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On products of distributions involving delta function and its partial derivatives

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Abstract. Li Banghe and Li Yaqing defined in [1] the product $S \circ T$ as a hyperdistribution (we call hyperdistributions to the complex linear functional of $\mathscr{D}(\boldsymbol{R}^n)$ in ${}^{\rho}\boldsymbol{C}'$) for S and T in $D'(\boldsymbol{R}^n)$ and they calculated $\delta \circ \delta$. In this paper, we shall obtain expressions for the products $\delta \circ \frac{\partial}{\partial x_i}\delta$ and $\frac{\partial}{\partial x_i}\delta \circ \frac{\partial}{\partial x_i}\delta$ for $i = 1, \dots, n$, and for even n, these products have the Hadamard finite part nonzero.

1. Introduction

In this section we shall define the product "$\circ$". This definition involves a harmonic representation of distributions and Non-Standard Analysis. Let $\boldsymbol{C}^*$ and $\boldsymbol{R}^*$ be nonstandard models for the complex field and the real field respectively, and let ρ be a positive infinitesimal. Let Θ denote the set of all infinitesimal in $\boldsymbol{C}^*$. Let

$$ {}^{\rho}\boldsymbol{C} = \{x \in \boldsymbol{C}^* \colon \text{for some finite integer } n, |x| < \rho^{-n}\}, \quad {}^{\rho}\boldsymbol{C}' = {}^{\rho}\boldsymbol{C}/\Theta. $$

We call *hyperdistributions* the complex linear functional of $\mathscr{D}(\boldsymbol{R}^n)$ in ${}^{\rho}\boldsymbol{C}'$.

Definition 1.1. *Let $T \in \mathscr{D}'(\boldsymbol{R}^n)$ and $u(x, y)$ be a harmonic function in $\boldsymbol{R}_+^{n+1} = \{(x, y) : x \in \boldsymbol{R}^n, y > 0\}$ so that $\lim_{y \to 0_+} u(x, y) = T$ in the sense of $\mathscr{D}'(\boldsymbol{R}^n)$. Then $u(x, y)$ is called a harmonic function associated with T or* harmonic representation *of T.*

Lemma 1.2. *Let $S, T \in \mathscr{D}'(\boldsymbol{R}^n)$ and $\hat{S}, \hat{T}$ be harmonic representations of S and T respectively, and ${}^*\hat{S}, {}^*\hat{T}$ denote the nonstandard extensions of $\hat{S}, \hat{T}$ respectively. For any $\phi \in \mathscr{D}(\boldsymbol{R}^n)$,*

$$ \langle {}^*\hat{S}(x, \rho)\, {}^*\hat{T}(x, \rho), {}^*\phi(x) \rangle \in {}^{\rho}\boldsymbol{C}. $$

Lemma 1.3. *Let $S, T \in \mathscr{D}'(\boldsymbol{R}^n)$. If $\hat{S}_1, \hat{S}_2$ and $\hat{T}_1, \hat{T}_2$ are two harmonic representations of S and T respectively, then for any $\phi \in \mathscr{D}(\boldsymbol{R}^n)$, we have*

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$$\langle {}^*\hat{S}_2(x,\rho)^*\hat{T}_2(x,\rho), {}^*\phi(x)\rangle - \langle {}^*\hat{S}_1(x,\rho)^*\hat{T}_1(x,\rho), {}^*\phi(x)\rangle \approx 0$$

Let $\Phi : {}^\rho\boldsymbol{C} \to {}^\rho\boldsymbol{C}'$ be the homomorphism modulo Θ. We define

DEFINITION 1.4. *Let $S, T \in \mathscr{D}'(\boldsymbol{R}^n)$ and $\hat{S}, \hat{T}$ be harmonic representations of S and T respectively. We call the complex linear functional of $\mathscr{D}(\boldsymbol{R}^n) \to {}^\rho\boldsymbol{C}'$*

$$\langle S \circ T, \phi\rangle = \Phi(\langle {}^*\hat{S}(x,\rho)^*\hat{T}(x,\rho), {}^*\phi(x)\rangle)$$

the product of S and T.

