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Subgroups of $\pi_*(L_2T(1))$ at the prime two

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Abstract. Let $T(1)$ be the Ravenel spectrum whose BP_*-homology is $BP_*[t_1](\subset BP_*(BP))$, and let L_2 denote the Bousfield localization functor with respect to $v_2^{-1}BP$. In this paper, we show that the E_4-term of the Adams-Novikov spectral sequence for $\pi_*(L_2T(1))$ has horizontal vanishing line and is the E_∞-term. We also find subgroups of the homotopy groups $\pi_*(L_2T(1))$.

1. Introduction

In this paper, everything is localized at the prime two. Let BP denote the Brown-Peterson ring spectrum at the prime two. Then the homotopy groups $\pi_*(BP)$ turn to the polynomial algebra $BP_* = \boldsymbol{Z}_{(2)}[v_1, v_2, \dots]$ over the Hazewinkel generators v_i with $|v_i| = 2^{i+1} - 2$. The Ravenel spectrum $T(1)$ is characterized by the Brown-Peterson homology as $BP_*(T(1)) = BP_*[t_1] \subset BP_*(BP) = BP_*[t_1, t_2, \dots]$. We consider the spectrum $G = v_2^{-1}BP$. Let L_2 denote the Bousfield localization functor on the stable homotopy category of spectra with respect to G. One of the methods to determine the homotopy groups $\pi_*(L_2T(1))$ is the Adams-Novikov spectral sequence $E_2^* = H^*v_2^{-1}BP_*[t_1]$ $\Rightarrow \pi_*(L_2T(1))$, where $H^*- = \mathrm{Ext}^*_{G_*(G)}(G_*, -)$. We study the E_2-term by the chromatic spectral sequence $\sum_{i=0}^{2} H^*M_0^i[t_1] \Rightarrow H^*v_2^{-1}BP_*[t_1]$ and the mod 2 Bockstein spectral sequences $H^*M_1^0[t_1] \Rightarrow H^*M_0^1[t_1]$ and $H^*M_1^1[t_1] \Rightarrow H^*M_0^2[t_1]$. Here, $M_0^0 = 2^{-1}BP_*$, $M_1^0 = v_1^{-1}BP_*/(2)$, $M_0^1 = v_1^{-1}BP_*/(2^\infty)$, $M_1^1 = v_2^{-1}BP_*/(2, v_1^\infty)$ and $M_0^2 = v_2^{-1}BP_*/(2^\infty, v_1^\infty)$. The modules $H^*M_0^0[t_1]$ and $H^*M_1^0[t_1]$ are given by Ravenel in [7]. In [5], Mahowald and the second author determined $H^*M_2^0[t_1]$ as the tensor product of the polynomial algebra $K(2)_*[v_3, h_{20}]$ and the exterior algebra $\Lambda(h_{21}, h_{30}, h_{31}, \rho_2)$, where $K(2)_* = \boldsymbol{Z}/2[v_2^{\pm 1}]$. In [8], the second author determined $H^*M_1^1[t_1]$ by the v_1-Bockstein spectral sequence $H^*M_2^0[t_1] \Rightarrow H^*M_1^1[t_1]$ to be the tensor product of $\Lambda(\rho_2)$ and the direct sum of modules A_i:

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$$A_0 = \left(v_1^{-1}K/K \oplus \sum_{n>1} x_n K/(v_1^{a_n})[x_{n+1}] \otimes \Lambda(g_{n+1})\right) \otimes \Lambda(\widetilde{h_{20}}),$$

$$A_1 = v_3^2 K/(v_1^2)[x_2] \otimes \Lambda(h_{30}, h_{31}) \qquad \text{and}$$

$$A_2 = v_3 K(2)_*[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}).$$

Here $K = \boldsymbol{Z}/2[v_1, v_2^{\pm 1}]$, a_n denotes the integer $2^n + \frac{2}{3}(2^n - 2^{\varepsilon(n)})$ for $\varepsilon(n) = (1-(-1)^n)/2$, and the elements x_n, g_n, h_{ij} and $\widetilde{h_{20}}$ denote the cohomology classes represented by the cocycles of the cobar complex $\Omega^*_{G_*(G)} G_*[t_1]/(2, v_1^j)$ for a suitable $j > 0$, whose leading terms are $v_3^{2^n}, v_3^{4(2^{n-2}-2^{\varepsilon(n)})/3} t_3^{2^{\varepsilon(n)}}, t_i^{2^j}$ and $v_3^2 t_2$, respectively. Consider the submodule

$$A_{21} = v_3 K_*^2[v_3^2] \otimes \Lambda(h_{21}, h_{30}, h_{31}) \subset A_2,$$

and put $A_2^0 = A_2/A_{21}$ as a module. We see that there is a submodule

$$\widetilde{A_2} = v_2 v_3 K_*^2[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31})$$

of $H^*M_0^2[t_1]$, where $K_*^2 = \boldsymbol{Z}/2[v_2^{\pm 2}]$ and $x \in \widetilde{A_2}$ is considered to be $x/2v_1 \in H^*M_0^2[t_1]$. Then we show that the map $\varphi : H^*M_1^1[t_1] \to H^*M_0^2[t_1]$ given by $\varphi(x) = x/2$ is restricted to $\varphi : A_2^0 \to \widetilde{A_2}$ and then the sequence $0 \to (\widetilde{A_2})^{s-1} \xrightarrow{\delta} (A_2^0)^s \xrightarrow{\varphi} (\widetilde{A_2})^s \to 0$ for each $s > 3$ is exact, where $(M)^s$ denotes the submodule of M consisting of elements of cohomology dimension s, and δ is the connecting homomorphism associated to the short exact sequence $0 \to M_1^1[t_1] \to M_0^2[t_1] \to M_0^2[t_1] \to 0$. This shows our first result.

