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The structure of the Hecke algebras of $GL_2(F_q)$ relative to the split torus and its normalizer

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ABSTRACT. Let A be the subgroup of $G = GL_2(F_q)$ consisting of diagonal matrices. We study the structure of the Hecke algebra $\mathscr{H}(G, A)$ of G relative to A. In particular, we determine the multiplication table of $\mathscr{H}(G, A)$ with respect to the standard basis. As an application, we describe the multiplication table of the Hecke algebra $\mathscr{H}(G, H)$ where H is the normalizer of A in G.

1. Introduction

The Hecke algebra $\mathscr{H}(G, A)$ of a finite group G relative to its subgroup A is a generalization of the group algebra $\mathbf{C}G$ of G, whose structure and representations are interesting mathematical objects as well as those of $\mathbf{C}G$.

In particular, the Hecke algebra $\mathscr{H}(G, A)$ plays an important role in the study of vertex-transitive graphs with vertex set G/A. In fact, such a graph is constructed by giving a certain family of double cosets of G relative to A. Moreover the adjacency matrix and its powers of such a graph are described in terms of the elements of $\mathscr{H}(G, A)$ ([3]). Therefore if one knows the multiplicative structure and irreducible characters of $\mathscr{H}(G, A)$, one can find the spectra of vertex-transitive graphs over G/A.

Let $G = GL_2(F_q)$ be the general linear group of 2×2 non-singular matrices over the finite field F_q, and let A be the subgroup of diagonal matrices of G (a split torus of G) and H be the normalizer of A in G. In our previous paper ([4]), we have considered the irreducible characters of $\mathscr{H}(G, A)$ and described the character table of it with respect to the standard basis of $\mathscr{H}(G, A)$. In the present article, we study the multiplicative structure of both $\mathscr{H}(G, A)$ and $\mathscr{H}(G, H)$. In particular we determine the multiplication tables of both $\mathscr{H}(G, A)$ and $\mathscr{H}(G, H)$ with respect to their standard basis.

The paper is organized as follows. In §2 we consider the double coset spaces $A\backslash G/A$ and $H\backslash G/H$. Using Bruhat decomposition of G, we determine a complete set $\mathscr{R}$ of representatives of $A\backslash G/A$ in Theorem 2.1. Moreover decomposing an H double coset into A double cosets, we give a complete set of

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representatives of $H\backslash G/H$ in Theorem 2.2. Let $\mathrm{ind}(AgA)$ (resp. $\mathrm{ind}(HgH)$) be the number of left A-cosets (resp. H-cosets) in the double coset AgA (resp. HgH). Their actual values are given in Theorem 2.3.

In §3 we introduce the Hecke algebra $\mathscr{H}(G,A)$ (resp. $\mathscr{H}(G,H)$), which is defined by $\mathscr{H}(G,A)=\varepsilon\mathbf{C}G\varepsilon$ (resp. $\varepsilon'\mathbf{C}G\varepsilon'$) where ε (resp. ε') is the idempotent of $\mathbf{C}G$ given by

$$\varepsilon=|A|^{-1}\sum_{a\in A}a \qquad (\text{resp. } \varepsilon'=|H|^{-1}\sum_{h\in H}h).$$

We notice that $\mathscr{H}(G,H)$ is a subalgebra of $\mathscr{H}(G,A)$ since A is a normal subgroup of H. The elements $\varepsilon[g]=\mathrm{ind}(AgA)\varepsilon g\varepsilon$ $(g\in\mathscr{R})$ of $\mathscr{H}(G,A)$ form a linear basis $\mathscr{B}$ of $\mathscr{H}(G,A)$, which we call the standard basis of $\mathscr{H}(G,A)$. Similarly we introduce the standard basis $\mathscr{B}'$ of $\mathscr{H}(G,H)$. Each element of $\mathscr{B}'$ is expressed as a linear combination of elements of $\mathscr{B}$ in Theorem 3.1.

In §4 we describe the multiplication table of $\mathscr{H}(G,A)$ with respect to the standard basis $\mathscr{B}$ in Theorem 4.1.

In §5 we give the multiplication table of $\mathscr{H}(G,H)$ with respect to the standard basis $\mathscr{B}'$ of $\mathscr{H}(G,H)$, by applying Theorem 3.1 and Theorem 4.1.

2. The double coset spaces $A\backslash G/A$ and $H\backslash G/H$

Let $F=F_q$ be a finite field with q elements where q is a power of an odd prime p. Let $F^\times=F-\{0\}$ be the multiplicative group of F. Then $F^\times$ is a cyclic group of order $q-1$. Let $G=GL_2(F)$ be the general linear group of 2×2 nonsingular matrices over F. The order $|G|$ of G is known to be equal to $q(q+1)(q-1)^2$. Let A be the subgroup of G consisting of diagonal matrices, namely

$$A=\left\{a(x,y)=\begin{pmatrix}x&0\\0&y\end{pmatrix};x,y\in F^\times\right\}.$$

Note that A is a split torus of G and the order $|A|$ of A is equal to $(q-1)^2$. Let $H=N_G(A)$ be the normalizer of A in G. Then one can write

$$H=A\cup wA=A\cup Aw \tag{2.1}$$

where w is an element of G given by

$$w=\begin{pmatrix}0&-1\\1&0\end{pmatrix}. \tag{2.2}$$

Note that $|H|=2(q-1)^2$ and

$$wa(x,y)w^{-1}=a(y,x) \qquad \text{for } a(x,y)\in A. \tag{2.3}$$

Let $Z(G)$ be the center of G. Then

$$Z(G) = \left\{ a(x,x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}; x \in F^{\times} \right\},$$

so that $Z(G)$ is contained in A and every element $a \in A$ can be written uniquely as

