HIROSHIMA MATH. J.
33 (2003), 379–389

A note on the Hopf homomorphism of a Toda bracket and its application

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(Received May 22, 2003)

ABSTRACT. The purpose of the present note is to extend a formula between the Toda bracket and Hopf homomorphism. As an application, we show that the generator of the 2-primary component of the homotopy group $\pi_{12}(S^5)$ is taken as a representative of a specific Toda bracket. And we shall give a short proof of the existence of the unstable Adams map.

1. Introduction

In this note all spaces, maps and homotopies are based. For a space X, we denote by ΣX a suspension of X and by $X \wedge X$ a smash product of X and itself. The Toda bracket [8] has some properties relative to the Hopf homomorphism $H : [\Sigma K, S^{n+1}] \to [\Sigma K, S^{2n+1}]$, where K is a CW-complex and S^n is the n-sphere. We shall extend the n-sphere to a CW-complex with exactly one vertex, that is, we define a generalized Hopf homomorphism $H : [\Sigma K, \Sigma A] \to [\Sigma K, \Sigma(A \wedge A)]$ for a CW-complex A with one vertex and prove the properties between the Toda bracket and this Hopf homomorphism. Then we can apply them to spaces of suspensions of the real projective plane and the quasi-quaternionic projective plane. From this extension, we find out a roundabout approach to determine the 2-primary component π^7_{14} of the homotopy group $\pi_{14}(S^7)$ [8]. And we shall prove a short and intuitive proof of the existence of the unstable Adams map [6], [3].

The authors wish to thank Professors Kachi, Matsuda and Tamaki for useful comments and advices.

2. Generalizations of Toda formulas

We shall recall the James construction [1]. Let A be a Hausdorff space. Denote by A_∞ the reduced product space of A [1]. Let a_0 be the base point of

2000 *Mathematics Subject Classification*. Primary 55Q25; Secondary 55Q25, 55Q40.
Key words and phrases. Hopf homomorphism, Toda bracket.
* Partially supported by Grant-in-Aid for Scientific Research (No. 15540067 (c), (2)), Japan Society for the Promotion of Science.

A, then A_∞ is the free monoid with the unit element a_0 generated by A. A point of A_∞ is a formal product $a_1 \dots a_n$ of elements of A. For a Hausdorff space B and a map $f : A \to B$, a map $f_\infty : A_\infty \to B_\infty$ is given by $f_\infty(a_1 \dots a_n) = f(a_1) \dots f(a_n)$.

Let A_2 be a subspace of A_∞ which consists of formal products of two elements of A. Consider a map $h' : A_2 \to A \wedge A$ defined by $h'(a_1 a_2) = a_1 \wedge a_2$. By the method of James [1], we can construct a map $h : A_\infty \to (A \wedge A)_\infty$ such that the restriction $h|_A : A \to (A \wedge A)_\infty$ is the constant map to the base point of $(A \wedge A)_\infty$. This map h is defined directly by the formula

$$h(a_1 \dots a_n) = \prod_\sigma (a_{\sigma(1)} \wedge a_{\sigma(2)}),$$

where $\sigma : \{1, 2\} \to \{1, \dots, n\}$ is a map such that $\sigma(1) < \sigma(2)$ and the order of the product $\prod$ in $(A \wedge A)_\infty$ is lexicographic from the right. The map h satisfies the following.

Lemma 2.1. *Let A and B be Hausdorff spaces and $f : A \to B$ be a map. Then $h \circ f_\infty = (f \wedge f)_\infty \circ h$.*

We denote by ΩX a loop space of a space X. Let A be a CW-complex with exactly one vertex and let $j' : A \to \Omega\Sigma A$ be a map which is defined by the formula $j'(a)(t) = (a, t)$. The map j' can be extended to the reduced product space A_∞ of A such a way that $a_1 \dots a_n$ is mapped to a loop in ΣA which is represented by a suitably weighted sum of the loops $j'(a_1), \dots, j'(a_n)$ [1]. Denote by $j : A_\infty \to \Omega\Sigma A$ the resulted map. Then j is a week homotopy equivalence map [1] and hence $j_* : [K, A_\infty] \to [K, \Omega\Sigma A]$ is bijective for an arbitrary CW-complex K. Let $\Omega_0 : [\Sigma K, \Sigma A] \cong [K, \Omega\Sigma A]$ be the adjoint isomorphism. Define an isomorphism Ω_1 by

$$\Omega_1 = j_*^{-1} \circ \Omega_0 : [\Sigma K, \Sigma A] \cong [K, A_\infty].$$

Define a generalized Hopf homomorphism H by

$$H = \Omega_1^{-1} \circ h_* \circ \Omega_1 : [\Sigma K, \Sigma A] \to [\Sigma K, \Sigma(A \wedge A)].$$

From (1.11) of [8], the following diagram is commutative:

$$\begin{tikzcd} {[K, A]} \arrow[r, "\Sigma"] \arrow[dr, "i_*"'] & {[\Sigma K, \Sigma A]} \arrow[d, "\cong"', "\Omega_1"] \\ & {[K, A_\infty],} \end{tikzcd} \tag{1}$$

where $i : A \to A_\infty$ is the inclusion.