Theorem 1.1. *$H^s M_0^2[t_1]$ is isomorphic to $(\widetilde{A_2} \otimes \Lambda(\rho_2))^s$ for $s > 4$.*

Furthermore, we show that the mod 2 Bockstein spectral sequence splits (see Lemma 3.6). A summand of the spectral sequence is $A_2^0 \Rightarrow \widetilde{A_2}$. It seems very complicated to determine the other parts $\boldsymbol{A}_1 = (A_0 \oplus A_1 \oplus A_{21}) \otimes \Lambda(\rho_2) \Rightarrow \widetilde{\boldsymbol{A}_1}$ (*cf.* [6], [2], [9]).

Let W be the spectrum such that $BP_*(L_2 W) = M_0^2$. Indeed, W is the cofiber of the localization map $V \to L_1 V$, where V is the cofiber of the localization map $S^0 \to S\boldsymbol{Q}$. Then $H^*M_0^2[t_1]$ is isomorphic to the E_2-term of the Adams-Novikov spectral sequence for $\pi_*(L_2 W \wedge T(1))$. We consider the submodule

$$\widetilde{A_{21}} = v_3^3 K_*^2[v_3^4] \otimes \Lambda(h_{30}, h_{31}) \subset H^*M_0^2[t_1],$$

and see that $\widetilde{A_{21}} \otimes \Lambda(\rho_2) \subset \widetilde{\boldsymbol{A}_1}$ (see Corollary 4.4). We write $\widetilde{\boldsymbol{A}_1}^0 = \widetilde{\boldsymbol{A}_1}/(\widetilde{A_{21}} \otimes \Lambda(\rho_2))$ as a module. We compute the differentials of the Adams-Novikov spectral sequence on $\widetilde{A_2}$ and $\widetilde{A_{21}}$, and then show that the differentials on $\widetilde{\boldsymbol{A}_1}^0$ are zero after a modification of $\widetilde{\boldsymbol{A}_1}^0$ (see Corollary 4.8).

THEOREM 1.2. *The Adams-Novikov E_∞-term for the homotopy groups $\pi_*(L_2T(1)\wedge W)$ is isomorphic to the direct sum of $\widetilde{A_1}^0$ and $\widehat{A_2}\otimes\Lambda(\rho_2)$, where*

$$\widehat{A_2}=v_2v_3K_*^2[v_3^4]\otimes\Lambda(h_{20},h_{21},h_{30},h_{31})\oplus v_2v_3h_{20}^2K_*^2[v_3^4]\otimes\Lambda(h_{30},h_{31}).$$

Note that we do not determine the structure of $\widetilde{A_1}^0$ of the theorem, though we know that the Adams-Novikov differentials are trivial on it.

By the definition of W, we have the composite $\eta: W\to\Sigma V\to S^2$, which induces the composite of connecting homomorphisms $\eta_*: H^sM_0^2[t_1]\to H^{s+1}v_2^{-1}BP_*/(2^\infty)[t_1]\to H^{s+2}v_2^{-1}BP_*[t_1]$ in the long exact sequences

$$H^sM_0^1[t_1]\to H^sM_0^2[t_1]\xrightarrow{\delta}H^{s+1}v_2^{-1}BP_*/(2^\infty)[t_1]\to H^{s+1}M_0^1[t_1]\qquad\text{and}$$
$$H^sM_0^0[t_1]\to H^sv_2^{-1}BP_*/(2^\infty)[t_1]\xrightarrow{\delta}H^{s+1}v_2^{-1}BP_*[t_1]\to H^{s+1}M_0^0[t_1].$$

Since we see that both of $H^sM_0^0[t_1]$ and $H^sM_0^1[t_1]$ are zero for $s>0$ (Theorem 2.5), we see that the connecting homomorphisms are isomorphisms for $s>0$, and so is η_*. In Proposition 4.7, we show that the E_4-term is the E_∞-term. Since η_* is a map of spectral sequences, we have the results on $\pi_*(L_2T(1))$.

COROLLARY 1.3. *The Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2T(1))$ collapses from the E_4-term.*

COROLLARY 1.4. *The homotopy groups $\pi_*(L_2T(1))$ contain the subgroups isomorphic to $\widehat{A_2}\otimes\Lambda(\rho_2)$, which is the image of $\widehat{A_2}\otimes\Lambda(\rho_2)$ under the map $\eta_*:\pi_*(L_2T(1)\wedge W)\to\pi_*(L_2T(1))$.*

In the next section, we show that $H^sM_0^1[t_1]$ is zero for $s>0$ by determining it. In sections 3 and 4, we give proofs of Theorems 1.1 and 1.2, respectively. The authors would like to thank Professor Xiangjun Wang who pointed out mistakes in Lemmas 3.3 and 4.3 in a draft version of this paper.