The product $S \circ T$ is a generalization of the usual product for continuous functions on $\boldsymbol{R}^n$. In the following, we enunciate the fundamental properties of the product:

I. Commutativity: $S \circ T = T \circ S$, with $S, T \in \mathscr{D}'(\boldsymbol{R}^n)$.

II. Bilinearity:

$$(\alpha_1 S_1 + \alpha_2 S_2) \circ (\beta_1 T_1 + \beta_2 T_2) = \sum_{k,m=1}^{2} \alpha_k \beta_m (S_k \circ T_m)$$

with $\alpha_k, \beta_m \in \boldsymbol{C}$ and $S_k, T_m \in \mathscr{D}'(\boldsymbol{R}^n)$ for $k, m = 1, 2$.

III. Localizability: Let U be an open subset of $\boldsymbol{R}^n$ and let $S_k, T_m \in \mathscr{D}'(\boldsymbol{R}^n)$ for $k, m = 1, 2$ so that $S_1\backslash_U = S_2\backslash_U$ and $T_1\backslash_U = T_2\backslash_U$. Then $(S_1 \circ T_1)\backslash_U = (S_2 \circ T_2)\backslash_U$.

IV. Let $f \in C^\infty$, $T \in \mathscr{D}'(\boldsymbol{R}^n)$ and $f \cdot T$ be the usual multiplication. Then $f \circ T = f \cdot T$.

V. Let f and g be continuous functions on $\boldsymbol{R}^n$. Then $f \circ g = fg$.

VI. The Leibnitz formula: Let $S, T \in \mathscr{D}'(\boldsymbol{R}^n)$. Then

$$\frac{\partial}{\partial x_i}(S \circ T) = \frac{\partial S}{\partial x_i} \circ T + S \circ \frac{\partial T}{\partial x_i}.$$

2. The product $\delta \circ \frac{\partial \delta}{\partial x_i}$

Li Banghe and Li Yaqing obtained in [1], the following formula for $\delta \circ \delta$ with $\delta \in \mathscr{D}'(\boldsymbol{R}^n)$:

$$\delta \circ \delta = \sum_{j=0}^{[n/2]} \sum_{\substack{s_i \in \boldsymbol{N} \sqcup \{o\} \\ \sum s_i = j}} \rho^{2j-n} C_{s_1,\dots,s_n} \frac{\partial^{2j}\delta}{\partial x_1^{2s_1} \dots \partial x_n^{2s_n}}, \tag{1}$$

where

$$C_{s_1,\dots,s_n} = \frac{2}{\Gamma\left(\frac{n}{2} + j\right)} \prod_{i=1}^{n} \frac{\Gamma\left(s_i + \frac{1}{2}\right)}{(2s_i)!} \int_0^\infty \frac{t^{2j+n-1}}{c_n^2 (1+t^2)^{n+1}}\, dt$$

and $c_n = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$, ρ a positive infinitesimal.

We can simplify the expression for the coefficients $C_{s_1,\dots,s_n}$ by taking into account the following formula for the Beta function:

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy$$
$$= 2\cdot\int_0^\infty \frac{t^{2p-1}}{(1+t^2)^{p+q}}\,dt \tag{2}$$

where $p, q \in \boldsymbol{R}^+$. Next, we obtain by taking into account commutativity, the Leibnitz formula and (2), the following expression for $\delta \circ \frac{\partial\delta}{\partial x_i}$:

$$\delta \circ \frac{\partial\delta}{\partial x_i} = \frac{1}{2}\frac{\partial}{\partial x_i}\{\delta\circ\delta\}$$
$$= \sum_{j=0}^{[n/2]} \sum_{\substack{s_i\in \boldsymbol{N}\sqcup\{o\}\\ \sum s_i = j}} \rho^{2j-n} D_{s_1,\dots,s_n} \frac{\partial^{2j+1}\delta}{\partial x_1^{2s_1}\dots\partial x_i^{2s_i+1}\dots\partial x_n^{2s_n}}, \tag{3}$$

where

$$D_{s_1,\dots,s_n} = \frac{\Gamma\left(\frac{n}{2}-j+1\right)\prod_{i=1}^n \Gamma\left(s_i+\frac{1}{2}\right)}{2\prod_{i=1}^n (2s_i)!n!c_n^2}. \tag{4}$$