Now we obtain the following propositions.

Proposition 2.2. *Let A and B be CW-complexes with one vertex and let K and L be CW-complexes. Let $\alpha \in [\Sigma K, \Sigma A]$, $\beta \in [L, K]$ and $\gamma \in [A, B]$. Then $H(\alpha \circ \Sigma\beta) = H(\alpha) \circ \Sigma\beta$ and $H(\Sigma\gamma \circ \alpha) = \Sigma(\gamma \wedge \gamma) \circ H(\alpha)$.*

Proposition 2.3. *Let A be a CW-complex with one vertex and let K, L and M be CW-complexes. For $n \geq 1$, let $\alpha \in [\Sigma^n K, \Sigma A]$, $\beta \in [L, K]$ and $\gamma \in [M, L]$ satisfy the conditions that $\alpha \circ \Sigma^n\beta = 0$ and $\beta \circ \gamma = 0$. Then $H\{\alpha, \Sigma^n\beta, \Sigma^n\gamma\}_n \subset \{H(\alpha), \Sigma^n\beta, \Sigma^n\gamma\}_n$.*

Propositions 2.2 and 2.3 are partially generalized versions of Propositions 2.2 and 2.3 of [8]. The proofs are similar to [8].

For any spaces X, Y and an element $\alpha \in [X, Y]$, we denote by CX a cone of X and by $C_\alpha = Y \cup_\alpha CX$ a mapping cone of α. We use the identification $\Sigma C_\alpha = C_{\Sigma\alpha}$.

Let A be a CW-complex with one vertex and K be a CW-complex. Let $\partial : [C\Sigma K, \Sigma K; A_\infty, A] \to [\Sigma K, A]$ be the connecting homomorphism of the exact sequence of the pair (A_∞, A). Define a homomorphism Γ by

$$\Gamma = \Omega_1^{-1} \circ h_* \circ \partial^{-1} : \partial[C\Sigma K, \Sigma K; A_\infty, A] \to [\Sigma^3 K, \Sigma(A \wedge A)]/H[\Sigma^3 K, \Sigma A],$$

where $H[\Sigma^3 K, \Sigma A] = (\Omega_1^{-1} \circ h_*)(\mathrm{Ker}\, \partial)$.

We shall prove a partially generalized version of Proposition 2.6 of [8] which also generalizes Proposition 3.4 of [2].

Proposition 2.4. *Let A be a CW-complex with one vertex and L, M and K be CW-complexes. Consider the elements $\alpha \in [L, A]$, $\beta \in [\Sigma K, L]$ and $\gamma \in [M, \Sigma K]$ with the conditions that $\Sigma(\alpha \circ \beta) = 0$ and $\beta \circ \gamma = 0$. Then $H\{\Sigma\alpha, \Sigma\beta, \Sigma\gamma\}_1 = -\Gamma(\alpha \circ \beta) \circ \Sigma^2\gamma$.*

Proof. The proof is done by the parallel argument to that of Proposition 2.6 of [8]. Let $i : A \to A_\infty$ be the inclusion. By the relation $\Sigma(\alpha \circ \beta) = 0$ and (1), we have $i_*(\alpha) \circ \beta = 0$. Then there is an extension $\overline{i_*(\alpha)} \in [C_\beta, L; A_\infty, A]$ of $i_*(\alpha)$ and let $\bar{a} : (C_\beta, L) \to (A_\infty, A)$ be a representative of $\overline{i_*(\alpha)}$. Consider the exact sequence of the pair (C_β, L) and let $i' : L \to C_\beta$ be the inclusion. Since $i'_*(\beta) = 0$, there is an element $\bar{\beta} \in [C\Sigma K, \Sigma K; C_\beta, L]$ such that the restriction $\bar{b}|_{\Sigma K} : \Sigma K \to L$ of a representative $\bar{b}$ of $\bar{\beta}$ represents β. Then there exists a map $a' : \Sigma^2 K \to (A \wedge A)_\infty$ such that the following diagram is commutative:

$$\begin{tikzcd}
(C\Sigma K, \Sigma K) \arrow[r, "\bar{b}"] \arrow[dr, "p_1"'] & (C_\beta, L) \arrow[r, "\bar{a}"] \arrow[d, "p_2"] & (A_\infty, A) \arrow[d, "h"] \\
 & (\Sigma^2 K, k_0) \arrow[r, "a'"'] & ((A \wedge A)_\infty, w_0),
\end{tikzcd}$$

where p_1, p_2 are the shrinking maps and k_0, w_0 are the base points of $\Sigma^2 K$, $(A \wedge A)_\infty$, respectively.