2. $H^*M_0^1[t_1]$

Let BP denote the Brown-Peterson spectrum at the prime two. Then $BP_*=\mathbf{Z}_{(2)}[v_1,v_2,\dots]$ and $BP_*(BP)=BP_*[t_1,t_2,\dots]$, and $(BP_*,BP_*(BP))$ is a Hopf algebroid. Hereafter, we write

$$H^*M=\mathrm{Ext}^*_{BP_*(BP)}(BP_*,M)$$

for a $BP_*(BP)$-comodule M. Consider the $BP_*(BP)$-comodule $M_1^0=v_1^{-1}BP_*/(2)$. Then in [7, Th. 6.1.1 and Cor. 6.5.6], it is shown that

$$H^*M_1^0[t_1]=K(1)_*[v_2]\otimes\Lambda(h_{20}).$$

Here H^*M for a $BP_*(BP)$-comodule M denotes $\mathrm{Ext}^*_{BP_*(BP)}(BP_*,M)$, $K(1)_*=$

$\boldsymbol{Z}/2[v_1^{\pm 1}]$ and h_{20} is the element represented by a cocycle of the cobar complex whose leading term is t_2. Consider the Hopf algebroid $(A, \Gamma) = (BP_*, BP_*[t_2, t_3, \dots])$, whose structure maps are induced from those of $BP_*(BP)$ under the projection $BP_*(BP) \to \Gamma$. We then have the change of rings theorem

$$H^* M[t_1] = \operatorname{Ext}_\Gamma^*(A, M),$$

for a $BP_*(BP)$-comodule M.

Lemma 2.1. *In the Hopf algebroid* (A, Γ),

$$\begin{aligned} &\eta_R(v_1) = v_1, \\ &\eta_R(v_2) = v_2 + 2t_2 \qquad \text{and} \\ &\eta_R(v_3) = v_3 + v_1 t_2^2 + 2t_3 - 2v_1 v_2 t_2 - 2v_1 t_2^2 - v_1^4 t_2. \end{aligned}$$

Proof. This is based on the Hazewinkel's and the Quillen's formulas:

$$v_n = 2m_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{2^i} \in \boldsymbol{Q} \otimes BP_* = \boldsymbol{Q}[m_1, m_2, \dots] \qquad \text{and}$$

$$\eta_R(m_n) = \sum_{i=0}^{n} m_i t_{n-i}^{2^i} \in \boldsymbol{Q} \otimes BP_*(BP).$$

We consider it in $\boldsymbol{Q} \otimes BP_*[t_2, t_3, \dots]$. Then $\eta_R(v_1) = 2\eta_R(m_1) = 2m_1 = v_1$, and $\eta_R(v_2) = 2\eta_R(m_2) - m_1 v_1^2 = 2(m_2 + t_2) - m_1 v_1^2 = v_2 + 2t_2$. For $\eta_R(v_3)$, we compute

$$\begin{aligned} \eta_R(v_3) &= 2(m_3 + m_1 t_2^2 + t_3) - m_1(v_2 + 2t_2)^2 - (m_2 + t_2) v_1^4 \\ &= 2m_3 + v_1 t_2^2 + 2t_3 - m_1 v_2^2 - 2v_1 v_2 t_2 - 2v_1 t_2^2 - m_2 v_1^4 - v_1^4 t_2 \\ &= v_3 + v_1 t_2^2 + 2t_3 - 2v_1 v_2 t_2 - 2v_1 t_2^2 - v_1^4 t_2. \end{aligned}$$ □

We define $x_{1,n} \in v_1^{-1} A = v_1^{-1} BP_*$ by

$$x_{1,0} = v_2, \qquad x_{1,1} = x_{1,0}^2 + 2v_1^3 v_2 + 4v_1^{-1} v_3, \qquad \text{and} \qquad x_{1,n} = x_{1,n-1}^2.$$

Let $d : v_1^{-1} A \to v_1^{-1} A \otimes_A \Gamma$ denote $\eta_R - \eta_L$. Then we have

Lemma 2.2. *Let* $x_{1,i}$ *be the elements defined above. Then we see that* $d(x_{1,n}) \equiv 2^{n+1} X_n t_2 \mod(2^{n+2})$ *for* $n \geq 0$, *where* $X_0 = 1$ *and* $X_n = x_{1,0} x_{1,1} \dots x_{1,n-1}$ *for* $n > 0$.

Proof. For $n = 0$, it follows from Lemma 2.1. For $n = 1$, we obtain the equation from the computations:

$$d(v_2^2) = (v_2 + 2t_2)^2 - v_2^2 = 4v_2t_2 + \underline{4t_2^2}_1,$$

$$d(4v_1^{-1}v_3) \equiv 4v_1^{-1}(\underline{v_1t_2^2}_1 + \underline{v_1^4t_2}_2) \mod(8) \qquad \text{and}$$

$$d(2v_1^3v_2) = \underline{4v_1^3t_2}_2.$$

Here, the underlined terms with the same subscript cancel out.