We observe that the Hadamard finite part of $\delta\circ\frac{\partial\delta}{\partial x_i}$ is nonzero for even n, because it is obtained for $j = \frac{n}{2}$ in the second term of the formula (3), so then

$$Hpf\left(\delta\circ\frac{\partial\delta}{\partial x_i}\right) = \sum_{\substack{s_i\in \boldsymbol{N}\sqcup\{o\}\\ \sum s_i = n/2}} D_{s_1,\dots,s_n} \frac{\partial^{n+1}\delta}{\partial x_1^{2s_1}\dots\partial x_i^{2s_i+1}\dots\partial x_n^{2s_n}}, \tag{5}$$

where

$$D_{s_1,\dots,s_n} = \frac{\prod_{i=1}^n \Gamma\left(s_i+\frac{1}{2}\right)}{2\prod_{i=1}^n (2s_i)!n!c_n^2}. \tag{6}$$

3. The product $\frac{\partial\delta}{\partial x_i}\circ\frac{\partial\delta}{\partial x_i}$

An expression for the product $\frac{\partial\delta}{\partial x_i}\circ\frac{\partial\delta}{\partial x_i}$ can be obtained by evaluating the following integral:

$$\int_{\boldsymbol{R}^n} \widehat{\frac{\partial\delta}{\partial x_i}}(x,\rho)\cdot\widehat{\frac{\partial\delta}{\partial x_i}}(x,\rho)\cdot\phi(x)dx, \tag{7}$$

here, we consider

$$\widehat{\frac{\partial\delta}{\partial x_i}} = \frac{-\rho(n+1)x_i}{c_n(|x|^2+\rho^2)^{(n+3)/2}}. \tag{8}$$

a harmonic representation for $\frac{\partial \delta}{\partial x_i} \in \mathscr{D}'(\boldsymbol{R}^n)$, $\phi \in \mathscr{D}(\boldsymbol{R}^n)$ and ρ the positive infinitesimal considered in section 1. We know that

$$\phi(x) = \sum_{j=0}^{n+2} \sum_{\substack{(k_1,\dots,k_n) \in \boldsymbol{N}_0^n \\ \sum k_i = j}} \frac{x_1^{k_1} \dots x_n^{k_n}}{k_1! \dots k_n!} \frac{\partial^j \phi}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}(0) + |x|^{n+3} \psi(x) \tag{9}$$

is the Taylor expansion of ϕ to $n+2$ th order, where ψ is continuous for $x \neq 0$ and bounded near $x = 0$. Then, we obtain in (7), by taking into account (8), (9) and *supp* $\phi \subset \{x : |x| < a\}$ that

$$\begin{aligned} &\int_{\boldsymbol{R}^n} \widehat{\frac{\partial \delta}{\partial x_i}}(x,\rho) \cdot \widehat{\frac{\partial \delta}{\partial x_i}}(x,\rho) \cdot \phi(x) dx \\ &\quad = \int_{|x|<a} \frac{\rho^2 (n+1)^2 x_i^2}{c_n^2 (|x|^2 + \rho^2)^{n+3}} \left\{ \sum_{j=0}^{n+2} \sum_{\substack{(k_1,\dots,k_n) \in \boldsymbol{N}_0^n \\ \sum k_i = j}} \frac{x_1^{k_1} \dots x_n^{k_n}}{k_1! \dots k_n!} \frac{\partial^j \phi}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}(0) \right\} dx \\ &\qquad + \int_{|x|<a} \frac{\rho^2 (n+1)^2 x_i^2}{c_n^2 (|x|^2 + \rho^2)^{n+3}} |x|^{n+3} \psi(x) dx. \end{aligned} \tag{10}$$

The last term of the second member of the equality (10) is an infinitesimal number and we can omit it, in fact:

Let $M = sup\{\psi(x) : |x| < a\}$ and we consider the polar coordinate in $\boldsymbol{R}^n$:

$$\begin{aligned} x_1 &= r \cos \theta_1 \\ x_2 &= r \sin \theta_1 \quad \cos \theta_2 \\ &\vdots \\ x_{n-1} &= r \sin \theta_1 \quad \sin \theta_2 \dots \sin \theta_{n-2} \quad \cos \theta_{n-1} \\ x_n &= r \sin \theta_1 \quad \sin \theta_2 \dots \sin \theta_{n-2} \quad \sin \theta_{n-1} \end{aligned}$$

where

$$0 \leq \theta_1, \theta_2, \dots, \theta_{n-2} \leq \pi, \qquad 0 \leq \theta_{n-1} \leq 2\pi. \tag{11}$$