Let $\tilde{\gamma} \in [\Sigma M, C_\beta]$ be a coextension of γ. From the above diagram and the property of the coextension, we obtain $h_*(\overline{i_*(\alpha)} \circ \tilde{\gamma}) = (a'_* \circ p_{2*})(\tilde{\gamma}) = a'_*(\Sigma\gamma)$. By Proposition 1.7 of [8], $h_*(\overline{i_*(\alpha)} \circ \tilde{\gamma}) \in h_*\{i_*(\alpha), \beta, \gamma\}$. Since the restriction $(\bar{a} \circ \bar{b})|_{\Sigma K}$ represents $\alpha \circ \beta$, we have that $a' \circ p_1$ and also a' represent an element of $h_*(\partial^{-1}(\alpha \circ \beta))$. Then $a'_*(\Sigma\gamma) \in h_*(\partial^{-1}(\alpha \circ \beta)) \circ \Sigma\gamma$. Hence $h_*(\overline{i_*(\alpha)} \circ \tilde{\gamma})$ is a common element of $h_*\{i_*(\alpha), \beta, \gamma\}$ and $h_*(\partial^{-1}(\alpha \circ \beta)) \circ \Sigma\gamma$. By Propositions 1.2, 1.3 of [8] and (1), we have

$$\begin{aligned} H\{\Sigma\alpha, \Sigma\beta, \Sigma\gamma\}_1 &= (\Omega_1^{-1} \circ h_* \circ j_*^{-1} \circ \Omega_0)\{\Sigma\alpha, \Sigma\beta, \Sigma\gamma\}_1 \\ &= -(\Omega_1^{-1} \circ h_* \circ j_*^{-1})\{(\Omega_0 \circ \Sigma)(\alpha), \beta, \gamma\} \\ &\supset -(\Omega_1^{-1} \circ h_*)\{(\Omega_1 \circ \Sigma)(\alpha), \beta, \gamma\} \\ &= -\Omega_1^{-1}(h_*\{i_*(\alpha), \beta, \gamma\}). \end{aligned}$$

And from (1.12) of [8],

$$\begin{aligned} -\Gamma(\alpha \circ \beta) \circ \Sigma^2\gamma &= -\Omega_1^{-1}(h_*(\partial^{-1}(\alpha \circ \beta))) \circ \Sigma^2\gamma \\ &= -\Omega_1^{-1}(h_*(\partial^{-1}(\alpha \circ \beta)) \circ \Sigma\gamma). \end{aligned}$$

Then it follows that $H\{\Sigma\alpha, \Sigma\beta, \Sigma\gamma\}_1$ and $-\Gamma(\alpha \circ \beta) \circ \Sigma^2\gamma$ have the common element $-\Omega_1^{-1}(h_*(\overline{i_*(\alpha)} \circ \tilde{\gamma}))$. From Lemma 1.1 of [8] and Proposition 2.2, $H\{\Sigma\alpha, \Sigma\beta, \Sigma\gamma\}_1$ is a coset of the subgroup

$$H([\Sigma^3 K, \Sigma A] \circ \Sigma^2\gamma + \Sigma\alpha \circ \Sigma[\Sigma M, L]) = H[\Sigma^3 K, \Sigma A] \circ \Sigma^2\gamma,$$

and, by the definition of Γ, $-\Gamma(\alpha \circ \beta) \circ \Sigma^2\gamma$ is a coset of the same subgroup. Then we obtain $H\{\Sigma\alpha, \Sigma\beta, \Sigma\gamma\}_1 = -\Gamma(\alpha \circ \beta) \circ \Sigma^2\gamma$ and the proof of the proposition is completed. $\square$

Let A be an m-connected CW-complex with one vertex and let K be a CW-complex. By the theorem of Blakers-Massey, the map h induces an isomorphism

$$h_* : [C\Sigma K, \Sigma K; A_\infty, A] \cong [\Sigma^2 K, (A \wedge A)_\infty]$$

for $\dim K \leq 3m - 1$. Under the condition that $\dim K \leq 3m - 1$, define a homomorphism Δ by

$$\Delta = \partial \circ h_*^{-1} \circ \Omega_1 : [\Sigma^3 K, \Sigma(A \wedge A)] \to [\Sigma K, A].$$

From the definitions of the maps Γ and Δ, we have

$$\alpha \in \Gamma(\Delta(\alpha)) \tag{2}$$

for $\alpha \in [\Sigma^3 K, \Sigma(A \wedge A)]$.

Finally we obtain a partially generalized version of Proposition 2.5 of [8]. We can prove it by the parallel argument to [8].

Proposition 2.5. *Let A be an m-connected CW-complex with one vertex and let K, L be CW-complexes with the conditions that* $\dim K \leq 3m-1$ *and* $\dim L \leq 3m-1$. *Let $\alpha \in [\Sigma^3 K, \Sigma(A \wedge A)]$ and $\beta \in [\Sigma L, \Sigma K]$. Then $\Delta(\alpha \circ \Sigma^2 \beta) = \Delta(\alpha) \circ \beta$.*

3. The generators of π_{n+7}^n for $5 \leq n \leq 7$ represent Toda brackets

In this section and the following section, we sometimes identify a map with its homotopy class. We denote by ι_X the homotopy class of the identity map of a space X and let $\iota_n = \iota_{S^n}$. By Proposition 1.9 of [8], we obtain the following.