Inductively, suppose that $d(x_{1,n}) \equiv 2^{n+1}X_nt_2 \mod(2^{n+2})$. Then

$$\begin{aligned} d(x_{1,n}^2) &\equiv (x_{1,n} + 2^{n+1}X_nt_2)^2 - x_{1,n}^2 \mod(2^{n+3}) \\ &\equiv 2^{n+2}x_{1,n}X_nt_2 \mod(2^{n+3}), \end{aligned}$$

and obtain the congruence for $n+1$. □

Lemma 2.3. *$H^0M_0^1[t_1]$ is the tensor product of $\boldsymbol{Z}_{(2)}[v_1, v_1^{-1}]$ and the direct sum of $\boldsymbol{Q}/\boldsymbol{Z}_{(2)}$ and $\boldsymbol{Z}/(2^{n+1})$ generated by $x_{1,n}^s/2^{n+1}$ for each $n \geq 0$ and odd $s > 0$.*

Proof. Let B denote the module of the lemma. Then we have a sequence $H^*M_1^0[t_1] \xrightarrow{\varphi} B \xrightarrow{2} B$ fitting in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0M_1^0[t_1] & \xrightarrow{\varphi} & H^0M_0^1[t_1] & \xrightarrow{2} & H^0M_0^1[t_1] & \xrightarrow{\delta} & H^1M_1^0[t_1] \\ & & \| & & \uparrow i & & \uparrow i & & \| \\ 0 & \longrightarrow & H^0M_1^0[t_1] & \xrightarrow{\varphi} & B & \xrightarrow{2} & B & \xrightarrow{\delta} & H^1M_1^0[t_1]. \end{array}$$

Here $\varphi(x) = x/2$. If the bottom sequence is exact, then the inclusion i is an isomorphism by [4, Remark 3.11]. To see the exactness, it suffices to show that $\operatorname{Ker}\delta \subset \operatorname{Im} 2$, which is seen by $\delta(x_{1,n}^s/2^{n+1}) = v_2^{2^n(s-1)+2^n-1}h_{20}$ for odd $s > 0$. □

Corollary 2.4. *The image of $\varphi : H^1M_1^0[t_1] \to H^1M_0^1[t_1]$ is zero.*

Proof. Note that each integer $s \geq 0$ is expressed uniquely as $2^{n+1}t + 2^n - 1$ for some $t, n \geq 0$. Therefore, each generator $v_2^sh_{20} \in H^1M_1^0[t_1]$ for $s \geq 0$ is the image of $x_{1,n}^{2t+1}/2^{n+1}$ under δ. □

Theorem 2.5. *$H^sM_0^1[t_1] = 0$ for $s > 0$.*

3. Proof of Theorem 1.1

We will study $H^sM_0^2[t_1]$ for $s \geq 0$ by using the exact sequence

$$(3.1) \qquad \cdots \to H^sM_1^1[t_1] \xrightarrow{\varphi} H^sM_0^2[t_1] \xrightarrow{2} H^sM_0^2[t_1] \xrightarrow{\delta} H^{s+1}M_1^1[t_1] \to \cdots$$

associated to the short exact sequence

$$0 \to M_1^1[t_1] \xrightarrow{\varphi} M_0^2[t_1] \xrightarrow{2} M_0^2[t_1] \to 0, \tag{3.2}$$

where $\varphi(x) = x/2$. Here, $H^*M = \mathrm{Ext}^*_{BP_*(BP)}(BP_*, M)$ as before. Consider the submodules

$$A_2 = v_3K(2)_*[v_3^2, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}) \qquad \text{and}$$

$$A_{21} = v_3K_*^2[v_3^2] \otimes \Lambda(h_{21}, h_{30}, h_{31})$$

of $H^*M_1^1[t_1]$, where $K(2)_* = \boldsymbol{Z}/2[v_2^{\pm 1}]$, $K_*^2 = \boldsymbol{Z}/2[v_2^{\pm 2}]$ and an element x of the modules is considered to be an element x/v_1 of $H^*M_1^1[t_1]$. Put $A_2^0 = A_2/A_{21}$ as a module. Then, it is shown in [8, Th. 6.13] that

$$H^sM_1^1[t_1] = (A_2^0 \otimes \Lambda(\rho_2))^s$$

for $s > 4$ and $H^4M_1^1[t_1] = (A_2^0 \otimes \Lambda(\rho_2))^4 \oplus v_3K_*^2[v_3^2]\{h_{21}h_{30}h_{31}\rho_2\}$, where $(M)^s$ denotes the submodule of M consisting of elements of cohomology dimension s.

The exact sequence (3.1) defines the Bockstein spectral sequence $H^*M_1^1[t_1] \Rightarrow H^*M_0^2[t_1]$. The differential d_1 is defined to be $d_1 = \delta\varphi : H^sM_1^1[t_1] \to H^{s+1}M_1^1[t_1]$ for the maps δ and φ in (3.1). Then we have the following lemma.

LEMMA 3.3. *The differential d_1 of the Bockstein spectral sequence acts on A_2^0 as follows:*

$$d_1(v_2^{2u+1}x) = v_2^{2u}xh_{20}$$

for an integer u and $x \in A_2^0$ with $v_2 \nmid x$.

