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Integrality of varifolds in the singular limit of reaction-diffusion equations

Yoshihiro Tonegawa

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Abstract. We answer a question posed by Ilmanen on the integrality of varifolds which appear as the singular perturbation limit of the Allen-Cahn equation. We show that the density of the limit measure is integer multiple of the surface constant almost everywhere at almost all time. This shows that limit measures obtained via the Allen-Chan equation and those via Brakke's construction share the same integrality property as well as being weak solutions for the mean curvature flow equation.

1. Introduction

The Allen-Cahn equation was proposed to describe the macroscopic motion of phase boundaries driven by surface tension [2]. It is

$$\varepsilon \frac{\partial u^\varepsilon}{\partial t} = \varepsilon \Delta u^\varepsilon - \varepsilon^{-1} W'(u^\varepsilon), \tag{1.1}$$

where W is a bi-stable potential with two wells of equal depth at ± 1 and the real-valued function u indicates the phase state at each point. Several authors studied the equation to the conclusion that the zero level set of u^ε approaches a hypersurface with its normal velocity determined by the mean curvature as $\varepsilon \to 0$. The phase boundaries should have the thickness of order ε.

The formal derivation was given by Fife [14], Rubinstein, Sternberg and Keller [20], and others. The rigorous proof for radially symmetric case was given by Bronsard and Kohn [4]. With the assumption that the classical solution for the mean curvature flow exists, the general case was proved by de Mottoni and Schatzman [10], Chen [6], Chen and Elliott [8] and others. Evans, Soner and Souganidis [11] showed that the limit of the level set of the Allen-Cahn equation is contained in the viscosity solution for the mean curvature flow studied by Evans and Spruck [13] and Chen, Giga and Goto [9]. Ilmanen [17] showed with a technique from geometric measure theory that the limit is a mean curvature flow in the sense of Brakke [3]. Subsequently,

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Soner [22] gave proofs that more general initial data may be admitted in Ilmanen's work. There are numerous articles related to the general subject of various Allen-Cahn type equations with modifications and those coupled with other field variables such as temperature. We cite only the most relevant articles and refer the reader to, for example, Soner's paper [22] for more complete references.

The purpose of this paper is to answer one technical question posed by Ilmanen [17, Section 13.2]. We show that the $(n-1)$-dimensional density of the limit measure μ_t of the Allen-Cahn equation is an integer multiple of the surface constant $\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\, ds$ for a.e. $t > 0$ and $\mathscr{H}^{n-1}$ a.e. x. Here, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$. The more heuristic interpretation is that there is no fractional interface appearing as $\varepsilon \to 0$, and the interface profiles for a.e. points are close to integer multiples of the 1-D standing wave profile at worst. Higher multiplicities can occur in fact, indicated in the existence results by Bronsard and Stoth [5]. Note that weak varifold solutions for the mean curvature flow constructed by Brakke [3] have such integrality property for a.e. $t > 0$. Accordingly, we conclude that the solutions obtained as the limit of the Allen-Cahn equation have all the measure-theoretic properties of Brakke's solutions. As the byproducts, all of the results on the weak varifold mean curvature flow due to Brakke hold for the limit of the Allen-Cahn equation, such as his clearing-out lemma, perpendicularity of the mean curvature, etc.

Another interest of this paper is our remark that the results due to Ilmanen, where the domain was $\mathbf{R}^n$, may be localized to a bounded domain. This is due to a local estimate of the so-called discrepancy measure, which in turn yields the local monotonicity formula for the properly scaled energy identity.

The proof of the stated results is accomplished through appropriate parabolic modification of the corresponding elliptic results due to Hutchinson and the author [16, Section 5]. There, we showed that finite energy equilibrium converges to a varifold with a locally constant mean curvature and an integer density modulo division by σ.

In Section 2, we state our assumptions and main results. In Section 3, we discuss the derivation of the local monotonicity formula, and in the last Section 4, show the integrality of the limit measure. Even though many parts of the proof in Section 4 are similar to those in [16, Section 5], we present the detail for the reader's convenience.

2. Assumptions and main results

2.1. Assumptions. Throughout this paper, we assume

A: The function $W : \mathbf{R} \to [0, \infty)$ is C^3 and $W(\pm 1) = W'(\pm 1) = 0$. For some $\gamma \in (-1, 1)$, $W' < 0$ on $(\gamma, 1)$ and $W' > 0$ on $(-1, \gamma)$. For some $\alpha \in (0, 1)$ and $\kappa > 0$, $W''(x) \geq \kappa$ for all $|x| \geq \alpha$.

B: $U \subset \mathbf{R}^n$ is a bounded open set with Lipschitz boundary ∂U and $0 < T \leq \infty$. A sequence of functions $\{u^i\}_{i=1}^{\infty}$, with $u^i_{tx_j}, u^i_{x_j x_k x_l} \in C(U \times (0, T))$, $1 \leq j, k, l \leq n$, satisfies

$$\varepsilon_i u^i_t = \varepsilon_i \Delta u^i - \varepsilon_i^{-1} W'(u^i) \tag{2.1}$$

on $U \times (0, T)$. Here, $\lim_{i\to\infty} \varepsilon_i = 0$, and we assume there exist c_0 and E_0 such that $\sup_{U\times(0,T)} |u^i| \leq c_0$ and

$$\int_{U\times\{t\}} \frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \leq E_0 \tag{2.2}$$

for all $t \in (0, T)$ and i. Moreover,

$$\int_{U\times(0,T)} \varepsilon_i |u^i_t|^2 \leq E_0 \tag{2.3}$$

for all i.

Assumption **B** is satisfied, for example, when we consider the following initial value problem

$$\begin{cases} \varepsilon u_t = \varepsilon \Delta u - \varepsilon^{-1} W'(u) & \text{on } U \times (0, \infty), \\ u(x, 0) = \phi_\varepsilon(x) & \text{on } U, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial U \times (0, \infty), \end{cases}$$

where the initial data have the sup norm and energy bounded uniformly with respect to ε as $\varepsilon \to 0$. Since the equation is a gradient flow of the energy in (2.2), assumptions (2.2) and (2.3) are satisfied with E_0 being the bound of the energy for the initial data. The sup norm bound of u follows from the standard maximum principle. The boundary Neumann condition may be also replaced by Dirichlet data $u = \phi_\varepsilon$ on $\partial U \times [0, \infty)$, where we also obtain (2.2) and (2.3). Our results are local in nature, so we take above assumptions as our starting point in this paper.

With this setting, for $t \in [0, T)$, define the Radon measures by

$$\mu^i_t(\phi) = \int_U \phi(x) \varepsilon_i \frac{|\nabla u^i(x, t)|^2}{2} \, dx \tag{2.4}$$

for $\phi \in C_c(U)$.

We also recall the notion of rectifiability for Radon measure.

Definition ([17, 1.7]). We call a Radon measure μ $(n-1)$-rectifiable if either of the following equivalent conditions is met:

(a) $\mu = \mathscr{H}^{n-1}\lfloor X \lfloor \theta$, where X is an $(n-1)$-rectifiable $\mathscr{H}^{n-1}$-measurable set and $\theta \in L^1_{loc}(\mathscr{H}^{n-1}\lfloor X, (0,\infty))$. Here, we denote the restriction of measure to X by $\lfloor X$. In particular, $\mu(A) = \int_{A\cap X} \theta \, d\mathscr{H}^{n-1}$ for Borel set A.