We denote $d\Omega$ the volume element of the unitary sphere S^{n-1}. Then we have $dx = r^{n-1}\, dr d\Omega$ with

$$d\Omega = \sin^{n-2} \theta_1 \cdot \sin^{n-3} \theta_2 \dots \sin \theta_{n-2}\, d\theta_1 \dots d\theta_{n-1}.$$

Let $B_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$ be the $(n-1)$-dimensional measure of the $(n-1)$-sphere unitary S^{n-1}. We obtain

$$\left|\int_{|x|<a} \frac{\rho^2(n+1)^2 x_i^2}{c_n^2(|x|^2+\rho^2)^{n+3}} |x|^{n+3} \psi(x) dx\right|$$
$$\leq \frac{\rho^2(n+1)^2}{c_n^2} M \int_{|x|<a} \frac{|x|^{n+5}}{(|x|^2+\rho^2)^{n+3}} \, dx$$
$$= \frac{\rho^2(n+1)^2}{c_n^2} M \int_{S^{n-1}} d\Omega \int_0^a \frac{r^{n+5}}{(r^2+\rho^2)^{n+3}} r^{n-1} \, dr$$
$$\overset{r=\rho t}{=} \rho M' \int_0^{a/\rho} \frac{t^{2n+4}}{(1+t^2)^{n+3}} \, dt, \tag{12}$$

where $M' = \frac{(n+1)^2 M B_n}{c_n^2}$, $M' \in \boldsymbol{R}$.

Now, we have

$$\int_0^{a/\rho} \frac{t^{2n+4}}{(1+t^2)^{n+3}} \, dt = \int_0^1 \frac{t^{2n+4}}{(1+t^2)^{n+3}} \, dt + \int_1^{a/\rho} \frac{t^{2n+4}}{(1+t^2)^{n+3}} \, dt. \tag{13}$$

Moreover

$$\left|\int_1^{a/\rho} \frac{t^{2n+4}}{(1+t^2)^{n+3}} \, dt\right| < \int_1^{a/\rho} \frac{1}{t^2} \, dt < \infty. \tag{14}$$

Next, from (12) we obtain:

$$\rho M' \int_0^{a/\rho} \frac{t^{2n+4}}{(1+t^2)^{n+3}} \, dt \approx 0$$

since $\rho \approx 0$. Then, we consider in (10) the terms in the Taylor expansion of ϕ until $n+2$ th order

$$\int_{|x|<a} \frac{\rho^2(n+1)^2 x_i^2}{c_n^2(|x|^2+\rho^2)^{n+3}} \left\{ \sum_{j=0}^{n+2} \sum_{\substack{(k_1,\dots,k_n) \in \boldsymbol{N}_0^n \\ \sum k_i = j}} \frac{x_1^{k_1} \dots x_n^{k_n}}{k_1! \dots k_n!} \frac{\partial^j \phi}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}(0) \right\} dx$$
$$= \sum_{j=0}^{n+2} \sum_{\substack{(k_1,\dots,k_n) \in \boldsymbol{N}_0^n \\ \sum k_i = j}} \frac{\rho^2(n+1)^2}{c_n^2 k_1! \dots k_n!} \int_{|x|<a} \frac{x_1^{k_1} \dots x_i^{k_i+2} \dots x_n^{k_n}}{(|x|^2+\rho^2)^{n+3}} \, dx$$
$$\cdot \frac{\partial^j \phi}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}(0). \tag{15}$$

We consider for each $j \leq n+2$, $(k_1, \dots, k_n) \in \boldsymbol{N}_0^n$ with $k_1 + \dots + k_n = j$, the following terms:

$$\frac{\rho^2(n+1)^2}{c_n^2 k_1!\dots k_n!}\int_{|x|<a}\frac{x_1^{k_1}\dots x_i^{k_i+2}\dots x_n^{k_n}}{(|x|^2+\rho^2)^{n+3}}\,dx. \tag{16}$$

First, we consider $i=n$. For that, we calculate the following terms:

$$\frac{\rho^2(n+1)^2}{c_n^2 k_1!\dots k_n!}\int_{|x|<a}\frac{x_1^{k_1}\dots x_i^{k_i}\dots x_n^{k_n+2}}{(|x|^2+\rho^2)^{n+3}}\,dx_1\dots dx_n. \tag{17}$$