PROOF. Each cohomology class is represented as follows:

$$h_{20} = [t_2], \qquad h_{21} = [t_2^2], \qquad h_{30} = [t_3] \qquad \text{and} \qquad h_{31} = [t_3^2].$$

For the diagonal map Δ, Quillen's formula $\Delta(t_n) = \Psi_0(n) + \sum_{k=1}^n m_k(\Psi_k(n) - \Delta(t_{n-k})^{p^k})$ together with Hazewinkel's formula shows that $\Delta(t_2) = \Psi_0(2) = t_2 \otimes 1 + 1 \otimes t_2$ and $\Delta(t_3) = \Psi_0(3) - v_1t_2 \otimes t_2 \equiv t_3 \otimes 1 + 1 \otimes t_3 \mod(4, v_1)$, where $\Psi_k(l) = \sum_{i=0}^l t_i^{p^k} \otimes t_{l-i}^{p^{k+i}}$ and $t_0 = 1$. Thus this together with Lemma 2.1 shows that

$$d(v_2) \equiv 2t_2, \quad d(v_3) \equiv 2t_3, \quad d(t_2^2) \equiv 2t_2 \otimes t_2 \quad \text{and} \quad d(t_3^2) \equiv 2t_3 \otimes t_3 \tag{3.4}$$

$\mathrm{mod}(4, v_1)$ in $\Omega^*v_2^{-1}BP_*$. By the definition of the differential of the cobar complex, the element $d(v_2^{2u+1}x/4)$ of $\Omega^*M_0^2[t_1]$ is computed

$$\begin{aligned} d(v_2^{2u+1}x/4) &= d(v_2^{2u+1})x/4 + v_2^{2u+1}d(x/4) \\ &= v_2^{2u}t_2x/2 + v_2^{2u+1}d(x)/4 \\ &= v_2^{2u}xh_{20}/2 + v_2^{2u+1}y/2, \end{aligned}$$

where y is an element of $\Omega^* v_2^{-1}BP_*/(4, v_1^\infty)[t_1]$ such that $d(x) = 2y$. We see that $y \not\equiv \pm v_2^{-1}xh_{20} \mod(4, v_1)$ by (3.4). Note here that $t_3 \otimes t_3$ represents the cohomology class $h_{31}(v_2^{-1}h_{20} + v_2^{-2}h_{21}) + v_2^{-3}v_3^2h_{20}h_{21}$ (see [5, p. 243, (1)]). $\square$

The lemma indicates that h_{21} is redefined as

$$h_{21} = [t_2^2 + v_2t_2]$$

and gives rise to the differential pattern on A_2^0:

$$\begin{array}{cccccccccc}
0 & & 1 & & 2 & & 3 & & 4 & \cdots \\
\hline
 & & v_2v_3h_{20} & \mapsto & v_3h_{20}^2 & & v_2v_3h_{20}^3 & \mapsto & v_3h_{20}^4 & \\
v_2v_3 & \mapsto & v_3h_{20} & & v_2v_3h_{20}^2 & \mapsto & v_3h_{20}^3 & & v_2v_3h_{20}^4 & \mapsto \\
 & & v_2v_3h_{21} & \mapsto & v_3h_{20}h_{21} & & v_2v_3h_{20}^2h_{21} & \mapsto & v_3h_{20}^3h_{21} & \\
 & & & & v_2v_3h_{20}h_{21} & \mapsto & v_3h_{20}^2h_{21} & & v_2v_3h_{20}^3h_{21} & \mapsto \\
 & & & & & & & & & \\
 & & & & v_2v_3h_{20}h_{3i} & \mapsto & v_3h_{20}^2h_{3i} & & v_2v_3h_{20}^3h_{3i} & \mapsto \\
 & & v_2v_3h_{3i} & \mapsto & v_3h_{20}h_{3i} & & v_2v_3h_{20}^2h_{3i} & \mapsto & v_3h_{20}^3h_{3i} & \\
 & & & & v_2v_3h_{21}h_{3i} & \mapsto & v_3h_{20}h_{21}h_{3i} & & v_2v_3h_{20}^2h_{21}h_{3i} & \mapsto \\
 & & & & & & v_2v_3h_{20}h_{21}h_{3i} & \mapsto & v_3h_{20}^2h_{21}h_{3i} & \\
 & & & & & & & & & \\
 & & & & & & v_2v_3h_{20}h_{30}h_{31} & \mapsto & v_3h_{20}^2h_{30}h_{31} & \\
 & & & & v_2v_3h_{30}h_{31} & \mapsto & v_3h_{20}h_{30}h_{31} & & v_2v_3h_{20}^2h_{30}h_{31} & \mapsto \\
 & & & & & & v_2v_3h_{21}h_{30}h_{31} & \mapsto & v_3h_{20}h_{21}h_{30}h_{31} & \\
 & & & & & & & & v_2v_3h_{20}h_{21}h_{30}h_{31} & \mapsto
\end{array}$$

in which $x \mapsto y$ denotes the $d_1(x/v_1) = y/v_1$ for $x/v_1, y/v_1 \in A_2^0$.

Observe the long exact sequence (3.1), and note that the module $\widetilde{A_2}$ given in the introduction is $\operatorname{Im} \varphi$. Then the above differential pattern shows that δ is a monomorphism on $\widetilde{A_2}$, since $\widetilde{A_2}$ is generated by the elements at the tails of the arrows.

Lemma 3.5. *The module $\widetilde{A_2}$ given in the introduction fits in the short exact sequence*

$$0 \to (\widetilde{A_2})^{s-1} \xrightarrow{\delta} (A_2^0)^s \xrightarrow{\varphi} (\widetilde{A_2})^s \to 0$$

for $s > 3$.