(2.4) $$a = a(x,x)a(y,1) \qquad \text{where } x, y \in F^{\times}.$$

Let U be the subgroup of G, which is defined by

$$U = \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in F \right\}.$$

Then one can check

(2.5) $$a(x,y)u(z)a(x^{-1},y^{-1}) = u(xy^{-1}z) \qquad \text{for } x, y \in F^{\times} \text{ and } z \in F,$$

so that A normalizes U. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \qquad \text{where } c \in F^{\times}.$$

Then one can verify

(2.6) $$g = u(ac^{-1})wu(cd(\det g)^{-1})a(c, c^{-1}\det g)$$

and therefore

(2.7) $$G = UA \cup UwUA \qquad \text{(Bruhat decomposition of } G\text{)}.$$

From (2.7), it follows that the coset space G/A is given by

$$G/A = \{u(x)A; x \in F\} \cup \{u(y)wu(z)A; y,z \in F\}.$$

Now we consider the double coset space $A\backslash G/A$.

Theorem 2.1. *Let $\mathscr{R}$ be the subset of G defined by*

$$\mathscr{R} = \{e, w, u(1), wu(1), u(1)wu(r) \ \ (r \in F)\}$$

where e is the identity matrix. Then $\mathscr{R}$ is a complete set of representatives of $A\backslash G/A$, that is,

$$A\backslash G/A = \{AgA; g \in \mathscr{R}\}$$

and consequently $|A\backslash G/A| = q+4$.

Proof. Since AgA $(g \in \mathscr{R})$ are all distinct, it is enough to see $A\backslash G/A \subset \{AgA; g \in \mathscr{R}\}$. Assume $g = u(x)a(s,t) \in UA$. Then $AgA = Au(x)A$. If $x = 0$, then $AgA = A$. While if $x \neq 0$, then by (2.5) we have $u(x) = a(x,1)u(1) \cdot a(x^{-1},1)$ and hence $AgA = Au(1)A$. Assume $g = u(y)wu(z)a(s,t) \in UwUA$. Then $AgA = Au(y)wu(z)A$. If $y = z = 0$, then $AgA = AwA$. If $y = 0$ and $z \neq 0$, then $AgA = Awu(z)A$. Since $u(z) = a(z,1)u(1)a(z^{-1},1)$, it follows that $AgA = Awa(z,1)u(1)A$. But by (2.3) we have $wa(z,1) = a(1,z)w$ and hence $AgA = Awu(1)A$. Similarly if $y \neq 0$ and $z = 0$, then we have $AgA = Au(1)wA$. Finally assume $y \neq 0$ and $z \neq 0$. Since $u(y) = a(y,1)u(1)a(y^{-1},1)$ and $a(y^{-1},1)w = wa(1,y^{-1})$, we have $Au(y)wu(z)A = Au(1)wa(1,y^{-1})u(z)A$. Using (2.5), we obtain $a(1,y^{-1})u(z) = u(yz)a(1,y^{-1})$ and hence

(2.8) $$Au(y)wu(z)A = Au(1)wu(yz)A \qquad \text{for } y,z \in F^{\times}.$$

Since $G = UA \cup UwUA$, our assertion is now clear.

Next we consider the double coset space $H\backslash G/H$.

Theorem 2.2. *The double coset space $H\backslash G/H$ is given by*

$$\{H, Hu(1)H, Hu(1)wu(2^{-1})H, Hu(1)wu(r)H = Hu(1)wu(1-r)H \ \ (r \in F')\}$$

where we put $F' = F - \{0,1,2^{-1}\}$ and consequently $|H\backslash G/H| = (q+3)/2$.

Proof. Since A is a subgroup of H, it follows that $HgH = HAgAH$ for $g \in G$. Therefore we conclude from Theorem 2.1 that $H\backslash G/H = \{HgH; g \in \mathscr{R}\}$. But by (2.1), we have

(2.9) $$HgH = AgA \cup AwgA \cup AgwA \cup AwgwA \qquad (g \in \mathscr{R}).$$

Assume $g = e$ or w. Since $w^2 = a(-1,-1) \in Z(G)$, it follows from (2.9) that

(2.10) $$H = A \cup AwA = HwH.$$

Next assume $g = u(1)$. Since $wu(1)w = u(-1)wu(-1)$ by (2.6) and hence $Awu(1)wA = Au(1)wu(1)A$ by (2.8), it follows from (2.9) that

(2.11) $$Hu(1)H = Au(1)A \cup Awu(1)A \cup Au(1)wA \cup Au(1)wu(1)A.$$

Similar argument yields that

(2.12) $$Hu(1)H = Hwu(1)H = Hu(1)wH = Hu(1)wu(1)H.$$

Finally assume $g = u(1)wu(r)$ with $r \in F - \{0,1\}$. Then by (2.6), we have $wg = u(-1)wu(r-1)$, $gw = u((r-1)r^{-1})wu(-r)a(r,r^{-1})$ and $wgw = u(-r(r-1)^{-1})wu(1-r)a(r-1,(r-1)^{-1})$ and hence by (2.8) $AwgA = Au(1)wu(1-r)A$, $AgwA = Au(1)wu(1-r)A$ and $AwgwA = Au(1)wu(r)A$. Therefore we have

(2.13) $$Hu(1)wu(r)H = Au(1)wu(r)A \cup Au(1)wu(1-r)A \qquad (r \in F - \{0,1\}),$$

from which we can deduce

$$Hu(1)wu(r)H = Hu(1)wu(1-r)H \qquad \text{for } r \in F - \{0,1\}. \tag{2.14}$$

In particular if $r = 2^{-1}$, then

$$Hu(1)wu(2^{-1})H = Au(1)wu(2^{-1})A. \tag{2.15}$$

Thus the theorem follows from (2.10), (2.12), (2.14) and (2.15).

