qquad t \geq T. \tag{3.22}$$

Repeating the same argument as above with (3.19) replaced by (3.22), we are led to the conclusion that there exists a positive solution $w(t)$ of the integral equation

$$w(t) = u(T) + \int_T^t \int_s^\infty \left[\int_r^\infty \int_\sigma^\infty F(h^{-1}(l(\rho)), w(l(\rho))) \cdot \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho d\sigma \right]^{1/\alpha} dr ds, \qquad t \geq T, \tag{3.23}$$

which clearly gives a positive solution of the differential equation (3.6). This completes the proof of Lemma 3.2.

C) *Oscillation criteria.* We first give a sufficient condition for all solutions of (A) in the sub-half-linear case to be oscillatory.

THEOREM 3.1. *Let $\alpha \geq 1 > \beta$. Suppose that there exists a continuously differentiable function $h : [0, \infty) \to (0, \infty)$ such that $h'(t) > 0$, $\lim_{t\to\infty} h(t) = \infty$, and*

$$\min\{t, g(t)\} \geq h(t) \qquad \textit{for all large } t. \tag{3.24}$$

If

$$\int_0^\infty (h(t))^{(2+1/\alpha)\beta} q(t)dt = \infty, \tag{3.25}$$

then all solutions of (A) *are oscillatory.*

Proof. Let us consider the equations

$$(|z''(t)|^\alpha \operatorname{sgn} z''(t))'' + q(t)|z(h(t))|^\beta \operatorname{sgn} z(h(t)) = 0, \tag{3.26}$$

$$(|w''(t)|^\alpha \operatorname{sgn} w''(t))'' + \frac{q(h^{-1}(t))}{h'(h^{-1}(t))}|w(t)|^\beta \operatorname{sgn} w(t) = 0. \tag{3.27}$$

Since

$$\int^\infty t^{(2+1/\alpha)\beta} \frac{q(h^{-1}(t))}{h'(h^{-1}(t))}\, dt = \int^\infty (h(\tau))^{(2+1/\alpha)\beta} q(\tau)d\tau = \infty$$

by (3.25), Theorem W-(i) implies that all solutions of (3.27) are oscillatory. Application of Lemma 3.2 then shows that all solutions of (3.26) are oscillatory, and the conclusion of the theorem follows from comparison of (A) with (3.26) by means of Lemma 3.1.

It will be shown below that there is a class of sub-half-linear equations of the type (A) for which the oscillation situation can be completely characterized.

Theorem 3.2. *Let* $\alpha \geq 1 > \beta$ *and suppose that*

$$\limsup_{t\to\infty} \frac{g(t)}{t} < \infty. \tag{3.28}$$

Then, all solutions of (A) *are oscillatory if and only if*

$$\int_0^\infty (g(t))^{(2+1/\alpha)\beta} q(t)dt = \infty. \tag{3.29}$$

Proof. That the oscillation of (A) implies (3.29) is an immediate consequence of Theorem 2.1.

Assume now that (3.29) is satisfied. The condition (3.28) means that there exists a constant $c > 1$ such that

$$g(t) \leq ct \qquad \text{for all sufficiently large } t.$$

Consider the ordinary differential equation

$$(|z''(t)|^\alpha \operatorname{sgn} z''(t))'' + \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))}|z(t)|^\beta \operatorname{sgn} z(t) = 0. \tag{3.30}$$

Since by (3.29)

$$\int^{\infty} t^{(2+1/\alpha)\beta} \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))}\, dt = \int^{\infty} \left(\frac{g(t)}{c}\right)^{(2+1/\alpha)\beta} q(t)dt = \infty,$$

all solutions of (3.30) are oscillatory according to Wu's theorem: Theorem W-(i). From Lemma 3.1 it follows that the equation

$$(|u''(t)|^{\alpha} \operatorname{sgn} u''(t))'' + \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))} |u(ct)|^{\beta} \operatorname{sgn} u(ct) = 0 \tag{3.31}$$

has all of its solutions oscillatory. Comparison of (A) with (3.31) via Lemma 3.2 then leads to the desired conclusion of the theorem. This completes the proof.

An oscillation criterion for the equation (A) in the super-half-linear case is given in the following theorem.

Theorem 3.3. *Let $\alpha \le 1 < \beta$ and suppose that*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0. \tag{3.32}$$

Then, all solutions of (A) *are oscillatory if and only if either* (3.2) *or* (3.3) *holds.*

Proof. We need only to prove the "if" part of the theorem, since the "only if" part follows immediately from Theorem 2.4.