Proof of Theorem 1.1. Since $H^sM_1^1[t_1] = (A_2^0 \otimes \Lambda(\rho_2))^s$ for $s > 4$, we have the commutative diagram

$$\begin{array}{ccccccccc}
(\widetilde{A_2} \otimes \Lambda(\rho_2))^{s-1} & \xrightarrow{\delta} & (A_2^0 \otimes \Lambda(\rho_2))^s & \xrightarrow{\varphi} & (\widetilde{A_2} \otimes \Lambda(\rho_2))^s & \xrightarrow{2=0} & (\widetilde{A_2} \otimes \Lambda(\rho_2))^s & \xrightarrow{\delta} & (A_2^0 \otimes \Lambda(\rho_2))^{s+1} \\
\downarrow & & \| & & \downarrow g & & \downarrow g & & \| \\
H^{s-1}M_0^2[t_1] & \xrightarrow{\delta} & H^sM_1^1[t_1] & \xrightarrow{\varphi} & H^sM_0^2[t_1] & \xrightarrow{2} & H^sM_0^2[t_1] & \xrightarrow{\delta} & H^{s+1}M_1^1[t_1]
\end{array}$$

of exact sequences by Lemma 3.5. If we show that the images of the left δ's agree, then the map g is an isomorphism by [4, Remark 3.11]. We denote the maps δ and φ in the top sequence by δ' and φ'. Then $\operatorname{Im}\delta' \subset \operatorname{Im}\delta$. For any $x \notin \operatorname{Im}\delta'$, $\varphi'(x) = x/2 \neq 0$ and $\delta'(x/2) \neq 0$, which shows $g(x/2) \neq 0$ since $\delta' = \delta g$. Therefore, $\varphi(x) = g(\varphi'(x)) = g(x/2) \neq 0$, and $x \notin \operatorname{Im}\delta$. □

Lemma 3.6. *The Bockstein spectral sequence* $H^*M_1^1[t_1] \Rightarrow H^*M_0^2[t_1]$ *splits into two spectral sequences* $\boldsymbol{A}_1 = (A_0 \oplus A_1 \oplus A_{21}) \otimes \Lambda(\rho_2) \Rightarrow \widetilde{\boldsymbol{A}_1}$ *and* $A_2^0 \otimes \Lambda(\rho_2) \Rightarrow \widetilde{A_2} \otimes \Lambda(\rho_2)$. *Here, the module* $\widetilde{\boldsymbol{A}_1}$ *denotes a module fitting in the long exact sequence*

$$0 \to (\boldsymbol{A}_1)^0 \xrightarrow{\varphi} (\widetilde{\boldsymbol{A}_1})^0 \xrightarrow{2} (\widetilde{\boldsymbol{A}_1})^0 \xrightarrow{\delta} (\boldsymbol{A}_1)^1 \xrightarrow{\varphi} \cdots$$
$$\xrightarrow{\delta} (\boldsymbol{A}_1)^s \xrightarrow{\varphi} (\widetilde{\boldsymbol{A}_1})^s \xrightarrow{2} (\widetilde{\boldsymbol{A}_1})^s \xrightarrow{\delta} (\boldsymbol{A}_1)^{s+1} \xrightarrow{\varphi} \cdots.$$

Proof. By Lemma 3.5, we have the subspectral sequence $A_2^0 \otimes \Lambda(\rho_2) \Rightarrow \widetilde{A_2} \otimes \Lambda(\rho_2)$. Furthermore, Lemma 3.5 implies that all elements of $A_2^0 \otimes \Lambda(\rho_2)$ do not survive to the E_2-term of the Bockstein spectral sequence. It follows that the differential d_r acts on $\boldsymbol{A}_1$. Now $\widetilde{\boldsymbol{A}_1}$ is generated by elements $\tilde{x}_r$ such that $2^{r-1}\tilde{x}_r = \tilde{x}_1 = \varphi(x)$ and $\delta(\tilde{x}_r)$'s are linearly independent. □

Remark. $\widetilde{\boldsymbol{A}_1}$ is not determined here. Even the 0-dimensional part $(\widetilde{\boldsymbol{A}_1})^0$ of it is very complicated (see. [6], [9]), though $(\widetilde{\boldsymbol{A}_1})^s = 0$ for $s > 4$.

4. Proof of Theorem 1.2

Recall [8] the spectrum C such that $BP_*(C) = BP_*/(2, v_1^\infty)[t_1]$. Then C fits in the cofiber sequence

$$C \xrightarrow{\varphi} W \wedge T(1) \xrightarrow{2} W \wedge T(1) \to \Sigma C,$$

which induces the short exact sequence

$$0 \to M_1^1[t_1] \xrightarrow{\varphi} M_0^2[t_1] \xrightarrow{2} M_0^2[t_1] \to 0$$

by applying $BP_*(L_2-)$. Let $E_r^{s,t}(X)$ denote the E_r-term of the $v_2^{-1}BP$ based Adams spectral sequence converging to $\pi_{t-s}(L_2X)$. Then the E_2-term is $\operatorname{Ext}^{*,*}_{v_2^{-1}BP_*(v_2^{-1}BP)}(v_2^{-1}BP_*, v_2^{-1}BP_*(X))$, which is isomorphic to $H^*v_2^{-1}BP_*(X)$ by the change of rings theorem of Hovey and Sadofsky [1, Th. 3.1]. Indeed, we use the modified one [3, Th. 3.3]. In our case, we consider the spectral sequences $E_2^*(C) = H^*M_1^1[t_1] \Rightarrow \pi_*(L_2C)$ and $E_2^*(W \wedge T(1)) = H^*M_0^2[t_1] \Rightarrow \pi_*(L_2W \wedge T(1))$.