We denote by $\operatorname{ind}(AgA)$ (resp. $\operatorname{ind}(HgH)$) the number of left A-cosets (resp. H-cosets) in AgA (resp. HgH). Then $\operatorname{ind}(AgA) = |AgA|/|A| = |A|/|A_g|$ where $A_g = A \cap gAg^{-1}$ (resp. $\operatorname{ind}(HgH) = |HgH|/|H| = |H|/|H_g|$ where $H_g = H \cap gHg^{-1}$).

Theorem 2.3. *For the double cosets AgA given in Theorem* 2.1 *and HgH given in Theorem* 2.2, *we have*

$$\operatorname{ind}(AgA) = \begin{cases} 1 & (g = e, w), \\ q-1 & (g \in \mathscr{R} - \{e, w\}) \end{cases}$$

and

$$\operatorname{ind}(HgH) = \begin{cases} 1 & g = e, \\ 2(q-1) & g = u(1), \\ (q-1)/2 & g = u(1)wu(2^{-1}), \\ q-1 & g = u(1)wu(r) \quad (r \in F'). \end{cases}$$

Proof. By simple matrix computations, we get

$$A_g = A \quad (g = e, w), \qquad A_g = Z(G) \quad (g \in \mathscr{R} - \{e, w\})$$

and

$$H_e = H, \qquad H_{u(1)} = Z(G),$$

$$H_{u(1)wu(2^{-1})} = Z(G) \cup a(1,-1)Z(G) \cup wZ(G) \cup wa(1,-1)Z(G),$$

$$H_{u(1)wu(r)} = Z(G) \cup wa((1-r)^{-1}, r^{-1})Z(G) \qquad (r \in F').$$

This implies the theorem immediately.

3. The Hecke algebras $\mathscr{H}(G,A)$ and $\mathscr{H}(G,H)$

Let $\mathbf{C}G$ be the group algebra of G over $\mathbf{C}$. Let ε (resp. ε') be the idempotent of $\mathbf{C}G$, which is defined by

$$\varepsilon = |A|^{-1} \sum_{a \in A} a \qquad (\text{resp. } \varepsilon' = |H|^{-1} \sum_{h \in H} h). \tag{3.1}$$

Then $\mathscr{H}(G,A)=\varepsilon\mathbf{C}G\varepsilon$ (resp. $\mathscr{H}(G,H)=\varepsilon'\mathbf{C}G\varepsilon'$) is a semisimple subalgebra of $\mathbf{C}G$, which we call the Hecke algebra of G relative to A (resp. H). Clearly $\mathscr{H}(G,A)$ (resp. $\mathscr{H}(G,H)$) is spanned by $\varepsilon g\varepsilon$ (resp. $\varepsilon' g\varepsilon'$) for $g\in G$ and $\varepsilon g_1\varepsilon=\varepsilon g_2\varepsilon$ (resp. $\varepsilon' g_1\varepsilon'=\varepsilon' g_2\varepsilon'$) for $g_1,g_2\in G$ if and only if $Ag_1A=Ag_2A$ (resp. $Hg_1H=Hg_2H$). Put

$$\varepsilon[g]=\mathrm{ind}(AgA)\varepsilon g\varepsilon \qquad (\text{resp. } \varepsilon'[g]=\mathrm{ind}(HgH)\varepsilon' g\varepsilon') \tag{3.2}$$

for $g\in G$. Then it is not difficult to see ([6]) that

$$\varepsilon[g]=|A|^{-1}\sum_{k\in AgA}k \qquad (\text{resp. } \varepsilon'[g]=|H|^{-1}\sum_{k\in HgH}k). \tag{3.3}$$

Note that $\varepsilon[e]=\varepsilon$ (resp $\varepsilon'[e]=\varepsilon'$). Furthermore the set $\mathscr{B}=\{\varepsilon[g];g\in\mathscr{R}\}$ is a linear basis of $\mathscr{H}(G,A)$ over $\mathbf{C}$ by Theorem 2.1 and the set

$$\mathscr{B}'=\{\varepsilon',\varepsilon'[u(1)],\varepsilon'[u(1)wu(2^{-1})],\varepsilon'[u(1)wu(r)]=\varepsilon'[u(1)wu(1-r)]\ \ (r\in F')\}$$

forms a linear basis of $\mathscr{H}(G,H)$ over $\mathbf{C}$ by Theorem 2.2. We call $\mathscr{B}$ (resp. $\mathscr{B}'$) the standard basis of $\mathscr{H}(G,A)$ (resp. $\mathscr{H}(G,H)$). Note that $\dim_{\mathbf{C}}\mathscr{H}(G,A)=q+4$ (resp. $\dim_{\mathbf{C}}\mathscr{H}(G,H)=(q+3)/2$).

Theorem 3.1. *The Hecke algebra $\mathscr{H}(G,H)$ is a commutative subalgebra of the Hecke algebra $\mathscr{H}(G,A)$. Moreover the standard basis elements of $\mathscr{H}(G,H)$ are expressed in terms of the standard basis elements of $\mathscr{H}(G,A)$ as follows.*

$$\varepsilon'=2^{-1}(\varepsilon+\varepsilon[w]), \tag{3.4}$$

$$\varepsilon'[u(1)]=2^{-1}(\varepsilon[u(1)]+\varepsilon[wu(1)]+\varepsilon[u(1)w]+\varepsilon[u(1)wu(1)]), \tag{3.5}$$

$$\varepsilon'[u(1)wu(2^{-1})]=2^{-1}\varepsilon[u(1)wu(2^{-1})], \tag{3.6}$$

$$\varepsilon'[u(1)wu(r)]=\varepsilon'[u(1)wu(1-r)]=2^{-1}(\varepsilon[u(1)wu(r)]+\varepsilon[u(1)wu(1-r)]) \tag{3.7}$$

for $r\in F'$.