(b) The measure-theoretic approximate tangent plane $T_x\mu$ exists μ-a.e. (see also [1, 3, 21]).

In [17], Ilmanen proved, among other things (with $U = \mathbf{R}^n$),

THEOREM 2.1 ([17]). *There is a subsequence of $\{\varepsilon_i\}$ and Radon measures μ_t on U for all $t \in [0,\infty)$ such that*

(i) *$\mu^i_t \to \mu_t$ for all $t > 0$ as Radon measures on U.*

(ii) *For a.e. $t > 0$, μ_t is $(n-1)$-rectifiable.*

(iii) *μ_t satisfies the mean curvature flow equation in the sense of Brakke, namely, for any $\phi \in C^2_c(U)$, $\phi \geq 0$,*

$$\overline{D}_t \int \phi \, d\mu_t \leq \int -\phi|H|^2 + \nabla\phi \cdot (T_x\mu_t)^{\perp} \cdot H \, d\mu_t \tag{2.5}$$

for each $t \in [0,\infty)$. Here, $\overline{D}_t$ is the upper derivative, and H is the generalized mean curvature vector of μ_t. The right-hand side is understood to be $-\infty$ whenever μ_t is not $(n-1)$-rectifiable, the first variation of μ_t is not absolutely continuous with respect to μ_t, or $|H|^2$ is not μ_t integrable. $T_x\mu$ denotes the weak tangent space (and the corresponding projection) of μ_t, and $(T_x\mu)^{\perp}$ denotes the normal subspace of $T_x\mu$ (and the corresponding projection).

Note that the first variation is defined usually for varifolds ([1, 3, 21]), while it is understood here that one may define the unique varifold from a given rectifiable Radon measure and the first variation is defined through this identification. In these regards, we follow Ilmanen's notations in [17].

Define the $(n-1)$-dimensional density $\theta(x)$ by

$$\theta(x) = \lim_{r\to 0} \frac{1}{\omega_{n-1} r^{n-1}} \mu_t(B_r(x))$$

whenever the limit exists. Here, ω_{n-1} is the volume of the unit ball in $\mathbf{R}^{n-1}$. What we prove is the following:

THEOREM 2.2. *For a.e. $t > 0$ and μ_t a.e. $x \in U$, $\theta(x) = N\sigma$ for some positive $N \in \mathbf{N}$, where $\sigma = \int_{-1}^{1} \sqrt{W(s)/2} \, ds$.*

Thus, for a.e. $t > 0$, $\mu_t = \mathscr{H}^{n-1}\lfloor X_t \lfloor \sigma N(x,t)$, where X_t is an $(n-1)$-rectifiable set and $N(x,t)$ is integer-valued $\mathscr{H}^{n-1}$-measurable function.

Due to the perpendicularity of the mean curvature vector for integral varifolds [3], we conclude that $H(x,t) \perp T_x\mu_t$ holds for a.e. $t > 0$ and μ_t a.e. $x \in U$. Hence, we show that the mean curvature equation (2.5) is satisfied in the following form as well:

$$\bar{D}_t \int \phi \, d\mu_t \le \int -\phi |H|^2 + \nabla\phi \cdot H \, d\mu_t. \tag{2.5$'$}$$

We note that if $N(x,t) = 1$ for a.e. $t > 0$ and $\mathscr{H}^{n-1}$-a.e. on X_t, then Brakke's partial regularity results apply to the measure μ_t and one may obtain the smoothness of the flow for a.e. sense.

3. Local monotonicity formula

In this section we assume that the function $u : U \to \mathbf{R}$ satisfies assumption **B** with u^i and ε_i there replaced by u and ε respectively. We assume $\tilde{U} \subset\subset U$ and $0 < \tilde{t} < T$.

Here, we show the local monotonicity formula in Proposition 3.3, which is the local version of [17, Section 4.1]. The key point for the extension is the local upper bound of the discrepancy function for all sufficiently small ε (Lemma 3.2).

For any $(y,s) \in \tilde{U} \times (\tilde{t}, T)$ and $(x,t) \in U \times (0,T)$ with $t < s$, denote

$$\rho = \rho_{y,s}(x,t) = \frac{1}{(4\pi(s-t))^{(n-1)/2}} e^{-|x-y|^2/4(s-t)}.$$

For $\phi \in C_c^2(U, \mathbf{R}^+)$, the computation (see [17, Section 3.2]) shows

Lemma 3.1.

$$\begin{aligned} \frac{d}{dt}\int \phi \, d\mu_t^\varepsilon = \int & -\varepsilon\phi\left(-\Delta u + \frac{W'(u)}{\varepsilon^2} - \frac{\nu\cdot\nabla\phi}{\phi}\right)^2 dx \\ & + \left(\frac{(\nu\cdot\nabla\phi)^2}{\phi} + \phi_{x_i x_i} - \nu_i\nu_j\phi_{x_i x_j} + \phi_t\right) d\mu_t^\varepsilon \\ & + \left(-\nu_i\nu_j\phi_{x_i x_j} + \frac{(\nu\cdot\nabla\phi)^2}{\phi}\right) d\xi_t^\varepsilon. \end{aligned} \tag{3.1}$$

Here, $\nu = \frac{\nabla u}{|\nabla u|}$ (where it is understood that $\nu = 0$ on $|\nabla u| = 0$) and $d\xi_t^\varepsilon = \left(\frac{\varepsilon|\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}\right) dx$. The summantion of the indices is also customary. To localize the monotonicity formula, we fix $\hat{U}$ with $\tilde{U} \subset\subset \hat{U} \subset\subset U$ and $\varphi \in C_c^\infty(U)$ such that $\varphi \equiv 1$ on $\hat{U}$. Insert $\phi = \varphi\rho$ in (3.1). Direct calculations show that (see [17])

$$\frac{(\nabla\rho\cdot\nu)^2}{\rho} + \rho_{x_i x_i} - \nu_i\nu_j\rho_{x_i x_j} + \rho_t \equiv 0,$$

$$-\nu_i\nu_j\rho_{x_i x_j} + \frac{(\nu\cdot\nabla\rho)^2}{\rho} = \frac{\rho}{2(s-t)}.$$

Thus, dropping the first term, (3.1) with this choice gives

$$\frac{d}{dt}\int \varphi\rho_{y,s}\,d\mu_t^\varepsilon \le c(\varphi)E_0 \sup_{x\in U\setminus\hat{U}} |\rho_{y,s}(x,t)| + \int \frac{\varphi\rho_{y,s}}{2(s-t)}\,d\xi_t^\varepsilon.$$

The first term arises from the differentiations of φ in (3.1). It is exponentially small when $s \approx t$. To control the second term, we need

LEMMA 3.2. *There exist constants c_2 and ε_2 which depend only on c_0,* $\mathrm{dist}(\tilde{U}\times(\tilde{t},T), \partial_0(U\times(0,T)))$ *and W such that*

$$\sup_{\tilde{U}\times(\tilde{t},T)}\left(\frac{\varepsilon|\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}\right) \le c_2 \tag{3.2}$$

for all $\varepsilon < \varepsilon_2$. Here, $\partial_0\cdot$ denotes the usual parabolic bounday.