Lemma 3.1. *For any element $\alpha \in [X, Y]$, let $i : Y \to C_\alpha$ and $p : C_\alpha \to \Sigma X$ be the inclusion and collapsing maps, respectively. Then the Toda bracket $\{\Sigma\alpha, p, i\} \subset [\Sigma Y, \Sigma Y]$ is represented by $-\iota_{\Sigma Y}$.*

Let $\mathbf{R}P^2$ be the real projective plane and set $M^n = \Sigma^{n-2}\mathbf{R}P^2$ for $n \geq 2$. We denote by $i_n : S^{n-1} \to M^n$ and $p_n : M^n \to S^n$ the inclusion and collapsing maps, respectively. Let $\eta_2 \in \pi_3(S^2)$ be the Hopf map. Set $\eta_n = \Sigma^{n-2}\eta_2$ and $\eta_n^2 = \eta_n \circ \eta_{n+1}$ for $n \geq 2$. Since $\eta_3 \circ 2\iota_4 = 0$ and $2\iota_3 \circ \eta_3 = 0$, there are an extension $\bar{\eta}_3 \in [M^5, S^3]$ and a coextension $\tilde{\eta}_3 \in \pi_5(M^4)$ of η_3. We set $\bar{\eta}_n = \Sigma^{n-3}\bar{\eta}_3$ and $\tilde{\eta}_n = \Sigma^{n-3}\tilde{\eta}_3$ for $n \geq 3$. Note that there exists a lift $\tilde{\eta}_2 \in \pi_4(M^3)$ of η_3 such that $\Sigma\tilde{\eta}_2 = \tilde{\eta}_3$ [5]. We note the following.

$$[M^{n+2}, S^n] = \mathbf{Z}_4\{\bar{\eta}_n\} \qquad \text{and} \qquad 2\bar{\eta}_n = \eta_n^2 p_{n+2} \quad \text{for } n \geq 3. \tag{3}$$

$$\pi_{n+2}(M^{n+1}) = \mathbf{Z}_4\{\tilde{\eta}_n\} \qquad \text{and} \qquad 2\tilde{\eta}_n = i_{n+1}\eta_n^2 \quad \text{for } n \geq 2. \tag{4}$$

Let ν' be a generator of $\pi_6^3 \cong \mathbf{Z}_4$ [8] and let $Q_2 = S^3 \cup_{\nu'} e^7$. We denote by $i_Q : S^3 \to Q_2$ and $p_Q : Q_2 \to S^7$ the inclusion and collapsing maps, respectively. We recall [8] that

$$\{2\iota_n, \eta_n, 2\iota_{n+1}\} = \eta_n^2 \qquad \text{for } n \geq 3 \tag{5}$$

and

$$\bar{\eta}_3\tilde{\eta}_4 \in \{\eta_3, 2\iota_4, \eta_4\} \ni \nu' \qquad \bmod 2\nu' = \eta_3^3. \tag{6}$$

A Toda bracket $\{\eta_3, \Sigma\nu', p_Q\} \subset [\Sigma Q_2, S^3]$ is well-defined because $\eta_3\Sigma\nu' = 0$ and $(\Sigma\nu')p_Q = 0$. By Lemma 3.1, we obtain $-\iota_4 \in \{\Sigma\nu', p_Q, i_Q\}$. Then the

Toda bracket $\{\eta_3, \Sigma\nu', p_Q\}$ is represented by an extension $\bar{\eta}_3'$ of η_3. We set $\bar{\eta}_n' = \Sigma^{n-3}\bar{\eta}_3'$ for $n \geq 3$. The first relation of the following lemma is pointed out by Oda.

Lemma 3.2. *$\{2\iota_3, \eta_3, \Sigma\nu'\} = 0$ and $\bar{\eta}_n'$ is of order 2 for $n \geq 3$.*

Proof. By [8], $\pi_5(S^3) = \mathbf{Z}_2\{\eta_3^2\}$, $\pi_8(S^3) = \mathbf{Z}_2\{\nu'\eta_6^2\}$ and $\Sigma^2\nu' = 2\nu_5$. Then the indeterminacy of a Toda bracket $\{2\iota_3, \eta_3, \Sigma\nu'\} \subset \pi_8(S^3)$ is

$$\pi_5(S^3) \circ \Sigma^2\nu' + 2\iota_3 \circ \pi_8(S^3) = 0.$$

So $\{2\iota_3, \eta_3, \Sigma\nu'\}$ consists of a single element 0 or $\nu'\eta_6^2$. By the fact that $\pi_6(S^4) = \mathbf{Z}_2\{\eta_4^2\}$, $\eta_5\nu_6 = 0$ [8] and by Propositions 1.2, 1.3 of [8] and (5), we have

$$\begin{aligned}
-\Sigma\{2\iota_3, \eta_3, \Sigma\nu'\} &\subset \{2\iota_4, \eta_4, 2\nu_5\}_1 \\
&\supset \{2\iota_4, \eta_4, 2\iota_5\}_1 \circ \nu_6 \\
&= \eta_4^2\nu_6 \\
&= 0 \qquad \bmod \pi_6(S^4) \circ 2\nu_6 + 2\iota_4 \circ \Sigma\pi_8(S^3) = 0.
\end{aligned}$$

Hence $\Sigma\{2\iota_3, \eta_3, \Sigma\nu'\} = 0$. From the fact that $\Sigma : \pi_8(S^3) \to \pi_9(S^4)$ is a monomorphism, the first half of the lemma is proved.