Proof. By the criterion of the commutativity of Hecke algebras ([6]), it is enough to see $Hg^{-1}H=HgH$ for $g\in G$. For that purpose, we have only to check it for $g=u(1)$ and $u(1)wu(r)$ $(r\in F-\{0,1\})$. Since $u(1)^{-1}=a(1,-1)u(1)a(1,-1)$ and $(u(1)wu(r))^{-1}=u(-r)wu(-1)a(-1,-1)$, it follows that $Hu(1)^{-1}H=Hu(1)H$ and $H(u(1)wu(r))^{-1}H=Hu(-r)wu(-1)H=Hu(1)wu(r)H$. Thus $\mathscr{H}(G,H)$ is commutative. Since A is a normal subgroup of H, it follows that $\varepsilon\varepsilon'=\varepsilon'=\varepsilon'\varepsilon$ and hence $\mathscr{H}(G,H)$ is a subalgebra of $\mathscr{H}(G,A)$. Applying (2.10), (2.11), (2.15) and (2.13) to (3.3), we obtain (3.4), (3.5), (3.6) and (3.7) respectively.

4. The multiplication table of $\mathscr{H}(G, A)$

The multiplication table of $\mathscr{H}(G, A)$, we mean, is the matrix

$$(\varepsilon[g]\varepsilon[h])_{(g,h)\in\mathscr{R}\times\mathscr{R}}$$

where $\{\varepsilon[g]; g \in \mathscr{R}\}$ is the standard basis of $\mathscr{H}(G, A)$.

THEOREM 4.1. *The Hecke algebra $\mathscr{H}(G, A)$ is not commutative and its multiplication table with respect to the standard basis $\{\varepsilon[g]; g \in \mathscr{R}\}$ is given as follows. Here we omit the contribution of $\varepsilon = \varepsilon[e]$ because it is the identity element of $\mathscr{H}(G, A)$.*

Table I.

	$\varepsilon[w]$	$\varepsilon[u(1)]$
$\varepsilon[w]$	ε	$\varepsilon[wu(1)]$
$\varepsilon[u(1)]$	$\varepsilon[u(1)w]$	$(q-1)\varepsilon + (q-2)\varepsilon[u(1)]$
$\varepsilon[wu(1)]$	$\varepsilon[u(1)wu(1)]$	$(q-1)\varepsilon[w] + (q-2)\varepsilon[wu(1)]$
$\varepsilon[u(1)w]$	$\varepsilon[u(1)]$	$\varepsilon[u(1)wu(1)] + S$
$\varepsilon[u(1)wu(1)]$	$\varepsilon[wu(1)]$	$\varepsilon[u(1)w] + S$
$\varepsilon[u(1)wu(s)]$ $(s \in F^\times - \{1\})$	$\varepsilon[u(1)wu(1-s)]$	$\varepsilon[u(1)wu(1)] + \varepsilon[u(1)w] + S_s$

	$\varepsilon[wu(1)]$	$\varepsilon[u(1)w]$
$\varepsilon[w]$	$\varepsilon[u(1)]$	$\varepsilon[u(1)wu(1)]$
$\varepsilon[u(1)]$	$\varepsilon[u(1)wu(1)] + S$	$(q-1)\varepsilon[w] + (q-2)\varepsilon[u(1)w]$
$\varepsilon[wu(1)]$	$\varepsilon[u(1)w] + S$	$(q-1)\varepsilon + (q-2)\varepsilon[u(1)wu(1)]$
$\varepsilon[u(1)w]$	$(q-1)\varepsilon + (q-2)\varepsilon[u(1)]$	$\varepsilon[wu(1)] + S$
$\varepsilon[u(1)wu(1)]$	$(q-1)\varepsilon[w] + (q-2)\varepsilon[wu(1)]$	$\varepsilon[u(1)] + S$
$\varepsilon[u(1)wu(s)]$ $(s \in F^\times - \{1\})$	$\varepsilon[u(1)wu(1)] + \varepsilon[u(1)w] + S_{1-s}$	$\varepsilon[u(1)] + \varepsilon[wu(1)] + S_{1-s}$

	$\varepsilon[u(1)wu(1)]$	$\varepsilon[u(1)wu(t)]$ $(t \in F^\times - \{1\})$
$\varepsilon[w]$	$\varepsilon[u(1)w]$	$\varepsilon[u(1)wu(1-t)]$
$\varepsilon[u(1)]$	$\varepsilon[wu(1)] + S$	$\varepsilon[wu(1)] + \varepsilon[u(1)wu(1)] + S_t$
$\varepsilon[wu(1)]$	$\varepsilon[u(1)] + S$	$\varepsilon[u(1)] + \varepsilon[u(1)w] + S_{1-t}$
$\varepsilon[u(1)w]$	$(q-1)\varepsilon[w] + (q-2)\varepsilon[u(1)w]$	$\varepsilon[wu(1)] + \varepsilon[u(1)wu(1)] + S_{1-t}$
$\varepsilon[u(1)wu(1)]$	$(q-1)\varepsilon + (q-2)\varepsilon[u(1)wu(1)]$	$\varepsilon[u(1)] + \varepsilon[u(1)w] + S_t$
$\varepsilon[u(1)wu(s)]$ $(s \in F^\times - \{1\})$	$\varepsilon[u(1)] + \varepsilon[wu(1)] + S_s$	$E(s, t)$

where we put

$$(4.1) \quad S = \sum_{x \in F - \{0,1\}} \varepsilon[u(1)wu(x)] \qquad \text{and} \qquad S_r = \sum_{x \in F - \{0,1,r\}} \varepsilon[u(1)wu(x)]$$

for $r \in F - \{0, 1\}$.