For the sake of simplicity, we compute differentials by setting $v_2^2 = 1$. In [8, Lemma 7.4], it is shown that for any $v_3^{4t+3}x/v_1 \in E_2^{s,u}(C) \cap A_2$,

$$d_3(v_3^{4t+3}x/v_1) = v_3^{4t+1}xh_{20}^3/v_1 \in E_2^{s+3,u+2}(C). \tag{4.1}$$

The other differentials on $E_r^*(C)$ are trivial except for the differentials

$$d_3(x_n^s\widetilde{h_{20}}/v_1^{a_n}) = \begin{cases} v_3^{2^n(s-1)+4(2^{n-2}-1)/3+1}h_{20}^2h_{21}h_{30}/v_1 & n \text{ is even} \\ v_2v_3^{2^n(s-1)+8(2^{n-3}-1)/3+1}h_{20}^2h_{21}h_{31}/v_1 & n \text{ is odd} \end{cases} \quad \text{and} \tag{4.2}$$

$$d_3(x_n^sg_{n+1}\widetilde{h_{20}}/v_1^{a_n}) = \begin{cases} v_3^{2^ns-3}h_{20}^2h_{21}h_{30}h_{31}/v_1 & n \text{ is even} \\ v_2v_3^{2^ns-3}h_{20}^2h_{21}h_{30}h_{31}/v_1 & n \text{ is odd} \end{cases}$$

for $n \geq 2$ and odd $s > 0$, and a v_2-multiple of them ([8, Lemmas 7.6 and 7.8]). Here $\widetilde{h_{20}}$ is defined as the class represented by the cocycle $\widetilde{t_2}$ in the congruence $d(v_3^4) \equiv 2v_1^2\widetilde{t_2} \bmod (4)$, whose leading term is $v_2^3v_3^2t_2$.

Lemma 4.3. *In the Adams-Novikov E_3^*-term for $\pi_*(L_2W \wedge T(1))$,*

$$d_3(v_3^3x/2v_1) = v_2v_3xh_{21}h_{20}^2/2v_1 \qquad \text{and} \qquad d_3(v_2v_3^3y/2v_1) = v_2v_3yh_{20}^3/2v_1$$

for $x \in K_*^2[v_3^4] \otimes \Lambda(h_{30}, h_{31})$ and $y \in K_*^2[v_3^4, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31})$, and

$$d_3(x_n^s\widetilde{h_{20}}/2v_1^{a_n}) = \begin{cases} v_2v_3^{2^n(s-1)+4(2^{n-2}-1)/3+1}h_{20}^3h_{30}/2v_1 & n \text{ is even} \\ v_2v_3^{2^n(s-1)+8(2^{n-3}-1)/3+1}h_{20}^2h_{21}h_{31}/2v_1 & n \text{ is odd}, \end{cases}$$

$$d_3(x_n^sg_{n+1}\widetilde{h_{20}}/2v_1^{a_n}) = \begin{cases} v_2v_3^{2^ns-3}h_{20}^3h_{30}h_{31}/2v_1 & n \text{ is even} \\ v_2v_3^{2^ns-3}h_{20}^2h_{21}h_{30}h_{31}/2v_1 & n \text{ is odd}, \end{cases}$$

$$d_3(v_2x_n^s\widetilde{h_{20}}/2v_1^{a_n}) = \begin{cases} v_2v_3^{2^n(s-1)+8(2^{n-3}-1)/3+1}h_{20}^2h_{21}h_{31}/2v_1 & n \text{ is even} \\ v_2v_3^{2^n(s-1)+4(2^{n-2}-1)/3+1}h_{20}^3h_{30}/2v_1 & n \text{ is odd} \end{cases} \quad \textit{and}$$

$$d_3(v_2x_n^sg_{n+1}\widetilde{h_{20}}/2v_1^{a_n}) = \begin{cases} v_2v_3^{2^ns-3}h_{20}^2h_{21}h_{30}h_{31}/2v_1 & n \text{ is even} \\ v_2v_3^{2^ns-3}h_{20}^3h_{30}h_{31}/2v_1 & n \text{ is odd} \end{cases}$$

for positive integers s and n with $n > 1$. Here the equations are all up to sign.

Proof. Note that $v_3xh_{20}^3/2v_1 = v_2v_3xh_{21}h_{20}^2/2v_1$ in $E_3^*(W \wedge T(1))$, since $\delta(v_2v_3xh_{20}^2/2v_1) = v_3xh_{20}^3/v_1 + v_2v_3xh_{21}h_{20}^2/v_1$ by Lemma 3.3. In the same manner as this, we have the relations $v_3^{2^n(s-1)+4(2^{n-2}-2^{\varepsilon(n)})/3+1}h_{20}^2h_{21}h_{3\varepsilon(n)}/2v_1 = v_2v_3^{2^n(s-1)+4(2^{n-2}-2^{\varepsilon(n)})/3+1}h_{20}^3h_{3\varepsilon(n)}/2v_1$ and $v_3^{2^ns-3}h_{20}^2h_{21}h_{30}h_{31}/2v_1 = v_2v_3^{2^ns-3}h_{20}^3h_{30}h_{31}/2v_1$, since $h_{21}^2 = h_{20}^2$. Then the differentials in (4.1) and (4.2) of the form $d_3(x) = y$ (resp. $d_3(x) = v_2y$) yield differentials $d_3(x/2) = v_2z/2$ and $d_3(v_2x/2) = v_2y/2$ (resp. $d_3(x/2) = v_2y/2$ and $d_3(v_2x/2) = v_2z/2$) of $E_3^*(W \wedge T(1))$, where z is an element such that $\delta(w) = y - z \in H^*M_1^1[t_1]$ for an element w of $H^*M_0^2[t_1]$. □