In view of (3.32) there exists a positive constant $c < 1$ such that

$$g(t) \ge ct \qquad \text{for all sufficiently large } t. \tag{3.33}$$

Consider the ordinary differential equation

$$(|z''(t)|^{\alpha} \operatorname{sgn} z''(t))'' + \frac{1}{c} q\left(\frac{t}{c}\right) |z(t)|^{\beta} \operatorname{sgn} z(t) = 0. \tag{3.34}$$

Using the assumptions on $q(t)$, we see that either

$$\int_0^{\infty} \frac{t}{c} q\left(\frac{t}{c}\right) dt = c \int_0^{\infty} tq(t)dt = \infty$$

or

$$\int_0^{\infty} t\left[\int_t^{\infty} (s-t)\frac{1}{c} q\left(\frac{s}{c}\right) ds\right]^{1/\alpha} dt = c^{1+1/\alpha} \int_0^{\infty} \frac{t}{c} \left[\int_{t/c}^{\infty} \left(s - \frac{t}{c}\right) q(s) ds\right]^{1/\alpha} dt = \infty,$$

which implies that all the solutions of (3.34) are oscillatory. We now apply one of the comparison principles, Lemma 3.2, to compare (3.34) with the equation

$$(3.35)\qquad (|u''(t)|^{\alpha} \operatorname{sgn} u''(t))'' + q(t)|u(ct)|^{\beta} \operatorname{sgn} u(ct) = 0,$$

and conclude that (3.35) has the same oscillatory behavior as (3.34). Since (3.33) holds, applying another comparison principle, Lemma 3.1, we conclude that all the solutions of (A) are necessarily oscillatory. This completes the proof.

From the proof of Theorems 3.2 and 3.3 we see that in case $\alpha \geq 1 > \beta$ or $\alpha \leq 1 < \beta$, the oscillation of the functional differential equation

$$(|y''(t)|^{\alpha} \operatorname{sgn} y''(t))'' + q(t)|y(ct)|^{\beta} \operatorname{sgn} y(ct) = 0$$

is equivalent to that of the ordinary differential equation (B). This observation combined with our comparison principles (Lemmas 3.2 and 3.3) will lead to the following result.

Corollary. *Let either $\alpha \geq 1 > \beta$ or $\alpha \leq 1 < \beta$ and suppose that $g(t)$ in* (A) *satisfies*

$$0 < \liminf_{t\to\infty} \frac{g(t)}{t}, \qquad \limsup_{t\to\infty} \frac{g(t)}{t} < \infty.$$

Then all solutions of the equation (A) *are oscillatory if and only if the same is true for the equation* (B).

Example. We present here an example which illustrates oscillation and nonoscillation theorems proven in Sections 1 and 2.

Consider the equation

$$(3.36)\qquad (|y''(t)|^{\alpha} \operatorname{sgn} y''(t))'' + t^{-\lambda}|y(t^{\gamma})|^{\beta} \operatorname{sgn} y(t^{\gamma}) = 0,$$

where α, β, γ are fixed positive constants and λ is a varying parameter.

It is easy to check that, written for (3.36),

(2.7) is equivalent to $\lambda > 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma$;

(2.13) is equivalent to $\lambda > 2 + 2\beta\gamma$;

(2.18) is equivalent to $\lambda > 2 + \alpha + \beta\gamma$;

(2.23) is equivalent to $\lambda > 2 + 2\alpha$,

so that from Theorems 2.1–2.4 we see that

(3.36) has a type-I_1 solution if and only if $\lambda > 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma$;

(3.36) has a type-I_3 solution if and only if $\lambda > 2 + 2\beta\gamma$;

(3.36) has a type-II_1 solution if and only if $\lambda > 2 + \alpha + \beta\gamma$;

(3.36) has a type-II_3 solution if and only if $\lambda > 2 + 2\alpha$.

It follows that (3.36) has solutions of all types $\mathrm{I}_1, \mathrm{I}_3, \mathrm{II}_1$ and II_3 if either

$$\alpha \le \beta\gamma \qquad \text{and} \qquad \lambda > 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma$$

or

$$\alpha > \beta\gamma \qquad \text{and} \qquad \lambda > 2 + 2\alpha.$$

It is easy to see that for (3.36) the conditions $\{(2.28), (2.29)\}$ and $\{(2.34), (2.35)\}$ guaranteeing the existence of solutions of "intermediate" types I_2 and II_2 may be realized only when $\alpha > \beta\gamma$. The conclusions which follow from Theorems 2.5 and 2.6 are:

(i) (3.36) has a type-I_2 solution if

$$\alpha > \beta\gamma \qquad \text{and} \qquad 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma < \lambda \le 2 + 2\beta\gamma. \tag{3.37}$$

(ii) (3.36) has a type-II_2 solution if

$$\alpha > \beta\gamma \qquad \text{and} \qquad 2 + \alpha + \beta\gamma < \lambda \le 2 + 2\alpha. \tag{3.38}$$

We note that if (3.37) holds, then (3.36) has no solutions of types $\mathrm{I}_3, \mathrm{II}_1$ and II_3, and that if (3.38) holds, then (3.36) has no solution of type II_3.

We now want oscillation criteria for (3.36).

Suppose that $\alpha \ge 1 > \beta$. If $\gamma \le 1$, then from Theorem 3.2 we conclude that all solutions of (3.36) are oscillatory if and only if

$$\lambda \le 1 + \left(2 + \frac{1}{\alpha}\right)\beta\gamma.$$

If $\gamma > 1$, then, applying Theorem 3.1, we see that all solutions of (3.36) are oscillatory if

$$\lambda \le 1 + \left(2 + \frac{1}{\alpha}\right)\beta.$$

Suppose that $\alpha \le 1 < \beta$. Then, Theorem 3.3 applies to (3.36) with $\gamma \ge 1$ and leads to the conclusion that all of its solutions are oscillatory if and only if

$$\lambda \le 2 + 2\alpha.$$

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