Moreover for $s, t \in F - \{0, 1\}$ *the product* $E(s,t) = \varepsilon[u(1)wu(s)]\varepsilon[u(1)wu(t)]$ *is given by*

$$E(s,t) = \begin{cases} (q-1)\varepsilon + (q-1)\varepsilon[w] + S(2^{-1}, 2^{-1}) & (t = s = 2^{-1}), \\ (q-1)\varepsilon + \varepsilon[wu(1)] + \varepsilon[u(1)w] + S(s,s) & (t = s \neq 2^{-1}), \\ (q-1)\varepsilon[w] + \varepsilon[u(1)] + \varepsilon[u(1)wu(1)] + S(s, 1-s) & (t = 1-s \neq 2^{-1}), \\ \varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)] + S(s,t) & (t \neq s, t \neq 1-s). \end{cases}$$

Here we set

$$S(s,t) = \sum_{x \in F - J_{s,t}} \varepsilon[u(1)wu(\psi_{s,t}(x))] \tag{4.2}$$

where $J_{s,t} = \{0, 1, s, s(1-t)^{-1}, (s-t)(1-t)^{-1}\}$ *and*

$$\psi_{s,t}(x) = (x-1)((t-1)x+s)(x-s)^{-1} \qquad \text{for } x \in F - \{s\}. \tag{4.3}$$

Before proving Theorem 4.1, we need the following lemma.

Lemma 4.2. *In* $\mathscr{H}(G, A)$, *the following identities hold.*

$$\varepsilon a(x,y) = \varepsilon = a(x,y)\varepsilon \qquad \text{for } x, y \in F^{\times}. \tag{4.4}$$

$$\varepsilon u(x)\varepsilon = \varepsilon u(1)\varepsilon, \qquad \varepsilon wu(x)\varepsilon = \varepsilon wu(1)\varepsilon, \tag{4.5}$$

$$\varepsilon u(x)w\varepsilon = \varepsilon u(1)w\varepsilon \qquad \text{for } x \in F^{\times}.$$

$$\varepsilon u(y)wu(z)\varepsilon = \varepsilon u(1)wu(yz)\varepsilon \qquad \text{for } y, z \in F^{\times}. \tag{4.6}$$

$$\varepsilon[g]\varepsilon[h] = \operatorname{ind}(AgA)\operatorname{ind}(AhA)(q-1)^{-1} \sum_{y \in F^{\times}} \varepsilon g a(y,1) h \varepsilon \qquad \text{for } g, h \in G. \tag{4.7}$$

Proof. (4.4) is clear from the definition of ε. (4.5) and (4.6) are also obvious from the proof of Theorem 2.1. Since $\varepsilon^2 = \varepsilon$,

$$\varepsilon[g]\varepsilon[h] = \operatorname{ind}(AgA)\operatorname{ind}(AhA)\varepsilon g \varepsilon h \varepsilon.$$

By (2.4) and (3.1), we can write

$$\varepsilon = (q-1)^{-2} \sum_{x, y \in F^{\times}} a(x,x) a(y,1),$$

so that

$$\varepsilon g \varepsilon h \varepsilon = (q-1)^{-2} \sum_{x, y \in F^{\times}} \varepsilon g a(x,x) a(y,1) h \varepsilon.$$

Since $a(x,x) \in Z(G)$, it follows that

$$\varepsilon g\varepsilon h\varepsilon = (q-1)^{-1}\sum_{y\in F^\times}\varepsilon g a(y,1)h\varepsilon.$$

Thus we obtain (4.7).

Proof of Theorem 4.1. Here we will verify the last column in Table I. The products in the other part are caluculated in a similar and simpler way. Applying $h = u(1)wu(t)$ $(t \in F - \{0,1\})$ to (4.7) and using $\mathrm{ind}(Au(1)wu(t)A) = q-1$, we have

$$\varepsilon[g]\varepsilon[u(1)wu(t)] = \mathrm{ind}(AgA)\sum_{y\in F^\times}\varepsilon g a(y,1)u(1)wu(t)\varepsilon \qquad \text{for } g\in\mathscr{R}.$$

Since $a(y,1)u(1)wu(t) = u(y)wu(ty^{-1})a(1,y)$, it follows that

$$\varepsilon[g]\varepsilon[u(1)wu(t)] = \mathrm{ind}(AgA)\sum_{y\in F^\times}\varepsilon g u(y)wu(ty^{-1})\varepsilon. \tag{4.8}$$

Case 1. $g = w$. Since $\mathrm{ind}(AwA) = 1$ and $wu(y)wu(ty^{-1}) = u(-y^{-1})\cdot wu(y(t-1))a(y,y)$, it follows from (4.8) that

$$\varepsilon[w]\varepsilon[u(1)wu(t)] = \sum_{y\in F^\times}\varepsilon u(-y^{-1})wu(y(t-1))\varepsilon.$$

Using (4.6), we get

$$\varepsilon[w]\varepsilon[u(1)wu(t)] = \sum_{y\in F^\times}\varepsilon u(1)wu(1-t)\varepsilon = (q-1)\varepsilon u(1)wu(1-t)\varepsilon.$$

Since $\mathrm{ind}(Au(1)wu(1-t)A) = q-1$, we have

$$\varepsilon[w]\varepsilon[u(1)wu(t)] = \varepsilon[u(1)wu(1-t)].$$

Case 2. $g = u(1)$. Since $\mathrm{ind}(Au(1)A) = q-1$ and $u(1)u(y)wu(ty^{-1}) = u(1+y)wu(ty^{-1})$, it follows from (4.8) that

$$\varepsilon[u(1)]\varepsilon[u(1)wu(t)] = (q-1)\sum_{y\in F^\times}\varepsilon u(1+y)wu(ty^{-1})\varepsilon.$$