Corollary 4.4. *The module* $\widetilde{A_{21}}$ *given in Introduction is a submodule of* $H^*M_0^2[t_1]$. *In other words, the map sending an element* $x \in \widetilde{A_{21}}$ *to* $x/2v_1 \in H^*M_0^2[t_1]$ *is a monomorphism.*

Proof. It suffices to show that $x/2v_1 \neq 0 \in H^*M_0^2[t_1]$ for $x \in \widetilde{A_{21}}$. The first equation of Lemma 4.3 shows $d_3(x/2v_1) \neq 0$. □

Corollary 4.5. *After a suitable modification of* $\widetilde{\boldsymbol{A}_1}^0$, *the* $v_2^{-1}BP$ *based Adams differentials* d_3 *originating in* $\widetilde{\boldsymbol{A}_1}^0$ *are all zero.*

Proof. The only non-trivial differentials originating in $\widetilde{\boldsymbol{A}_1}^0$ are given in Lemma 4.3, and their targets are all in the image of d_3 originating in $(\widetilde{A_2} \oplus \widetilde{A_{21}}) \otimes \Lambda(\rho_2)$. □

Remark. This modification of $\widetilde{\boldsymbol{A}_1}^0$ does not change the additive structure of $\widetilde{\boldsymbol{A}_1}^0$ nor the E_2-term $H^*M_0^2[t_1]$. In fact, each generator $x \in \widetilde{\boldsymbol{A}_1}^0$ is just replaced by $x + y$ for some $y \in (\widetilde{A_2} \oplus \widetilde{A_{21}}) \otimes \Lambda(\rho_2)$.

Theorem 4.6. *The* E_4*-term of the* $v_2^{-1}BP$ *based Adams spectral sequence contains* $\widehat{A_2} \otimes \Lambda(\rho_2)$, *which is obtained from the subgroup* $\widetilde{A_2} \otimes \Lambda(\rho_2)$ *of the* E_2*-term. Here,* $\widehat{A_2}$ *is the module given in Theorem* 1.2.

Proof. The $v_2^{-1}BP$ based Adams differential d_3 makes $(\widetilde{A_2}, d_3)$ a differential module by Lemma 4.3, whose homology is

$$\widehat{A_2}' = v_2v_3K_*^2[v_3^4, h_{20}]/(h_{20}^3) \otimes \Lambda(h_{21}, h_{30}, h_{31}).$$

We decompose $\widehat{A_2}'$ into the direct sum of the two modules

$$\widehat{A_{21}}' = v_2v_3K_*^2[v_3^4] \otimes \Lambda(h_{20}, h_{21}, h_{30}, h_{31}) \oplus v_2v_3h_{20}^2K_*^2[v_3^4] \otimes \Lambda(h_{30}, h_{31}) \qquad \text{and}$$

$$\widehat{A_{22}}' = v_2v_3h_{20}^2h_{21}K_*^2[v_3^4] \otimes \Lambda(h_{30}, h_{31}).$$

The first differential in Lemma 4.3 gives the isomorphism $d_3 : \widetilde{A_{21}} \cong \widehat{A_{22}}'$, and we obtain the theorem by setting

$$\widehat{A_2} = \widehat{A_{21}}'.$$ □

Proposition 4.7. *The* $v_2^{-1}BP$ *based Adams spectral sequence converging to* $\pi_*(L_2W \wedge T(1))$ *collapses from the* E_4*-term. That is,* $E_4^* = E_\infty^*$.

Proof. Since $(\widetilde{\boldsymbol{A}_1})^s = 0$ for $s > 4$ and $(\widehat{A_2} \otimes \Lambda(\rho_2))^s = 0$ for $s > 5$, we see that $E_5^s = 0$ for $s > 5$. Therefore, the differentials d_r are all trivial for $r > 5$. Suppose that $d_5(x/2^l) = y/2$ for $x/2^l \in \widetilde{\boldsymbol{A}_1}$. Then $y/2 \in \widehat{A_{21}}'$, and so $\delta(y/2) \neq 0 \in E_2^6(C)$. Send the relation $d_5(x/2^l) = y/2$ by δ, and we see that $d_5(\delta(x/2^l)) = \delta(y/2) \in E_5^6(C)$. Since $E_5^6(C) = 0$ by [8, Corollary 7.9], there is an element $z \neq 0 \in E_3^3$ such that $d_3(z) = \delta(y/2)$. Then, $\varphi_*(z)$ must be hit by $x/2^{l+1}$ under d_3. By Lemma 4.3, there is no such differential. $\square$

From the proof of this together with Corollary 4.5, we obtain the following:

Corollary 4.8. *The differentials* d_r *of the* $v_2^{-1}BP$ *based Adams spectral sequence for* $\pi_*(L_2W \wedge T(1))$ *are trivial on* $\widetilde{A_1}^0 \subset E_2^*$.

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