By Proposition 1.4 of [8] and by the first half, we see that

$$2\bar{\eta}_3' = 2\iota_3 \circ \bar{\eta}_3' \in 2\iota_3 \circ \{\eta_3, \Sigma\nu', p_Q\} = -\{2\iota_3, \eta_3, \Sigma\nu'\} \circ \Sigma p_Q = 0.$$

So the order of $\bar{\eta}_n'$ is 2 for $n \geq 3$. This leads to the second half, completing the proof. □

By Lemma 3.2, we can define a Toda bracket $\{\eta_3, 2\iota_4, \bar{\eta}_4'\} \subset [\Sigma^3 Q_2, S^3]$. By (6), $\{\eta_3, 2\iota_4, \bar{\eta}_4'\}$ is represented by $\overline{\nu'}$ or $-\overline{\nu'}$, where $\overline{\nu'}$ is an extension of ν'. Let ν_n be a generator of $\pi_{n+3}^n \cong \mathbf{Z}_8$ for $n \geq 5$ [8]. By Lemma 3.1, we obtain that a Toda bracket $\{\nu_5, \Sigma^5\nu', \Sigma^4 p_Q\} \subset [\Sigma^5 Q_2, S^5]$ is represented by an extension $\bar{\nu}_5'$ of ν_5. We set $\bar{\nu}_n' = \Sigma^{n-5}\bar{\nu}_5'$ for $n \geq 5$. By (5), $2\iota_3 \circ \{\eta_3, 2\iota_4, \bar{\eta}_4'\} = \eta_3^2\bar{\eta}_5'$. Then we obtain

$$2\overline{\nu'} = \eta_3^2\bar{\eta}_5'. \tag{7}$$

Let σ''' be the generator of $\pi_{12}^5 \cong \mathbf{Z}_2$ [8]. By making use of the cofibration starting with ν', we obtain

$$2\bar{\nu}_5' \equiv \Sigma^2\overline{\nu'} \quad \bmod \sigma'''\Sigma^5 p_Q \qquad \text{and} \qquad 4\bar{\nu}_5' = \eta_5^2\bar{\eta}_7', \tag{8}$$

and so $\bar{\nu}_n'$ is of order 8 for $n \geq 5$.

We consider the homotopy exact sequence of a pair (Q_2, S^3):

$$\pi_7(S^3) \xrightarrow{i_{Q*}} \pi_7(Q_2) \to \pi_7(Q_2, S^3) \xrightarrow{\partial} \pi_6(S^3).$$

By the Blakers-Massey theorem, we obtain $\pi_7(Q_2, S^3) \cong \pi_7(S^7) \cong \mathbf{Z}$ and $\operatorname{Im}\partial = \mathbf{Z}_4\{\nu'\}$. Since $\pi_7(S^3) = \mathbf{Z}_2\{\nu'\eta_6\}$ [8] and $i_{Q*}(\nu') = 0$, $i_{Q*}(\pi_7(S^3)) = 0$. So we get that $\pi_7(Q_2) = \mathbf{Z}\{\widetilde{4\iota_6}\}$, where $\widetilde{4\iota_6}$ is a coextension of $4\iota_6$. We set $\widetilde{4\iota_n} = \Sigma^{n-6}\widetilde{4\iota_6}$ for $n \geq 6$. By the similar argument, we obtain the following.

Lemma 3.3. *$\pi_{n+7}(\Sigma^n Q_2) = \mathbf{Z}\{\widetilde{4\iota_{n+6}}\} \oplus \mathbf{Z}_2\{(\Sigma^n i_Q)\nu_{n+3}\eta_{n+6}\}$ for $n = 1, 2$ and $\pi_{n+7}(\Sigma^n Q_2) = \mathbf{Z}\{\widetilde{4\iota_{n+6}}\}$ for $n = 0$ and $n \geq 3$.*

We recall that $\pi_{10}(S^5) = \mathbf{Z}_2\{\nu_5\eta_8^2\}$, $\pi_{11}(S^6) = \mathbf{Z}\{\Delta(\iota_{13})\}$, $\pi_{n+5}(S^n) = 0$ for $n \geq 7$ and $\pi_{n+6}(S^n) = \mathbf{Z}_2\{\nu_n^2\}$ for $n \geq 5$ [8]. As is easily seen, we obtain the following.