The proof is a straightfoward modification of the elliptic case discussed in [16, Proposition 3.3], so we omit the proof. Then, (3.1) and (3.2) combined with $\int_{\mathbf{R}^n}\rho\,dx = (4\pi(s-t))^{1/2}$ give

$$\frac{d}{dt}\int \varphi\rho_{y,s}\,d\mu_t^\varepsilon \le c(\varphi)E_0 \sup_{x\in U\setminus\hat{U}} |\rho_{y,s}(x,t)| + c_2\pi^{1/2}/\sqrt{2(s-t)}.$$

By integrating above over t and choosing an appropriate constant, we obtain

PROPOSITION 3.3. *There exist constants c_3 and ε_3 depending only on φ, c_0, $\tilde{t}, T, E_0$ and W such that, for $0<\tilde{t}<t_1<t_2<s<T$ and $y\in\tilde{U}$,*

$$\int \varphi\rho_{y,s}\,d\mu_{t_2}^\varepsilon \le \int \varphi\rho_{y,s}\,d\mu_{t_1}^\varepsilon + c_3(\sqrt{s-t_1} - \sqrt{s-t_2}) \tag{3.3}$$

for all $\varepsilon < \varepsilon_3$.

Note that the last term may be made as small as we like by choosing $s-t_1$ small. Once we have (3.3), we may localize Ilmanen's argument in [17] which shows the rectifiability of the limit measure and the Brakke's flow equation under the assumption **A** and **B** on a bounded domain. This requires a careful re-evaluation of his proof, but we only point out that no part of Ilmanen's argument requires global properties and the estimates there go through with minor modifications coming from the small error term in (3.3). Since our main objective in this paper is the proof of the integrality, we omit the detail in this paper.

4. The proof of integrality

Here, we prove that the limit measure has the integral density property for a.e. point for a.e. time. The proof is similar to the time independent case, even though one needs to control the time derivative term. This is achieved, roughly speaking, by analyzing the measure at generic times when there is no sudden jump of mass. Also, one does not have a uniform density ratio lower bound on the support of the limit measure, which is different from the corresponding time-independent situation discussed in [16].

We describe the outline of the proof. Proposition 4.1 shows that there is only a little amount of energy on the region $\{|u| \geq 1 - b\}$, where $b > 0$ is a small fixed number, uniformly in t. This is intuitively clear, while one heavily depends on the monotonicity formula for this result. Two lemmas are used. Lemma 4.2 relates the value of u (being $1 - |u| \approx \varepsilon^{\beta}$) and the distance to the interface (being $\approx \beta\varepsilon|\ln \varepsilon|$). Lemma 4.3 shows that the r-neighborhood of the interface has volume of $O(r)$. Using these two lemmas, Proposition 4.1 is proved.

Next, while we look at generic points where blow-up limits are flat hypersurfaces, it is possible that several sheets of interfaces are piling up. Lemma 4.4 deals with separating these sheets into two groups, each having the "right" amount of energy. Though it is only analogous, the similar technical lemma has appeared in the compactness proof of integral varifolds ([1]) as well as integrality proof of Brakke's varifold mean curvature flow ([3]). Note that the term $\int (1 - (\nu_n)^2)\varepsilon|\nabla u|^2$ corresponds roughly to the "tilt-excess energy" for the corresponding sharp interface situation. It has inductive structure, so Proposition 4.5 follows from Lemma 4.4 by repeatedly separating sheets until all are separated. Proposition 4.5 deals with the ε-scale. With the control of the correct quantities, the solution is shown to be very close to the 1-D standing wave solution of the ODE in ε-scale. We use above three propositions to prove the integrality. Ilmanen has proved that the limit Radon measure is rectifiable for generic time. He also demonstrated that generic points satisfy the conditions (1)–(7) in the proof of integrality. Namely, condition (2) indicates that we look at time where there is no serious jump of mass, condition (3) is satisfied by the equi-partition of the energy and (4) follows from the monotonicity formula. A measure-theoretic argument shows that one may generically look at points where the term involving u_t in Proposition 4.5 may be controlled by the energy. This leads to the situation where Proposition 4.5 is applicable for most of the interface region, resulting a conclusion that the density has to be integer valued modulo division by σ.

Proposition 4.1. *Assume that Assumptions* **A** *and* **B** *are true with* u^i, ε_i *and* $U \times (0, T)$ *replaced by* u, ε *and* $B_3(0) \times (0, 2)$ *respectively and suppose* $s > 0$ *is given. Then there exist positive constants* b *and* ε_4 *depending only on* c_0, E_0, W *and* s *such that*

$$\int_{(B_1(0)\times\{t\})\cap\{|u|\geq 1-b\}} \frac{W(u)}{\varepsilon} \leq s$$

for all $t \in (1,2)$ *whenever* $\varepsilon \leq \varepsilon_4$.

To prove this, we need the following two lemmas.

Lemma 4.2. *Under the assumptions of Proposition* 4.1, *there exists a constant* c_4 *depending only on* κ *such that if* $(x_0, t_0) \in B_1(0) \times (1,2)$, $0 < \varepsilon < 1$ *and* $u(x_0, t_0) < 1 - \varepsilon^\beta$ *(or* $u(x_0, t_0) > -1 + \varepsilon^\beta$*), where* β *satisfies* $1 \leq \tilde{r} \equiv c_4\beta|\ln \varepsilon| \leq \varepsilon^{-1}$, *then*

$$\inf_{B_{\varepsilon\tilde{r}}(x_0)\times(t_0-\varepsilon^2\tilde{r}^2, t_0)} u < \alpha \qquad \left(resp. \sup_{B_{\varepsilon\tilde{r}}(x_0)\times(t_0-\varepsilon^2\tilde{r}^2, t_0)} u > -\alpha\right).$$

Proof. Rescale the domain by $x \mapsto \frac{x-x_0}{\varepsilon}$ and $t \mapsto \frac{t-t_0}{\varepsilon^2}$. For the comparison arugment, we need a function $\psi \geq 1$ with the following properties:

$$\begin{cases} \psi_t = \Delta\psi - \frac{\kappa}{4}\psi & \text{on } \mathbf{R}^n \times (-\infty, 0), \\ \psi(x,t) \geq e^{(|x|+|t|)/c_4} & \text{on } \mathbf{R}^n \times (-\infty, 0)\backslash B_1^{n+1}(0,0), \\ \psi(0,0) = 1 \end{cases}$$

for some $c_4 = c(\kappa)$. Such function is obtained by first defining $\tilde{\psi}$ on $\mathbf{R}^n$ as the entire radial solution of $\Delta\tilde{\psi} = \frac{\kappa}{8}\tilde{\psi}$, $\tilde{\psi}(0) = 1$, which grows exponentially as $|x| \to \infty$, and then by defining $\psi(x,t) = \tilde{\psi}(x)e^{-\kappa t/8}$. Fix such ψ and c_4. Let $\tilde{r} = c_4\beta|\ln \varepsilon|$. Note $1 - \varepsilon^\beta e^{\tilde{r}/c_4} = 0$. For a contradiction, assume that $u(0,0) < 1 - \varepsilon^\beta$ and $\inf_{B_{\tilde{r}}(0,0)\times(-\tilde{r}^2,0)} u > \alpha$. Define $\phi = 1 - \varepsilon^\beta\psi$. Then ϕ satisfies $\phi_t = \Delta\phi + \frac{\kappa}{4}(1-\phi)$, $\phi < 1 - \varepsilon^\beta e^{\tilde{r}/c_4} < \alpha < u$ on $\partial_0(B_{\tilde{r}}(0,0) \times (-\tilde{r}^2, 0))$ and $\phi(0,0) > u(0,0)$. Thus $u - \phi$ achieves a negative minimum away from the parabolic boundary. There, $(u-\phi)_t - \Delta(u-\phi) \leq 0$ and thus with the equation,

$$0 \geq -W'(u) + \frac{\kappa}{4}(\phi - 1) \geq -W'(\phi) + \frac{\kappa}{4}(\phi - 1) \geq \frac{\kappa}{2}\varepsilon^\beta\psi - \frac{\kappa}{4}\varepsilon^\beta\psi > 0,$$

which is a contradiction. This proves the desired estimate after rescaling back. The supremum estimate is similar. $\square$