Replacing $1+y$ by x, we get

$$\varepsilon[u(1)]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon wu(-t)\varepsilon + (q-1)\sum_{x\in F^\times-\{1\}}\varepsilon u(x)wu(t(x-1)^{-1})\varepsilon.$$

Using (4.5) and (4.6), we have

$$\varepsilon[u(1)]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon wu(1)\varepsilon + (q-1)\sum_{x\in F^\times-\{1\}} \varepsilon u(1)wu(tx(x-1)^{-1})\varepsilon.$$

Putting $z = tx(x-1)^{-1}$, we can deduce

$$\varepsilon[u(1)]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon wu(1)\varepsilon + (q-1)\sum_{z\in F^\times-\{t\}} \varepsilon u(1)wu(z)\varepsilon.$$

Since $\mathrm{ind}(Awu(1)A) = \mathrm{ind}(Au(1)wu(z)A) = q-1$, we get

$$\varepsilon[u(1)]\varepsilon[u(1)wu(t)] = \varepsilon[wu(1)] + \sum_{z\in F^\times-\{t\}} \varepsilon[u(1)wu(z)],$$

which is equal to

$$\varepsilon[u(1)]\varepsilon[u(1)wu(t)] = \varepsilon[wu(1)] + \varepsilon[u(1)wu(1)] + S_t.$$

Case 3. $g = wu(1)$. Since $\mathrm{ind}(Awu(1)A) = q-1$ and $wu(1)u(y)wu(ty^{-1}) = wu(1+y)wu(ty^{-1})$, we have, by putting $x = 1+y$,

$$\varepsilon[wu(1)]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon u(-t)\varepsilon + (q-1)\sum_{x\in F^\times-\{1\}} \varepsilon wu(x)wu(t(x-1)^{-1})\varepsilon.$$

Using (4.5), $wu(x)wu(t(x-1)^{-1}) = u(-x^{-1})wu(x(tx(x-1)^{-1}-1))a(x,x^{-1})$ and (4.6), we have

$$\varepsilon[wu(1)]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon u(1)\varepsilon + (q-1)\sum_{x\in F^\times-\{1\}} \varepsilon u(1)wu(1-tx(x-1)^{-1})\varepsilon.$$

Putting $z = 1-tx(x-1)^{-1}$, we can deduce

$$\varepsilon[wu(1)]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon u(1)\varepsilon + (q-1)\sum_{z\in F-\{1,1-t\}} \varepsilon u(1)wu(z)\varepsilon.$$

Since $\mathrm{ind}(Au(1)A) = \mathrm{ind}(Au(1)wu(z)A) = q-1$, we obtain

$$\varepsilon[wu(1)]\varepsilon[u(1)wu(t)] = \varepsilon[u(1)] + \varepsilon[u(1)w] + \sum_{z\in F-\{0,1,1-t\}} \varepsilon[u(1)wu(z)].$$

Case 4. $g = u(1)w$. Since $\mathrm{ind}(Au(1)wA) = q-1$ and $u(1)wu(y)wu(ty^{-1}) = u((y-1)y^{-1})wu(y(t-1))a(y,y^{-1})$, it follows from (4.8) that

$$\begin{aligned}\varepsilon[u(1)w]\varepsilon[u(1)wu(t)] &= (q-1)\varepsilon wu(t-1)\varepsilon \\ &\quad + (q-1)\sum_{y\in F^\times-\{1\}} \varepsilon u((y-1)y^{-1})wu(y(t-1))\varepsilon.\end{aligned}$$

By (4.5) and (4.6), we obtain

$$\varepsilon[u(1)w]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon wu(1)\varepsilon + (q-1)\sum_{y\in F^{\times}-\{1\}} \varepsilon u(1)wu((y-1)(t-1))\varepsilon.$$

Putting $z=(y-1)(t-1)$ and using $\mathrm{ind}(Awu(1)A)=\mathrm{ind}(Au(1)wu(z)A)=q-1$, we get

$$\varepsilon[u(1)w]\varepsilon[u(1)wu(t)] = \varepsilon[wu(1)] + \sum_{z\in F^{\times}-\{1-t\}} \varepsilon[u(1)wu(z)],$$

which yields

$$\varepsilon[u(1)w]\varepsilon[u(1)wu(t)] = \varepsilon[wu(1)] + \varepsilon[u(1)wu(1)] + S_{1-t}.$$

Case 5. $g=u(1)wu(1)$. Since $u(1)wu(1)u(y)wu(ty^{-1})=u(1)wu(1+y)wu(ty^{-1})$ and $\mathrm{ind}(Au(1)wu(1)A)=q-1$, it follows from (4.8) that

$$\varepsilon[u(1)wu(1)]\varepsilon[u(1)wu(t)] = (q-1)\sum_{y\in F^{\times}} \varepsilon u(1)wu(1+y)wu(ty^{-1})\varepsilon.$$

Putting $x=1+y$, we have

$$\begin{aligned}&\varepsilon[u(1)wu(1)]\varepsilon[u(1)wu(t)]\\ &\quad = (q-1)\varepsilon u(-t)\varepsilon + (q-1)\sum_{x\in F^{\times}-\{1\}} \varepsilon u(1)wu(x)wu(t(x-1)^{-1})\varepsilon.\end{aligned}$$

By (4.5), $u(1)wu(x)wu(t(x-1)^{-1}) = u((x-1)x^{-1})wu(x(tx(x-1)^{-1}-1))a(x,x^{-1})$ and (4.6), we can deduce

$$\varepsilon[u(1)wu(1)]\varepsilon[u(1)wu(t)] = (q-1)\varepsilon u(1)\varepsilon + (q-1)\sum_{x\in F^{\times}-\{1\}} \varepsilon u(1)wu((t-1)x+1)\varepsilon.$$