Lemma 3.4. *$[M^{11}, S^5] = \mathbf{Z}_2\{\nu_5\eta_8\bar{\eta}_9\} \oplus \mathbf{Z}_2\{\nu_5^2 p_{11}\}$ and $[M^{n+6}, S^n] = \mathbf{Z}_2\{\nu_n^2 p_{n+6}\}$ for $n \geq 6$.*

By the fact that $2\nu' = \eta_3^3$, $\Delta(\iota_9) = \pm(2\nu_4 - \Sigma\nu')$ [8] and by (3), we obtain

$$\eta_4^2\bar{\eta}_6'(\Sigma^4 i_Q)\bar{\eta}_7 = 2(\Sigma\nu')\bar{\eta}_7 = (\Sigma\nu')\eta_7^2 p_9 = \Delta(\iota_9) \circ \eta_7^2 p_9. \tag{9}$$

By the fact that $\Sigma^2\nu' = 2\nu_5$, $\Delta(\iota_{11}) = \nu_5\eta_8$ [8] and by (3), we obtain

$$\Sigma^2\overline{\nu'}(\Sigma^5 i_Q)\bar{\eta}_8 = 2\nu_5\bar{\eta}_8 = \nu_5\eta_8^2 p_{10} = \Delta(\iota_{11}) \circ \eta_9 p_{10}. \tag{10}$$

By the fact that $\Delta(\iota_{13}) \in \{\nu_6, \eta_9, 2\iota_{10}\} \bmod 2\Delta(\iota_{13})$ [8] and that $\bar{\eta}_9$ represents a Toda bracket $\{\eta_9, 2\iota_{10}, p_{10}\}$, we obtain

$$\bar{\nu}_6'(\Sigma^6 i_Q)\bar{\eta}_9 = \nu_6\bar{\eta}_9 \in \nu_6 \circ \{\eta_9, 2\iota_{10}, p_{10}\} \ni \Delta(\iota_{13}) \circ p_{11} \qquad \bmod 0. \tag{11}$$

Let σ'' and σ' be generators of $\pi_{13}^6 \cong \mathbf{Z}_4$ and $\pi_{14}^7 \cong \mathbf{Z}_8$, respectively [8]. We recall that the elements σ''', σ'' and σ' have the properties $H(\sigma''') = 4\nu_9$, $H(\sigma'') = \eta_{11}^2$ and $H(\sigma') = \eta_{13}$, respectively [8]. Now we show the following.

Theorem 3.5. (i) *$\sigma''' = \{\eta_5^2\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1$.*
(ii) *$\sigma'' \equiv \{\Sigma^3\overline{\nu'}, (\Sigma^6 i_Q)\bar{\eta}_9, \tilde{\eta}_{10}\}_1 \bmod \Sigma\sigma'''$.*
(iii) *$\sigma' \equiv \alpha \bmod \Sigma\sigma'$ for $\alpha \in \{\bar{\nu}_7', (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}_1$.*

Proof. Each Toda bracket is well-defined by use of (9), (10), (11) and (6). The indeterminacy of $\{\eta_5^2\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1$ is

$$[M^{11}, S^5] \circ \tilde{\eta}_{10} + \eta_5^2\bar{\eta}_7' \circ \Sigma\pi_{11}(\Sigma^4 Q_2).$$

By the fact that $\eta_8\nu_9 = \nu_8\eta_{11} = 0$ [8] and by (6), we obtain $\nu_5\eta_8\bar{\eta}_9\tilde{\eta}_{10} = \pm 2\nu_5\eta_8\nu_9 = 0$ and $\nu_5^2 p_{11}\tilde{\eta}_{10} = \nu_5^2\eta_{11} = 0$. Then, by Lemma 3.4, $[M^{11}, S^5] \circ \tilde{\eta}_{10} = 0$. From the fact that $\pi_{12}(S^7) = 0$, we have $\bar{\eta}'_7 \circ \Sigma\pi_{11}(\Sigma^4 Q_2) = 0$ and hence $\eta_5^2\bar{\eta}'_7 \circ \Sigma\pi_{11}(\Sigma^4 Q_2) = 0$. By Propositions 2.4, 2.5, (2) and (9), we have

$$\begin{aligned} H\{\eta_5^2\bar{\eta}'_7, (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1 &= -\Gamma(\eta_4^2\bar{\eta}'_6(\Sigma^4 i_Q)\bar{\eta}_7) \circ \tilde{\eta}_{10} \\ &= \eta_9^2 p_{11}\tilde{\eta}_{10} \\ &= 4\nu_9. \end{aligned}$$

Since $H : \pi_{12}^5 \to \pi_{12}^9$ is a monomorphism, we have (i).

Next the indeterminacy of $\{\Sigma^3\overline{\nu'}, (\Sigma^6 i_Q)\bar{\eta}_9, \tilde{\eta}_{10}\}_1$ is

$$[M^{12}, S^6] \circ \tilde{\eta}_{11} + \Sigma^3\overline{\nu'} \circ \Sigma\pi_{12}(\Sigma^5 Q_2).$$

By the relation $\nu_9\eta_{12} = 0$ and Lemma 3.4, $[M^{12}, S^6] \circ \tilde{\eta}_{11} = 0$. From the fact that $\pi_{10}^3 = 0$ [8] and by Lemma 3.3 and (7),

$$\Sigma^3\overline{\nu'} \circ \Sigma\pi_{12}(\Sigma^5 Q_2) = \Sigma^3(\overline{\nu'} \circ \pi_{10}(\Sigma^3 Q_2)) \subset \Sigma^3\pi_{10}^3 = 0.$$

By Propositions 2.4, 2.5, (2) and (10), we have

$$H\{\Sigma^3\overline{\nu'}, (\Sigma^6 i_Q)\bar{\eta}_9, \tilde{\eta}_{10}\}_1 = -\Gamma(\Sigma^2\overline{\nu'}(\Sigma^5 i_Q)\bar{\eta}_8) \circ \tilde{\eta}_{11} = \eta_{11} p_{12}\tilde{\eta}_{11} = \eta_{11}^2.$$

By the fact that $\pi_{12}^5 = \mathbf{Z}_2\{\sigma'''\}$ and by the EHP-sequence ((2.11) of [8]), we have (ii).