For $t \in (1,2)$ and $0 < r < 1$, define

$$Z_{r,t} = \left\{x \in B_1(0) \,|\, \inf_{B_r(x)\times(t-r^2,t)} |u| < \alpha\right\}.$$

Lemma 4.3. *Under the assumptions of Proposition* 4.1, *there exist constants* c_5 *and* ε_5 *depending only on* c_0, E_0 *and* W *such that if* $\varepsilon \leq r \leq 1$, *then*

$$\mathscr{L}^n(Z_{r,t}) \leq c_5 r,$$

provided $0 < \varepsilon < \varepsilon_5$ *and* $t \in (1,2)$.

Proof. Let $\varphi \in C_c^\infty(B_3(0))$ be as in Proposition 3.3 with $\varphi \equiv 1$ on $\tilde{U} = B_2(0)$, $\tilde{t} = 1$, $T = 2$ and let c_3 and ε_2 be the constants for the monotonicity formula under these conditions. We claim that there exist some constants c_6 and c_7 such that

$$\int_{B_{c_6 r}(x_0)\times\{t_0-2r^2\}} \frac{\varepsilon|\nabla u|^2}{2} + \frac{W}{\varepsilon} \geq c_7 r^{n-1} \tag{4.1}$$

whenever $x_0 \in Z_{r,t_0}$ and $\varepsilon \leq \varepsilon_3$. To see this, let $(x_1,t_1) \in B_r(x_0) \times (t_0 - r^2, t_0)$ with $|u(x_1,t_1)| < \alpha$. The change of variables $x \mapsto \frac{x-x_1}{\varepsilon}$, $t \mapsto \frac{t-t_1}{\varepsilon^2}$ with $\tilde{u}(x,t) = u(\varepsilon x + x_1, \varepsilon^2 t + t_1)$ shows

$$\int \varphi \rho_{x_1,t_1+\varepsilon^2}(\cdot,t_1)\frac{W(u)}{\varepsilon} \geq \int_{B_{\varepsilon^{-1}}(0)} \rho_{0,1}(\cdot,0)W(\tilde{u}). \tag{4.2}$$

Since $|\tilde{u}|_{C^1} \leq c(W)$ in this scale, $|\tilde{u}(0,0)| < \alpha$ implies $W(\tilde{u}) \geq c(W) > 0$ on some neighborhood determined by W and c_0. Thus (4.2) is bounded from below by some definite constant, say, c_8. By (3.3),

$$\int \varphi\rho_{x_1,t_1+\varepsilon^2}\,d\mu_{t_1}^\varepsilon \leq \int \varphi\rho_{x_1,t_1+\varepsilon^2}\,d\mu_{t_0-2r^2}^\varepsilon + c_3\sqrt{t_1+\varepsilon^2-t_0+2r^2}$$

when $\varepsilon < \varepsilon_3$. Since $|t_0 - t_1| \leq r^2$, by restricting r depending on c_3 and c_8, we have

$$\frac{c_8}{2} \leq \int \varphi\rho_{x_1,t_1+\varepsilon^2}\,d\mu_{t_0-2r^2}^\varepsilon.$$

Then, choosing an appropriate c_6 depending only on c_8 and E_0, we have

$$\frac{c_8}{4} \leq \int_{B_{c_6 r}(x)} \rho_{x_1,t_1+\varepsilon^2}\,d\mu_{t_0-2r^2}^\varepsilon.$$

On $B_{c_6 r}(x)$, $\rho_{x_1,t_1+\varepsilon^2}(\cdot,t_0-2r^2) \leq c(c_6)r^{n-1}$. Thus we obtain (4.1) with an appropriate choice of c_7. Once this is done, the Besicovitch covering theorem immediately yields the lemma. □

Proof of Proposition 4.1. With these two lemmas, the proof proceeds similarly to that of [16, Proposition 5.1].

First assume that $1-b > \alpha$ and $c_4|\ln b| \geq 1$, and choose an integer $J = J(\varepsilon,b) \geq 1$ such that $\varepsilon^{1/2^{J+1}} \in (b,\sqrt{b}]$. We also assume that $\varepsilon \leq \varepsilon_5$ and $c_4|\ln\varepsilon| \leq \varepsilon^{-1}$. Fix $t \in (1,2)$. For $j = 1,\dots,J$, define

$$A_j = \{x \in B_1(0) \mid 1 - \varepsilon^{1/2^{j+1}} \leq |u(x,t)| \leq 1 - \varepsilon^{1/2^j}\}.$$

Then Lemma 4.2 with $\beta = 1/2^j$ shows that

$$A_j \subset Z_{c_4 2^{-j}\varepsilon|\ln\varepsilon|,t},$$

and Lemma 4.3 shows

$$\mathscr{L}^n(A_j) \le c_5 c_4 2^{-j} \varepsilon |\ln \varepsilon| \qquad \text{for } j = 1, \dots, J.$$

On A_j, using $|u| \ge 1 - \varepsilon^{1/2^{j+1}}$,

$$\frac{W(u)}{\varepsilon} \le \max_{u \in [\alpha, 1]} W''(u) \cdot \varepsilon^{-1} (\varepsilon^{1/2^{j+1}})^2 / 2 \le c_9(W) \varepsilon^{2^{-j}-1}.$$

Let $Y = B_1(0) \cap \{1 - b \le |u| \le 1 - \sqrt{\varepsilon}\} \subset \bigcup_{j=1}^{J} A_j$. Since $\varepsilon^{1/2^{J+1}} < \sqrt{b}$ it now follows with $c_{10} = c_9 c_5 c_4$ (depending only on E_0 and W) that

$$\begin{aligned}\int_Y \frac{W(u)}{\varepsilon} &\le \sum_{j=1}^{J} \int_{A_j} \frac{W(u)}{\varepsilon} \le c_{10} |\ln \varepsilon| \sum_{j=1}^{J} 2^{-j} \varepsilon^{2^{-j}} \\ &\le c_{10} |\ln \varepsilon| \int_0^{J+1} 2^{-t} \varepsilon^{2^{-t}} = c_{10} (\varepsilon^{2^{-(J+1)}} - \varepsilon) / \ln 2 \le c_{10} \sqrt{b} / \ln 2.\end{aligned}$$

We restrict b so that the last term is less than $\frac{s}{2}$.