Putting $z=(t-1)x+1$ and using $\mathrm{ind}(Au(1)A)=\mathrm{ind}(Au(1)wu(z)A)=q-1$, we obtain

$$\varepsilon[u(1)wu(1)]\varepsilon[u(1)wu(t)] = \varepsilon[u(1)] + \varepsilon[u(1)w] + \sum_{z\in F^{\times}-\{1,t\}} \varepsilon[u(1)wu(z)],$$

which yields

$$\varepsilon[u(1)wu(1)]\varepsilon[u(1)wu(t)] = \varepsilon[u(1)] + \varepsilon[u(1)w] + S_t.$$

Case 6. $g=u(1)wu(s)$ $(s\in F-\{0,1\})$. Set $E(s,t)=\varepsilon[u(1)wu(s)]\varepsilon[u(1)wu(t)]$. Since $\mathrm{ind}(Au(1)wu(s)A)=q-1$, it follows from (4.8) that

$$E(s,t) = (q-1)\sum_{y\in F^\times} \varepsilon u(1)wu(s+y)wu(ty^{-1})\varepsilon.$$

Putting $x = s + y$, we have

$$E(s,t) = (q-1)\sum_{x\in F-\{s\}} \varepsilon u(1)wu(x)wu(t(x-s)^{-1})\varepsilon,$$

which equals

$$E(s,t) = (q-1)\varepsilon u((s-t)s^{-1})\varepsilon + (q-1)\sum_{x\in F^\times-\{s\}} \varepsilon u(1)wu(x)wu(t(x-s)^{-1})\varepsilon.$$

Since $u(1)wu(x)wu(t(x-s)^{-1}) = u((x-1)x^{-1})wu(x(tx(x-s)^{-1}-1))a(x,x^{-1})$, it follows from (4.6) that

$$\begin{aligned} E(s,t) &= (q-1)\varepsilon u((s-t)s^{-1})\varepsilon + (q-1)\varepsilon wu((s+t-1)(1-s)^{-1})\varepsilon \\ &\quad + (q-1)\sum_{x\in F^\times-\{1,s\}} \varepsilon u(1)wu((x-1)(tx(x-s)^{-1}-1))\varepsilon. \end{aligned}$$

Since $(x-1)(tx(x-s)^{-1}-1) = \psi_{s,t}(x)$, we have

$$\begin{aligned} E(s,t) &= (q-1)\varepsilon u((s-t)s^{-1})\varepsilon + (q-1)\varepsilon wu((s+t-1)(1-s)^{-1})\varepsilon \\ &\quad + (q-1)\sum_{x\in F^\times-\{1,s\}} \varepsilon u(1)wu(\psi_{s,t}(x))\varepsilon. \end{aligned} \tag{4.9}$$

If $t = s = 2^{-1}$, then (4.9) becomes

$$E(2^{-1},2^{-1}) = (q-1)\varepsilon + (q-1)\varepsilon[w] + \sum_{x\in F^\times-\{1,2^{-1}\}} \varepsilon[u(1)wu(\psi_{2^{-1},2^{-1}}(x))].$$

Since $J_{2^{-1},2^{-1}} = \{0,1,2^{-1}\}$, it follows that

$$E(2^{-1},2^{-1}) = (q-1)\varepsilon + (q-1)\varepsilon[w] + S(2^{-1},2^{-1}).$$

If $t = s \neq 2^{-1}$, then (4.9) becomes

$$E(s,s) = (q-1)\varepsilon + \varepsilon[wu(1)] + \sum_{x\in F^\times-\{1,s\}} \varepsilon[u(1)wu(\psi_{s,s}(x))].$$

Since $\psi_{s,s}^{-1}(0) = \{s(1-s)^{-1}\}$ and $\psi_{s,s}^{-1}(1)$ is empty, it follows that

$$E(s,s) = (q-1)\varepsilon + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \sum_{x\in F^\times-\{1,s,s(1-s)^{-1}\}} \varepsilon[u(1)wu(\psi_{s,s}(x))],$$

which implies

$$E(s,s) = (q-1)\varepsilon + \varepsilon[wu(1)] + \varepsilon[u(1)w] + S(s,s).$$

If $t = 1-s \neq 2^{-1}$, then (4.9) becomes

$$E(s,1-s) = \varepsilon[u(1)] + (q-1)\varepsilon[w] + \sum_{x\in F^\times-\{1,s\}} \varepsilon[u(1)wu(\psi_{s,1-s}(x))].$$

Since $\psi_{s,1-s}^{-1}(0)$ is empty and $\psi_{s,1-s}^{-1}(1) = \{(2s-1)s^{-1}\}$, it follows that

$$\begin{aligned} E(s,1-s) = {} & (q-1)\varepsilon[w] + \varepsilon[u(1)] + \varepsilon[u(1)wu(1)] \\ & + \sum_{x\in F^\times-\{1,s,(2s-1)s^{-1}\}} \varepsilon[u(1)wu(\psi_{s,1-s}(x))], \end{aligned}$$

which yields

$$E(s,1-s) = (q-1)\varepsilon[w] + \varepsilon[u(1)] + \varepsilon[u(1)wu(1)] + S(s,1-s).$$

If $t \neq s$ and $t \neq 1-s$, then (4.9) becomes

$$E(s,t) = \varepsilon[u(1)] + \varepsilon[wu(1)] + \sum_{x\in F^\times-\{1,s\}} \varepsilon[u(1)wu(\psi_{s,t}(x))].$$

Since $\psi_{s,t}^{-1}(0) = \{s(1-t)^{-1}\}$ and $\psi_{s,t}^{-1}(1) = \{(s-t)(1-t)^{-1}\}$, it follows that

$$E(s,t) = \varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)] + \sum_{x\in F-J_{s,t}} \varepsilon[u(1)wu(\psi_{s,t}(x))],$$

which implies

$$E(s,t) = \varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)] + S(s,t).$$