The indeterminacy of $\{\bar{\nu}'_7, (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}_1$ is

$$[M^{13}, S^7] \circ \tilde{\eta}_{12} + \bar{\nu}'_7 \circ \Sigma\pi_{13}(\Sigma^6 Q_2).$$

From the fact that $\nu_{10}\eta_{13} = 0$ and by Lemma 3.4, $[M^{13}, S^7] \circ \tilde{\eta}_{12} = 0$. By Lemma 3.3 and (8),

$$\bar{\nu}'_7 \circ \Sigma\pi_{13}(\Sigma^6 Q_2) = \Sigma(\bar{\nu}'_6 \circ \pi_{13}(\Sigma^6 Q_2)) \subset \Sigma\pi_{13}^6.$$

By Propositions 2.4, 2.5, (2) and (11), we have

$$H\{\bar{\nu}'_7, (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}_1 = -\Gamma(\bar{\nu}'_6(\Sigma^6 i_Q)\bar{\eta}_9) \circ \tilde{\eta}_{12} \ni p_{13}\tilde{\eta}_{12} = \eta_{13} \qquad \text{mod } 0.$$

By the fact that $\pi_{13}^6 = \mathbf{Z}_4\{\sigma''\}$ and by the EHP-sequence, we have (iii). This completes the proof. $\square$

Finally we show the following.

Corollary 3.6. *$2\sigma'' = \Sigma\sigma'''$ and $2\sigma' = \pm\Sigma\sigma''$.*

Proof. First we note that $2\iota_6 \circ \sigma'' = 2\sigma'' + [\iota_6, \iota_6] \circ \eta_{11}^2$ [9]. Since $\Sigma : \pi_{12}^6 \to \pi_{13}^7$ is an isomorphism, we obtain $[\iota_6, \iota_6] \circ \eta_{11} = 0$ and hence $2\iota_6 \circ \sigma'' = 2\sigma''$. By Theorem 3.5 and (7), we see that

$$\begin{aligned} 2\sigma'' &= 2\iota_6 \circ \sigma'' \\ &\in \{2\Sigma^3\overline{\nu'}, (\Sigma^6 i_Q)\bar{\eta}_9, \tilde{\eta}_{10}\}_1 \\ &= \{\eta_6^2\bar{\eta}_8', (\Sigma^6 i_Q)\bar{\eta}_9, \tilde{\eta}_{10}\}_1 \\ &\supset -\Sigma\{\eta_5^2\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\} \\ &\ni \Sigma\sigma''' \qquad \bmod [M^{12}, S^6] \circ \tilde{\eta}_{11} + \eta_6^2\bar{\eta}_8' \circ \Sigma\pi_{12}(\Sigma^5 Q_2) = 0. \end{aligned}$$

Thus the first half of the corollary is proved.

By Theorem 3.5, we have $\sigma' + k_1\Sigma\sigma'' \in \{\bar{\nu}_7', (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}_1$ for some $k_1 \in \{0, 1, 2, 3\}$. Then

$$2\sigma' + 2k_1\Sigma\sigma'' = 2\iota_7 \circ (\sigma' + k_1\Sigma\sigma'') \in \{2\bar{\nu}_7', (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}.$$

By (8), we obtain $2\bar{\nu}_5' = \Sigma^2\overline{\nu'} + k_2\sigma'''\Sigma^5 p_Q$ for some $k_2 \in \{0, 1\}$. So, by the first relation and Proposition 1.6 of [8], we see that

$$\begin{aligned} \{2\bar{\nu}_7', (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\} &= \{\Sigma^4\overline{\nu'} + 2k_2(\Sigma\sigma'')\Sigma^7 p_Q, (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\} \\ &\subset \{\Sigma^4\overline{\nu'}, (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\} + \{2k_2(\Sigma\sigma'')\Sigma^7 p_Q, (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}. \end{aligned}$$

By Theorem 3.5, $\pm\Sigma\sigma'' \in \{\Sigma^4\overline{\nu'}, (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}$ and by Corollary 3.4 of [7], $2k_2\Sigma\sigma'' \in \{2k_2(\Sigma\sigma'')\Sigma^7 p_Q, (\Sigma^7 i_Q)\bar{\eta}_{10}, \tilde{\eta}_{11}\}$. Furthermore, by Lemmas 3.3 and 3.4, both indeterminacies of these Toda brackets are 0. Hence we obtain $2\sigma' = \pm\Sigma\sigma'' + 2(-k_1 + k_2)\Sigma\sigma''$. This leads to the second assertion, completing the proof. □

4. Existence of the unstable Adams map

First we recall the following [3].

$$\pi_6(M^4) = \mathbf{Z}_4\{\delta\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5\} \qquad \text{and} \qquad 2\delta = i_4\nu'. \tag{12}$$

We show the following.