To estimate the integral on $\{1 - \sqrt{\varepsilon} \le |u|\}$ let

$$A_0 = \{x \in B_1(0) \mid 1 - \sqrt{\varepsilon} \le |u(x)| \le 1 - \varepsilon^{2/3}\}$$

and similarly estimate

$$\int_{A_0} \frac{W(u)}{\varepsilon} \le c_{10} \frac{2}{3} \varepsilon |\ln \varepsilon|.$$

Finally for $\{|u| \ge 1 - \varepsilon^{2/3}\}$, using $|u| \le 1 + \varepsilon$ (which can be proved by the parabolic maximum principle),

$$\int_{B_1(0) \cap \{1 - \varepsilon^{2/3} \le |u|\}} \frac{W(u)}{\varepsilon} \le c_{11}(c_0, W) \varepsilon.$$

Restricting ε again, we obtain the stated inequality. $\square$

In the following, define

$$e_\varepsilon = \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}, \qquad \xi_\varepsilon = \frac{\varepsilon |\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}.$$

Also define $P : \mathbf{R}^n \to \mathbf{R}^{n-1}$ by $P(x) = (x_1, \dots, x_{n-1})$, and $P^\perp : \mathbf{R}^n \to \mathbf{R}$ by $P^\perp(x) = x_n$, where $x = (x_1, \dots, x_n)$. Also define $\nu = (\nu_1, \dots, \nu_n) = \frac{\nabla u}{|\nabla u|}$ whenever $|\nabla u| \ne 0$ and $\nu = 0$ when $|\nabla u| = 0$.

A few pointers. We use the next lemma inductively in Proposition 4.5. In the initial step, $l_1 = -\infty$ and $l_2 = \infty$ so there is no condition (5) for the first step. For the following steps, condition (5) ensures that the monotonicity formula *restricted to* $\{x \mid l_1 < x_n < l_2\}$ holds. In fact, it is an error term for "cutting" the solution along the two hypersurfaces in the computation of the monotonicity identity. Also note that the quantity in condition (5) is generically controlled by that of (6) (see the computation after (4.3)) and we include (A) so that we may inductively continue the separation.

Lemma 4.4. *Suppose*

(1) *$N \geq 1$ is an interger, Y is a subset of $\mathbf{R}^n$, $0 < R < \infty$, $1 < M < \infty$, $0 < a < \infty$, $0 < \varepsilon < 1$, $0 < \eta < 1$, $0 < E_0 < \infty$ and $-\infty \leq l_1 < l_2 \leq \infty$.*

(2) *Y has no more than $N+1$ elements, $P(y) = 0$ for all $y \in Y$, $Y \subset \{x \mid l_1 + a < x_n < l_2 - a\}$ and $|y - z| > 3a$ for any distinct $y, z \in Y$.*

(3) *$(M+1)$ diameter $Y < R$, and put $\tilde{R} \equiv M$ diameter Y.*

(4) *On $\{x \in \mathbf{R}^n \mid \operatorname{dist}(x, Y) < R\}$, u satisfies (1.1) with $|u| \leq 2$ and $\xi_\varepsilon \leq \eta$.*

(5) *For each $x = (x_1, \dots, x_n) \in Y$,*

$$\int_0^R \frac{d\tau}{\tau^n} \int_{B_\tau(x) \cap \{y_n = l_j\}} |e_\varepsilon (y_n - x_n) - \varepsilon u_{x_n} (y - x) \cdot \nabla u| \, d\mathscr{H}^{n-1} y \leq \eta$$

for $j = 1, 2$.

(6) *For each $x \in Y$ and $a \leq r \leq R$,*

$$\int_{B_r(x)} \varepsilon |u_t| \, |\nabla u| + |\xi_\varepsilon| + (1 - (\nu_n)^2) \varepsilon |\nabla u|^2 \leq \eta r^{n-1} \qquad \text{and}$$

$$\int_{B_r(x)} \varepsilon |\nabla u|^2 \leq E_0 r^{n-1}.$$

Then the following hold:

(A): *There exists $l_3 \in (l_1, l_2)$ such that $|x_n - l_3| \geq a$ and*

$$\int_0^{\tilde{R}} \frac{d\tau}{\tau^n} \int_{B_\tau(x) \cap \{y_n = l_3\}} |e_\varepsilon (y_n - x_n) - \varepsilon u_{x_n} (y - x) \cdot \nabla u| \, d\mathscr{H}^{n-1} y$$
$$\leq 3(N+1) N M (\eta + E_0^{1/2} \eta^{1/2})$$

for each $x \in Y$.

(B): *Put*

$$Y_1 = Y \cap \{x \mid l_1 < x_n < l_3\}, \qquad Y_2 = Y \cap \{x \mid l_3 < x_n < l_2\},$$

$$S_0 = \{x \mid l_1 < x_n < l_2 \text{ and } \operatorname{dist}(Y, x) < R\},$$

$$S_1 = \{x \mid l_1 < x_n < l_3 \text{ and } \operatorname{dist}(Y_1, x) < \tilde{R}\},$$

$$S_2 = \{x \mid l_3 < x_n < l_2 \text{ and } \operatorname{dist}(Y_2, x) < \tilde{R}\}.$$

Then Y_1 and Y_2 are non-empty and

$$\frac{1}{\tilde{R}^{n-1}}\left\{\int_{S_1} e_\varepsilon + \int_{S_2} e_\varepsilon\right\} \le \left(1+\frac{1}{M}\right)^{n-1}\frac{1}{R^{n-1}}\int_{S_0} e_\varepsilon + c(n)\eta(R+1)$$

holds.

PROOF. Note the condition on $\varepsilon|u_t|\,|\nabla u|$ in (6) has the effect of keeping the time derivative term small, so that we may derive the desired results just as in the elliptic case.

Let $\zeta_2(y) : \mathbf{R}^n \to \mathbf{R}$ be a smooth approximation to the characteristic function of the set $S \equiv \{y \in \mathbf{R}^n \mid l_1 < y_n < l_2\}$ which depends only on y_n. Let $x \in Y$ (and change the coordinates so that $x = 0$) and let $\zeta_1(y)$ be a smooth approximation of the characteristic function $\chi_{B_r(0)}$, where $0 < r < R$. Multiply the equation (1.1) by $(y \cdot \nabla u)\zeta_1(y)\zeta_2(y)$. After integration by parts twice and letting $\zeta_1 \to \chi_{B_r(0)}$, we obtain

$$\frac{d}{dr}\left\{\frac{1}{r^{n-1}}\int_{B_r} e_\varepsilon\zeta_2\right\} + \frac{1}{r^n}\int_{B_r}(\zeta_\varepsilon + \varepsilon u_t(y\cdot\nabla u))\zeta_2$$
$$- \frac{\varepsilon}{r^{n+1}}\int_{\partial B_r}(y\cdot\nabla u)^2\zeta_2 - \frac{1}{r^n}\int_{B_r}\{e_\varepsilon y_n - \varepsilon u_{x_n}(y\cdot\nabla u)\}\zeta_2' = 0.$$

After integrating over $[r, R]$ and letting $\zeta_2 \to \chi_S$, and then using (4), (5) and (6), we obtain

$$\frac{1}{R^{n-1}}\int_{B_R\cap S} e_\varepsilon \ge \frac{1}{r^{n-1}}\int_{B_r\cap S} e_\varepsilon - c(n)\eta(R+1) \tag{4.3}$$

where $c(n)$ depends only on the dimension n.