Applying polar coordinate in $\boldsymbol{R}^n$ and by taking into account that $\sum_{i=1}^n k_i=j$, we obtain in (17) the following formula:

$$\begin{aligned}
&\frac{\rho^2(n+1)^2}{c_n^2 k_1!\dots k_n!}\int_{|x|<a}\frac{x_1^{k_1}\dots x_i^{k_i}\dots x_n^{k_n+2}}{(|x|^2+\rho^2)^{n+3}}\,dx_1\dots dx_n\\
&\quad=\frac{\rho^2(n+1)^2}{c_n^2 k_1!\dots k_n!}\int_0^a\frac{r^{k_1+\cdots+k_n+2}}{(r^2+\rho^2)^{n+3}}r^{n-1}\,dr\cdot\int_0^\pi\cos^{k_1}\theta_1\cdot\sin^{j-k_1+2+n-2}\theta_1\,d\theta_1\\
&\qquad\cdot\int_0^\pi\cos^{k_2}\theta_2\cdot\sin^{j-(k_1+k_2)+2+n-3}\theta_2\,d\theta_2\\
&\qquad\cdots\int_0^\pi\cos^{k_{n-2}}\theta_{n-2}\cdot\sin^{j-(k_1+k_2+\cdots+k_{n-2})+2+n-(n-2+1)}\theta_{n-2}\,d\theta_{n-2}\\
&\qquad\cdot\int_0^{2\pi}\cos^{k_{n-1}}\theta_{n-1}\cdot\sin^{j-(k_1+k_2+\cdots+k_{n-1})+2+n-(n-1+1)}\theta_{n-1}\,d\theta_{n-1}.
\end{aligned}\tag{18}$$

The integrals of the type $\int_0^\pi\cos^k\theta\sin^r\theta\,d\theta$ are zero for k odd, then we consider even $k_1,\dots,k_{n-2}$. Moreover, the integrals of type $\int_0^{2\pi}\cos^k\theta\sin^r\theta\,d\theta$ are nonzero only for even k and r. Then we consider even k_i for $i=1,\dots,n$. Let $k_i=2s_i$, then $\sum_{i=1}^n k_i=\sum_{i=1}^n 2s_i=j$, we obtain in (18)

$$\begin{aligned}
&\frac{\rho^2(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\int_{|x|<a}\frac{x_1^{2s_1}\dots x_i^{2s_i}\dots x_n^{2s_n+2}}{(|x|^2+\rho^2)^{n+3}}\,dx_1\dots dx_n\\
&\quad=\frac{\rho^2(n+1)^2}{c_n^2 k_1!\dots k_n!}\int_0^a\frac{r^{j+n+1}}{(r^2+\rho^2)^{n+3}}\,dr\cdot\frac{\Gamma\left(\frac{2s_1+1}{2}\right)\Gamma\left(\frac{j-2s_1+2+n-2+1}{2}\right)}{\Gamma\left(\frac{j+2+n-2}{2}+1\right)}\\
&\qquad\cdot\frac{\Gamma\left(\frac{2s_2+1}{2}\right)\Gamma\left(\frac{j-(2s_1+2s_2)+2+n-3+1}{2}\right)}{\Gamma\left(\frac{j-2s_1+2+n-3}{2}+1\right)}\\
&\qquad\cdots\frac{\Gamma\left(\frac{2s_{n-2}+1}{2}\right)\Gamma\left(\frac{j-(2s_1+2s_2+\cdots+2s_{n-2})+2+n-(n-2+1)+1}{2}\right)}{\Gamma\left(\frac{j-(2s_1+2s_2+\cdots+2s_{n-3})+2+n-(n-2+1)}{2}+1\right)}
\end{aligned}$$

$$\cdot 2 \frac{\Gamma\left(\frac{2s_{n-1}+1}{2}\right)\Gamma\left(\frac{j-(2s_1+2s_2+\cdots+2s_{n-1})+2+n-(n-1+1)+1}{2}\right)}{\Gamma\left(\frac{j-(2s_1+2s_2+\cdots+2s_{n-2})+2+n-(n-1+1)}{2}+1\right)}$$

$$= \frac{\rho^2(n+1)^2}{c_n^2 k_1! \dots k_n!} \int_0^a \frac{r^{j+n+1}}{(r^2+\rho^2)^{n+3}}\, dr$$

$$\cdot 2 \cdot \frac{\prod_{i=1}^{n-1}\left[\Gamma\left(s_i+\frac{1}{2}\right)\Gamma\left(\frac{j-\sum_{r=1}^{i} 2s_r+n-(i+1)+3}{2}\right)\right]}{\Gamma\left(\frac{j+2+n-2}{2}+1\right)\prod_{i=1}^{n-2}\Gamma\left(\frac{j-\sum_{r=1}^{i} 2s_r+n-(i+1+1)+2}{2}+1\right)}. \tag{19}$$