Lemma 4.1. $\tilde{\eta}_3\eta_5^2\bar{\eta}_7 = \tilde{\eta}_3\nu_5\eta_8 p_9$ *and* $\tilde{\eta}_4\eta_6^2\bar{\eta}_8 = 0$.

Proof. By the fact that $\eta_3^3 = 2\nu'$, $\nu'\eta_6 = \eta_3\nu_4$ [8], $\tilde{\eta}_3 \in \{i_4, 2\iota_3, \eta_3\}$ and by (3), we have

$$\begin{aligned}\tilde{\eta}_3\eta_5^2\bar{\eta}_7 &\in \{i_4, 2\iota_3, \eta_3\} \circ \eta_5^2\bar{\eta}_7\\ &\subset \{i_4, 2\iota_3, 2\nu'\bar{\eta}_6\}\\ &= \{i_4, 2\iota_3, \eta_3\nu_4\eta_7 p_8\}\\ &\supset \{i_4, 2\iota_3, \eta_3\} \circ \nu_5\eta_8 p_9\\ &\ni \tilde{\eta}_3\nu_5\eta_8 p_9 \qquad \bmod \pi_4(M^4) \circ 2(\Sigma\nu')\bar{\eta}_7 + i_4 \circ [M^9, S^3].\end{aligned}$$

Since $\pi_4(M^4) = \mathbf{Z}_2\{i_4\eta_3\}$, $\pi_4(M^4) \circ 2(\Sigma\nu')\bar{\eta}_7 = 0$. Obviously we obtain $[M^9, S^3] = \mathbf{Z}_2\{\nu'\eta_6\bar{\eta}_7\}$. By (12), $i_4\nu'\eta_6\bar{\eta}_7 = 2\delta \circ \eta_6\bar{\eta}_7 = 0$ and hence $i_4 \circ [M^9, S^3] = 0$. This leads to the first assertion. By the fact that $\nu_6\eta_9 = 0$ [8] and by the first half, we obtain $\tilde{\eta}_4\eta_6^2\bar{\eta}_8 = \tilde{\eta}_4\nu_6\eta_9 p_{10} = 0$. This leads to the second half, completing the proof. □

Now we show the existence of the unstable Adams map. By Lemma 4.1, we can define a Toda bracket

$$\{\tilde{\eta}_4\eta_6\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1 \subset \pi_{12}(M^5).$$

By Theorem 3.5,

$$p_5 \circ \{\tilde{\eta}_4\eta_6\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1 \subset \{\eta_5^2\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1 = \sigma'''.$$

So the Toda bracket $\{\tilde{\eta}_4\eta_6\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1$ is represented by a lift $[\sigma''']$ of σ'''. To show that $[\sigma''']$ is extendable to the unstable Adams map from M^{13} to M^5, it suffices to show the following.

LEMMA 4.2. *The order of* $[\sigma''']$ *is two.*

PROOF. By the fact that $\eta_6^3 = 4\nu_6$, $\nu_6\eta_9 = 0$ and $4\tilde{\eta}_4 = 0$, we see that

$$\begin{aligned}2[\sigma'''] &\in \{\tilde{\eta}_4\eta_6\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\} \circ 2\iota_{12}\\ &\subset \{\tilde{\eta}_4\eta_6\bar{\eta}_7', (\Sigma^5 i_Q)\bar{\eta}_8, i_{10}\eta_9^2\}\\ &\subset \{\tilde{\eta}_4, 4\nu_6, \eta_9^2\}\\ &\supset \{0, \nu_6, \eta_9^2\}\\ &\ni 0 \qquad \bmod \pi_{10}(M^5) \circ \eta_{10}^2 + \tilde{\eta}_4 \circ \pi_{12}(S^6).\end{aligned}$$

By [5], $\pi_{10}(M^5) = \mathbf{Z}_4\{i_5\nu_4^2\} \oplus \mathbf{Z}_2\{[\tilde{\eta}_4, i_5] \circ \eta_9\}$. So $\pi_{10}(M^5) \circ \eta_{10}^2 = 0$. By [4], $\tilde{\eta}_4\nu_6 = i_5\nu_4\eta_7^2$. Then $\tilde{\eta}_4\nu_6^2 = i_5\nu_4\eta_7^2\nu_9 = 0$ and hence $\tilde{\eta}_4 \circ \pi_{12}(S^6) = 0$. Then we obtain $2[\sigma'''] = 0$. This completes the proof. □

Finally we shall show that $H(\mu_3) = \sigma'''$, which is a part of Lemma 6.5 of [8]. Since $\Sigma(\bar{\eta}_3[\sigma''']) \in \{\eta_4, 2\iota_5, \sigma'''\}$, we have $\bar{\eta}_3[\sigma'''] \equiv \mu_3 \bmod \eta_3\varepsilon_4$. Then

$\mu_3 \in \{\nu'\eta_6\bar{\eta}'_7, (\Sigma^5 i_Q)\bar{\eta}_8, \tilde{\eta}_9\}_1$. Hence, by Proposition 2.3 of [8] and Theorem 3.5, $H(\mu_3) = \sigma'''$.

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