Next, choose $\tilde{y}, \tilde{z} \in Y$ such that $\tilde{z}_n - \tilde{y}_n \ge$ diameter Y/N and that there is no element of Y in $\{x \in \mathbf{R}^n \mid \tilde{y}_n < x_n < \tilde{z}_n\}$. Let $\tilde{l}_1 = \tilde{y}_n + \frac{\tilde{z}_n - \tilde{y}_n}{3}$ and $\tilde{l}_2 = \tilde{z}_n - \frac{\tilde{z}_n - \tilde{y}_n}{3}$. To choose an appropriate $l \in [\tilde{l}_1, \tilde{l}_2]$ which satisfies (A), we first observe, for $x \in Y$ and $y \in B_r(x)$,

$$\begin{aligned} I &\equiv |e_\varepsilon(y_n - x_n) - \varepsilon u_{x_n}(y-x)\cdot\nabla u| \\ &= |(-\zeta_\varepsilon)(y_n - x_n) + \varepsilon|\nabla u|^2((y_n - x_n) - \nu_n(y-x)\cdot\nu)| \\ &\le |\zeta_\varepsilon| r + \varepsilon|\nabla u|^2 r(1 - (\nu_n)^2 + \sqrt{1-(\nu_n)^2}). \end{aligned}$$

Using (6), we compute

$$\begin{aligned} \int_{\tilde{l}_1}^{\tilde{l}_2} dl \int_0^{\tilde{R}} \frac{d\tau}{\tau^n}\int_{B_\tau(x)\cap\{y_n = l\}} I\, d\mathscr{H}^{n-1}y &= \int_0^{\tilde{R}}\frac{d\tau}{\tau^n}\int_{B_\tau(x)\cap\{\tilde{l}_1 < y_n < \tilde{l}_2\}} I\, dy \\ &\le |\tilde{R}(\eta + E_0^{1/2}\eta^{1/2}). \end{aligned}$$

Thus, we may choose $l_3 \in [\tilde{l}_1, \tilde{l}_2]$ such that

$$\int_0^{\tilde{R}} \frac{d\tau}{\tau^n} \int_{B_\tau(x) \cap \{y_n = l_3\}} I \, d\mathscr{H}^{n-1} y \le \frac{(N+1)\tilde{R}(\eta + E_0^{1/2}\eta^{1/2})}{\tilde{l}_2 - \tilde{l}_1}$$

for all $x \in Y$. Since $\tilde{l}_2 - \tilde{l}_1 \le \text{diameter } Y/3N$, we have $\tilde{R}/(\tilde{l}_2 - \tilde{l}_1) \le 3MN$, and we obtain (A).

Define S_1 and S_2 as in (B). For any $x \in Y$, we have $S_1 \cup S_2 \subset B_{(\tilde{R} + \text{diam } Y)}(x) \cap S$, thus

$$\frac{1}{\tilde{R}^{n-1}} \left\{ \int_{S_1} e_\varepsilon + \int_{S_2} e_\varepsilon \right\} \le \frac{1}{\tilde{R}^{n-1}} \int_{B_{(\tilde{R}+\text{diam } Y)}(x) \cap S} e_\varepsilon$$
$$\le \left(1 + \frac{1}{M}\right)^{n-1} \left\{ \frac{1}{R^{n-1}} \int_{B_R(x) \cap S} e_\varepsilon + c(n)\eta(R+1) \right\}.$$

We used (4.3) in the last inequality. Finally, noting that $B_R(x) \cap S \subset S_0$, we obtain (B). □

Once we have the previous lemma, we obtain the following proposition by inductively using the lemma and separating each element of Y. We choose M very large and then choose η very small, depending on N. Note that the monotonicity formula restricted to the vertically separated region is available at the end. Again, the time variable here is fixed:

Proposition 4.5. *Corresponding to each R, E_0, s and N such that $0 < R < \infty$, $0 < E_0 < \infty$, $0 < s < 1$ and N is a positive integer, there exists $\eta > 0$ with the following property:*

Assume the following:

(1) *$Y \subset \mathbf{R}^n$ has no more than $N+1$ elements, $P(y) = 0$ for all $y \in Y$, $a > 0$, $|y - z| > 3a$ for all $y, z \in Y$ and* diameter $Y \le \eta R$.

(2) *On $\{x \in \mathbf{R}^n \mid \text{dist}(x, Y) < R\}$, u satisfies* (1.1) *with $|u| \le 2$ and $\xi_\varepsilon \le \eta$.*

(3) *For each $y \in Y$ and $a \le r \le R$,*

$$\int_{B_r(y)} \varepsilon|u_t| \, |\nabla u| + |\xi_\varepsilon| + (1 - (v_n)^2)\varepsilon|\nabla u|^2 dy \le \eta r^{n-1},$$

$$\int_{B_r(y)} \varepsilon|\nabla u|^2 \le E_0 r^{n-1}.$$

Then we have

$$\sum_{y \in Y} \frac{1}{a^{n-1}} \int_{B_a(y)} e_\varepsilon \le s + \frac{1+s}{R^{n-1}} \int_{\{x \mid \text{dist}(Y,x) < R\}} e_\varepsilon.$$

The next proposition is almost identical to [16, Proposition 5.6], which deals with the "ε-scale". Note that we do not have to include any condition on the time derivative term.

PROPOSITION 4.6. *Given $0 < s < 1$ and $0 < b < 1$, there exist $0 < \eta < 1$ and $1 < L < \infty$ (which also depend on W) with the following property:*

Assume $0 < \varepsilon < 1$ and u satisfies (1.1) and $\xi_\varepsilon \le \eta$ on $B_1(0) \times (-1, 1)$, $|u(0,0)| \le 1 - b$, and

$$\int_{B_{4\varepsilon L}(0)\times\{0\}} (|\xi_\varepsilon| + (1 - (\nu_n)^2)\varepsilon|\nabla u|^2) \le \eta(4\varepsilon L)^{n-1}. \tag{4.4}$$

Then, we have $P^{-1}(0) \cap \{x \in B_{3L\varepsilon}(0) \mid u(x,0) = u(0,0)\} = \{0\}$ and

$$\left| \frac{1}{\omega_{n-1}(L\varepsilon)^{n-1}} \int_{B_{L\varepsilon}(0)\times\{0\}} e_\varepsilon - 2\sigma \right| \le s. \tag{4.5}$$

PROOF. We rescale the domain by $x \mapsto x/\varepsilon$ and $t \mapsto t/\varepsilon^2$ for convenience. The rescaled function is still denoted by u.

Let $q : \mathbf{R} \to (-1, 1)$ be the unique solution of the ODE

$$\begin{cases} q'(t) = \sqrt{2W(q(t))} & \text{for } t \in \mathbf{R}, \\ q(0) = u(0,0). \end{cases}$$

We note that

$$\int_{-\infty}^{\infty} \frac{1}{2}|q'(t)|^2 dt = \int_{-\infty}^{\infty} \sqrt{\frac{W(q(t))}{2}}\, q'(t) dt = \int_{-1}^{1} \sqrt{\frac{W(s)}{2}}\, ds = \sigma.$$

We also identify q on $\mathbf{R}^n$ by $q(x_1, \dots, x_n) = q(x_n)$.