Now,

$$\int_0^a \frac{r^{j+n+1}}{(r^2+\rho^2)^{n+3}}\, dr \overset{r=\rho t}{=} \rho^{j-n-4} \int_0^{a/\rho} \frac{t^{j+n+1}}{(1+t^2)^{n+3}}\, dt \tag{20}$$

and,

$$\frac{\prod_{i=1}^{n-1}\left[\Gamma\left(s_i+\frac{1}{2}\right)\Gamma\left(\frac{j-\sum_{r=1}^{i} 2s_r+n-(i+1)+3}{2}\right)\right]}{\Gamma\left(\frac{j+2+n-2}{2}+1\right)\prod_{i=1}^{n-2}\Gamma\left(\frac{j-\sum_{r=1}^{i} 2s_r+n-(i+1+1)+2}{2}+1\right)}$$

$$= \frac{\left(s_n+\frac{1}{2}\right)\prod_{i=1}^{n}\left[\Gamma\left(s_i+\frac{1}{2}\right)\right]}{\Gamma\left(\frac{j+n}{2}+1\right)}. \tag{21}$$

Next, we obtain from (17) and the above calculations:

$$\frac{\rho^2(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\int_{|x|<a} \frac{x_1^{2s_1}\dots x_i^{2s_i}\dots x_n^{2s_n+2}}{(|x|^2+\rho^2)^{n+3}}\, dx_1\dots dx_n$$

$$= \frac{\rho^2(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\rho^{j-n-4}\int_0^{a/\rho}\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\, dt \cdot 2 \cdot \frac{\left(s_n+\frac{1}{2}\right)\prod_{i=1}^{n}\left[\Gamma\left(s_i+\frac{1}{2}\right)\right]}{\Gamma\left(\frac{j+n}{2}+1\right)}. \tag{22}$$

We observe that

$$\int_0^{a/\rho}\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\, dt = \int_0^{\infty}\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\, dt - \int_{a/\rho}^{\infty}\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\, dt,$$

and,

$$\left|\rho^{j-n-4}\int_{a/\rho}^{\infty}\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\, dt\right| \le \rho^{j-n-4}\int_{a/\rho}^{\infty} t^{j-n-5}\, dt = \frac{a^{j-n-4}}{n-j+4}.$$

Therefore, this last term is a finite number. Next, we obtain from (22),

$$
\begin{aligned}
&\frac{\rho^2(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\int_{|x|<a}\frac{x_1^{2s_1}\dots x_i^{2s_i}\dots x_n^{2s_n+2}}{(|x|^2+\rho^2)^{n+3}}\,dx_1\dots dx_n \\
&=\frac{(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\left[\rho^2\rho^{j-n-4}\int_0^\infty\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\,dt-\rho^2\rho^{j-n-4}\int_{a/\rho}^\infty\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\,dt\right] \\
&\quad\cdot 2\cdot\frac{\left(s_n+\frac{1}{2}\right)\prod_{i=1}^n\left[\Gamma\left(s_i+\frac{1}{2}\right)\right]}{\Gamma\left(\frac{j+n}{2}+1\right)}
\end{aligned}
\tag{23}
$$

and

$$
\rho^2\rho^{j-n-4}\int_{a/\rho}^\infty\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\,dt\approx 0.
$$

Next,

$$
\begin{aligned}
&\frac{\rho^2(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\int_{|x|<a}\frac{x_1^{2s_1}\dots x_i^{2s_i}\dots x_n^{2s_n+2}}{(|x|^2+\rho^2)^{n+3}}\,dx_1\dots dx_n \\
&=\frac{(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\rho^{j-n-2}\int_0^\infty\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\,dt\cdot 2\cdot\frac{\left(s_n+\frac{1}{2}\right)\prod_{i=1}^n\left[\Gamma\left(s_i+\frac{1}{2}\right)\right]}{\Gamma\left(\frac{j+n}{2}+1\right)}.
\end{aligned}
\tag{24}
$$