For given b and s, we fix a large enough $L > 1$ so that

$$\left| \frac{1}{\omega_{n-1}L^{n-1}} \int_{B_L(0)} \left(\frac{1}{2}|\nabla q|^2 + W(q) \right) - 2\sigma \right| \le \frac{s}{2} \tag{4.6}$$

whenever $|q(0)| \le 1 - b$. Next, using the pointwise assumption $\frac{1}{2}|\nabla u|^2 - W(u) \le \eta$ on $B_{4L}(0) \times \{0\}$ and $|u(0,0)| \le 1 - b$, we restrict η so that $|u| \le 1 - \tilde{b}$ on $B_{4L}(0) \times \{0\}$ for some $\tilde{b} = \tilde{b}(W, b, s) > 0$.

Define a function $z(x,t) : B_{4L}(0) \times (-1, 1) \to \mathbf{R}$ by setting

$$z(x,t) = q^{-1}(u(x,t)),$$

where $q^{-1} : (-1, 1) \to \mathbf{R}$ is the inverse function of q. Since $|u| \le 1 - \tilde{b}$, z is well-defined and $q'(z(x,t)) \ge \min_{|u| \le 1-\tilde{b}} \sqrt{2W(u)}$ for $x \in B_{4L}(0)$. Moreover, since we may use the equation (1.1) to estimate $\|u\|_{C^2(B_{3L}(0)\times\{0\})}$, $\|z\|_{C^2(B_{3L}(0)\times\{0\})}$ is uniformly bounded depending only on W, b and s by the lower bound of q'. Thus, with

$$\frac{1}{2}|\nabla u|^2 - W(u) = \frac{1}{2}(q'(z))^2(|\nabla z|^2 - 1),$$

$$|\nabla u|^2(1 - (v_n)^2) = (q'(z))^2(|\nabla z|^2 - (z_{x_n})^2)$$

and the inequality (4.4), we may obtain (with either $+$ or $-$)

$$\|z(x) \pm x_n\|_{C^1(B_{3L}(0)\times\{0\})} \le c(b, s, W)\eta^{1/(n+1)}.$$

This shows that $u(x)$ is C^1 close to $q(x_n)$ on $B_{3L}(0) \times \{0\}$. Combined with (4.6), by choosing η sufficiently small, we obtain (4.5). Also, $u_{x_n} = q'(z)z_{x_n} \neq 0$ on $B_{3L}(0) \times \{0\}$ implies the first assertion. □

Proof of integrality. By the argument in [17, Section 9.3, 9.5], for any $t = t_0 > 0$ with $\overline{D}_t\mu_t(\phi) > -\infty$, where $\phi \in C_c^2(U; \mathbf{R}^+)$ is a fixed function, we can choose a sequence $\{t_i\}_{i=1}^\infty$ such that (after choosing a suitable subsequence of $\{u^i\}_{i=1}^\infty$ and translating t_0 to 0)

(1) $t_i > 0$, $\lim_{i\to\infty} t_i = 0$,
(2) $\limsup_{i\to\infty} \int \varepsilon_i\phi|u_t^i|^2|_{t=t_i} < \infty$,
(3) $\lim_{i\to\infty} \int \phi|\xi^i|\,|_{t=t_i} = 0$, where $\xi^i = \frac{\varepsilon_i|\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i}\Big|_{t=t_i}$,
(4) $\int_{B_r(x)} d\mu_{t_i}^i \le E_0 r^{n-1}$ for all $x \in \operatorname{supp}\phi$ and $0 < r < \operatorname{dist}(\operatorname{supp}\phi, \partial U)/2$,
(5) $\mu_{t_i}^i \to \mu_0$ as Radon maesure in U,
(6) $\mathscr{H}^{n-1}(\operatorname{supp}\mu_0 \cap \{\phi > 0\}) < \infty$,
(7) $\frac{\varepsilon_i|\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i}\Big|_{t=t_i} \le c_2$ on $\{\phi > 0\}$ (by Lemma 3.2).

Note that (4) follows from the monotonicity formula (see [17, Section 5.1(2)]). Under these conditions, Ilmanen [17] proved the rectifiability of the limit μ_0 as well as the Brakke's inequality of varifold mean curvature flow equation (2.5) for μ_t. As a result, the convergence of $\mu_{t_i}^i$ to μ_0 is also in the sense of varifold ([17, Section 9]). Here we show that $\sigma^{-1}\mu_0$ is also integral. This is achieved by showing that the $(n-1)$-dimensional density of $\sigma^{-1}\mu_0$ is integer-valued for $\mathscr{H}^{n-1}$-a.e. for $t = 0$, which is a generic time.

For any $q \in \mathbf{N}$, define

$$A_{i,q} = \left\{x \in \operatorname{supp}\phi \mid \int_{B_r(x)} \varepsilon_i|u_t^i|\,|\nabla u^i|\phi\Big|_{t=t_i} \le q\int_{B_r(x)} \phi\, d\mu_{t_i}^i\right.$$
$$\left.\text{for all } 0 < r < \operatorname{dist}(\operatorname{supp}\phi, \partial U)/2\right\}.$$

Since

$$\limsup_{i\to\infty} \int \varepsilon_i|u_t^i|\,|\nabla u^i|\phi \le \limsup_{i\to\infty}\left(\int \varepsilon_i|u_t^i|^2\phi\right)^{1/2}\left(\int \varepsilon_i|\nabla u^i|^2\phi\right)^{1/2} \le:= c_{11} < \infty,$$

the Besicovitch covering theorem shows that

$$\int_{A_{i,q}^c} \phi \, d\mu_{t_i}^i \le q^{-1} c(n) c_{11}$$

for all large i. Here, we denote the complement of a set X by X^c. Let A_q be the set of points $x \in \operatorname{supp} \phi$ such that there are $x_i \in A_{i,q}$ for infinitely many i with $x = \lim_{i\to\infty} x_i$. By the definition, A_q is closed. Define $A = \bigcup_{q=1}^{\infty} A_q$. We want to see that $\int_{A^c} \phi \, d\mu_0 = 0$. If not true, then, we would have $\int_{A_q^c} \phi \, d\mu_0 > 0$ for any $q > 0$. Let $\tilde{\phi} \in C_c(A_q^c)$ be such that $0 \le \tilde{\phi} \le 1$ and $\int_{A_q^c} \phi\tilde{\phi} \, d\mu_0 > \frac{1}{2}\int_{A_q^c} \phi \, d\mu_0$. Since $\operatorname{supp} \tilde{\phi}$ is compact, we may choose (by the definition of A_q) $N(q) \in \mathbf{N}$ such that $\operatorname{supp} \tilde{\phi} \subset A_{i,q}^c$ for all $i \ge N(q)$. Hence,

$$\frac{1}{2}\int_{A_q^c} \phi \, d\mu_0 \le \limsup_{i\to\infty} \int_{A_{i,q}^c} \phi\tilde{\phi} \, d\mu_{t_i}^i \le q^{-1} c(n) c_{11}.$$

We then have, for any $q \in \mathbf{N}$, $\int_{A^c} \phi \, d\mu_0 \le q^{-1} c(n) c_{11}$, so $\int_{A^c} \phi \, d\mu_0 = 0$.

Next, since μ_0 is rectifiable, μ_0-a.e. point x (which we translate to the origin subsequently) has a weak tangent plane. Namely, let V be the rectifiable varifold with $\|V\| = \mu_0$. Then, at such point (after rotation), $\lim_{i\to\infty} (\Phi_{r_i})_\# V = \theta v(P)$, where $r_i \to 0$, $(\Phi_{r_i})_\#$ is the usual push-forward with $\Phi_{r_i}(x) = \frac{x}{r_i}$, $v(P)$ corresponds to the varifold associated with the $(n-1)$-dimensional plane $P = \{x_n = 0\}$, and θ is the density at the point. For μ_0 a.e., we may also assume that the point is in A_q for some $q \in \mathbf{N}$ and thus there exists a sequence $x_i \in A_{i,q}$ with $0 = \lim_{i\to\infty} x_i$. Let V^i be the varifold associated with $\mu_{t_i}^i$. After choosing a subsequence, we may assume that $(\Phi_{r_i})_\# V^i$ converge to $\theta v(P)$, $\lim_{i\to\infty} \frac{x_i}{r_i} = 0$ and $\lim_{i\to\infty} \frac{t_i}{r_i^2} = 0$. Rescale the coordinates by $\tilde{x} = \frac{x}{r_i}$, $\tilde{t} = \frac{t}{r_i^2}$ and $\tilde{\varepsilon}_i = \frac{\varepsilon_i}{r_i}$ (and subsequently drop $\tilde{\cdot}$). We preserve the form of the equation (1.1) under the scaling, and by the definition of $A_{i,q}$ and (4), we have

$$\int_{B_3(0)} \varepsilon_i |u_t^i| \, |\nabla u^i| \bigg|_{t=t_i} \le r_i q \to 0 \tag{4.7}$$

as $i \to \infty$. The condition (7) is, under the scaling,

$$\frac{\varepsilon_i |\nabla u^i|^2}{2} - \frac{W(u^i)}{\varepsilon_i} \bigg|_{t=t_i} \le c_2 r_i \to 0 \tag{4.8}$$

on $\{\phi > 0\}$ as $i \to \infty$. Since $(\Phi_{r_i})_\# V^i$ converges to $\theta v(P)$ in the varifold sense, we also have

$$\lim_{i\to\infty} \int_{B_3(0)} (1 - (v_n)^2) \varepsilon_i |\nabla u^i|^2 \bigg|_{t=t_i} = 0. \tag{4.9}$$

Suppose N is the smallest positive integer greater than $\sigma^{-1}\theta$. Fix an arbitrary small $s > 0$. Use Proposition 4.1 to choose $b > 0$, and then with (4.8) we have

$$\int_{B_3(0)\cap\{|u^i|\geq 1-b\}} \left(\frac{\varepsilon_i |\nabla u^i|^2}{2} + \frac{W(u^i)}{\varepsilon_i} \right) \bigg|_{t=t_i} \leq s \tag{4.10}$$

for all sufficiently large i. With these chioces of s, b and $R = 1$, we choose η and L via Proposition 4.5 and 4.6 (the smaller η should be chosen). For all large i, we define

$$\begin{aligned} G_i = {} & B_2(0) \times \{t_i\} \cap \{|u^i| \leq 1 - b\} \\ & \cap \left\{ x \,\middle|\, \int_{B_r(x)} \varepsilon_i |u_t^i|\,|\nabla u^i| + |\xi^i| + (1 - (v_n)^2)\varepsilon_i |\nabla u^i|^2 \bigg|_{t=t_i} \right. \\ & \qquad \left. \leq \eta E_0^{-1} \mu_{t_i}^i(B_r(x)) \text{ if } 4\varepsilon_i L \leq r \leq 1 \right\}. \end{aligned}$$

By repeating the argument leading to (4.3), one may prove that there exist constants c_{12} and c_{13} depending only on $n, \tilde{U}$ and W such that

$$\frac{1}{r^{n-1}} \mu_{t_i}^i(B_r(x)) \geq c_{12} \qquad \text{for all } \varepsilon_i \leq r \leq c_{13} \text{ and } x \in G_i. \tag{4.11}$$

By the Besicovitch covering theorem, one shows that

$$\begin{aligned} & \mu_{t_i}^i(B_2(0) \cap \{|u| \leq 1 - b\} \backslash G_i) \\ & \quad \leq c(n)\eta^{-1} E_0 \int_{B_3(0)} \varepsilon_i |u_t^i|\,|\nabla u^i| + |\xi^i| + (1 - v_n^2)\varepsilon_i |\nabla u^i|^2 \bigg|_{t=t_i}, \end{aligned} \tag{4.12}$$

which goes to 0 as $i \to \infty$ by (3), (4.7) and (4.9). Also $\operatorname{dist}(P, G_i) \to 0$ as $i \to \infty$, since $\mu_{t_i}^i \to \theta \|v(T)\|$ and by (4.11).

For any $x \in B_1^{n-1}(0) := (\mathbf{R}^{n-1} \times \{0\}) \cap B_1(0)$ and $|l| \leq 1 - b$, we let $Y = P^{-1}(x) \cap G_i \cap \{u^i = l\}$ and apply Proposition 4.5, where we set $a = L\varepsilon_i$. By Proposition 4.6, each element of Y is separated by at least $3L\varepsilon_i$, and all the assumptions are satisfied for sufficiently large i. We prove that Y does not contain more than $N - 1$ elements for any $x \in B_1^{n-1}(0)$ as follows. Since

$$\sup_{x \in B_1^{n-1}(0)} \frac{1}{\omega_{n-1}} \int_{B_1(x)} \left(\frac{\varepsilon_i}{2} |\nabla u^i|^2 + \frac{W(u^i)}{\varepsilon_i} \right) \leq 2\theta + s$$

for large i, Y having more than $N - 1$ elements would imply, by Proposition 4.5, that

$$2\sigma N \leq (N + 1)s + (1 + s)(2\theta + s).$$

This would be a contradiction to $\theta\sigma^{-1} < N$ for sufficiently small s depending only on N.

Finally, since $|\xi^i| \to 0$ as $i \to \infty$, we have $\left|\frac{\varepsilon_i|\nabla u^i|^2}{2} - |\nabla u^i|\sqrt{W(u^i)/2\varepsilon_i}\right| \to 0$ in L^1_{loc}. As the result and by (4.10) and (4.12), for $t = t_i$,

$$\omega_{n-1}\theta = \lim_{i\to\infty}\int_{B_1(0)} \frac{\varepsilon_i|\nabla u^i|^2}{2} \le \lim_{i\to\infty}\int_{B_1(0)\cap\{|u^i|\le 1-b\}\cap G_i} |\nabla u^i|\sqrt{W(u^i)/2} + s.$$

By the co-area formula, $\lim_{i\to\infty}\|P_\# V^i\| = \theta v(P)$ and the above discussion then implies

$$\omega_{n-1}\theta \le \lim_{i\to\infty}\int_{-1+b}^{1-b} \|P_\#(v(\{u^i = t\}\cap G_i))\|(B_1^{n-1}(0))\sqrt{W(t)/2}\,dt + s$$
$$\le \omega_{n-1}(N-1)\int_{-1+b}^{1-b}\sqrt{W(t)/2}\,dt + s \le \omega_{n-1}(N-1)\sigma + s.$$

Since s is arbitrary, we have $\theta = (N-1)\sigma$. □

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Yoshihiro Tonegawa
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810 *Japan*
E-mail: tonegawa@math.sci.hokudai.ac.jp