By taking into account the formula (2), we obtain:

$$
\int_0^\infty\frac{t^{j+n+1}}{(1+t^2)^{n+3}}\,dt=\frac{1}{2}\,\frac{\Gamma\left(\frac{j+n+2}{2}\right)\Gamma\left(\frac{n-j+4}{2}\right)}{\Gamma(n+3)}.
\tag{25}
$$

We obtain from (24) and (25) that:

$$
\begin{aligned}
&\frac{\rho^2(n+1)^2}{c_n^2(2s_1)!\dots(2s_n)!}\int_{|x|<a}\frac{x_1^{2s_1}\dots x_i^{2s_i}\dots x_n^{2s_n+2}}{(|x|^2+\rho^2)^{n+3}}\,dx_1\dots dx_n \\
&\qquad=\frac{(n+1)^2\rho^{j-n-2}\left(s_n+\frac{1}{2}\right)\Gamma\left(\frac{n-j+4}{2}\right)\prod_{i=1}^n\left[\Gamma\left(s_i+\frac{1}{2}\right)\right]}{c_n^2(2s_1)!\dots(2s_n)!(n+2)!}
\end{aligned}
\tag{26}
$$

Finally, we obtain from formulas (10) and (26) for the case $i=n$ that:

$$
\frac{\partial\delta}{\partial x_n}\circ\frac{\partial\delta}{\partial x_n}=\sum_{j=0}^{[(n+2)/2]}\rho^{2j-n-2}\sum_{\substack{s_i\in\boldsymbol{N}\cup\{0\}\\ \sum s_i=j}}C_{s_1,\dots,s_n}\frac{\partial^{2j}\delta}{\partial x_1^{2s_1}\dots\partial x_n^{2s_n}},
\tag{27}
$$

where

$$C_{s_1,\dots,s_n} = \frac{(n+1)^2 \Gamma\left(\frac{n-2j+4}{2}\right)\left(s_n + \frac{1}{2}\right) \prod_{r=1}^{n}\left[\Gamma\left(s_r + \frac{1}{2}\right)\right]}{\prod_{r=1}^{n}(2s_r)! c_n^2 (n+2)!}. \tag{28}$$

In the same way, the following formula is obtained for $i = 1, \dots, n-1$:

$$\frac{\partial \delta}{\partial x_i} \circ \frac{\partial \delta}{\partial x_i} = \sum_{j=0}^{[(n+2)/2]} \rho^{2j-n-2} \sum_{\substack{s_r \in \boldsymbol{N} \cup \{0\} \\ \sum s_r = j}} C_{s_1,\dots,s_n} \frac{\partial^{2j} \delta}{\partial x_1^{2s_1} \dots \partial x_n^{2s_n}}, \tag{29}$$

where

$$C_{s_1,\dots,s_n} = \frac{(n+1)^2 \Gamma\left(\frac{n-2j+4}{2}\right)\left(s_i + \frac{1}{2}\right) \prod_{r=1}^{n}\left[\Gamma\left(s_r + \frac{1}{2}\right)\right]}{\prod_{r=1}^{n}(2s_r)! c_n^2 (n+2)!}. \tag{30}$$

The Hadamard finite part is nonzero for even n. In fact, we obtain in formula (29) that:

$$Hpf\left(\frac{\partial \delta}{\partial x_i} \circ \frac{\partial \delta}{\partial x_i}\right) = \sum_{\substack{s_i \in \boldsymbol{N} \cup \{0\} \\ \sum s_i = (n+2)/2}} C_{s_1,\dots,s_n} \frac{\partial^{2j} \delta}{\partial x_1^{2s_1} \dots \partial x_n^{2s_n}}, \tag{31}$$

where

$$C_{s_1,\dots,s_n} = \frac{(n+1)^2 \left(s_i + \frac{1}{2}\right) \prod_{r=1}^{n}\left[\Gamma\left(s_r + \frac{1}{2}\right)\right]}{\prod_{r=1}^{n}(2s_r)! c_n^2 (n+2)!}. \tag{32}$$

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