5. The multiplication table of $\mathscr{H}(G,H)$

Using the multiplication table of $\mathscr{H}(G,A)$ given in §4, we describe the multiplication table of $\mathscr{H}(G,H)$ with respect to the basis

$$\mathscr{B}' = \{\varepsilon', \varepsilon'[u(1)], \varepsilon'[u(1)wu(2^{-1})], \varepsilon'[u(1)wu(r)] = \varepsilon'[u(1)wu(1-r)]\ \ (r\in F')\}.$$

To start with, we need some properties of the map $\psi_{s,t} : F-\{s\} \to F$ in (4.3) and the sum $S(s,t)$ in (4.2) where $s,t\in F-\{0,1\}$.

Lemma 5.1. *Let* $s,t\in F-\{0,1\}$. *Let* $\psi_{s,t} : F-\{s\} \to F$ *be the map defined by*

$$\psi_{s,t}(x) = (x-1)((t-1)x+s)(x-s)^{-1}$$

and let $S(s,t)$ *be the sum*

$$S(s,t) = \sum_{x \in F - J_{s,t}} \varepsilon[u(1)wu(\psi_{s,t}(x))]$$

where $J_{s,t} = \{0, 1, s, s(1-t)^{-1}, (s-t)(1-t)^{-1}\}$. *Then we have*

$$\psi_{1-s,1-t}(x) = \psi_{s,t}((tx+s-t)(1-t)^{-1}) \qquad \textit{for } x \in F - \{1-s\}, \tag{5.1}$$

$$\psi_{1-s,t}(x) = 1 - \psi_{s,t}(1-x) \qquad \textit{for } x \in F - \{1-s\}, \tag{5.2}$$

$$S(1-s, 1-t) = S(s,t), \tag{5.3}$$

$$S(s, 1-t) = S(1-s, t) = \sum_{x \in F^{\times} - J_{s,t}} \varepsilon[u(1)wu(1 - \psi_{s,t}(x))]. \tag{5.4}$$

Proof. (5.1) and (5.2) are proved by direct computations. Put $f(x) = (tx+s-t)(1-t)^{-1}$ for $x \in F$. Then by (5.1)

$$S(1-s, 1-t) = \sum_{x \in F - J_{1-s,1-t}} \varepsilon[u(1)wu(\psi_{s,t}(f(x)))].$$

Since the map f transforms $F - J_{1-s,1-t}$ bijectively onto $F - J_{s,t}$, it follows that

$$S(1-s, 1-t) = \sum_{y \in F - J_{s,t}} \varepsilon[u(1)wu(\psi_{s,t}(y))],$$

which equals $S(s,t)$. By (5.3), we have $S(s, 1-t) = S(1-s, t)$. Using (5.2), we can write

$$S(1-s, t) = \sum_{x \in F - J_{1-s,t}} \varepsilon[u(1)wu(1 - \psi_{s,t}(1-x))].$$

Since the map $g(x) = 1-x$ transforms $F - J_{1-s,t}$ bijectively onto $F - J_{s,t}$, it follows that

$$S(1-s, t) = \sum_{y \in F - J_{s,t}} \varepsilon[u(1)wu(1 - \psi_{s,t}(y))].$$

Thus (5.4) holds.

Lemma 5.2. *Let* $s, t \in F - \{0, 1\}$ *and put* $K_{s,t} = \{x \in F - \{s\}; \psi_{s,t}(x) = 2^{-1}\}$. *Then*

$$|K_{s,t}| = \begin{cases} 2 & (D_{s,t} \in F_0^{\times}), \\ 1 & (D_{s,t} = 0), \\ 0 & (D_{s,t} \in F_1^{\times}) \end{cases} \tag{5.5}$$

where $F_0^\times$ *(resp.* $F_1^\times$*) is the set of squares (resp. non-squares) in* $F^\times$ *and*

$$D_{s,t} = (s - 2^{-1})^2 + (t - 2^{-1})^2 - 2^{-2}.$$

In particular

$$|K_{2^{-1},2^{-1}}| = \begin{cases} 2 & (q \equiv 1 \pmod 4), \\ 0 & (q \equiv 3 \pmod 4). \end{cases} \tag{5.6}$$

Proof. It is clear that $\psi_{s,t}(x) = 2^{-1}$ if and only if

$$(t-1)x^2 + (s - t + 2^{-1})x - 2^{-1}s = 0.$$

Since $t \neq 1$, this gives a quadratic equation, whose discriminant is $D_{s,t}$. Hence (5.5) is valid. If $s = t = 2^{-1}$, then $D_{2^{-1},2^{-1}} = -2^{-2}$. Since $-1 \in F_0^\times$ (resp. $-1 \in F_1^\times$) if and only if $q \equiv 1 \bmod 4$ (resp. $q \equiv 3 \bmod 4$), (5.6) follows immediately.

Lemma 5.3. *Let* $F' = F - \{0, 1, 2^{-1}\}$. *Define the sums* S', S'_s $(s \in F')$ *and* $S'(s,t)$ $(s,t \in F - \{0,1\})$ *by*

$$S' = \sum_{x \in F'} \varepsilon'[u(1)wu(x)], \qquad S'_s = \sum_{x \in F' - \{s\}} \varepsilon'[u(1)wu(x)] \tag{5.7}$$

and

$$S'(s,t) = \sum_{x \in F - J_{s,t} \cup K_{s,t}} \varepsilon'[u(1)wu(\psi_{s,t}(x))]. \tag{5.8}$$

Then the sums S, S_s $(s \in F - \{0,1\})$ *in* (4.1) *and* $S(s,t)$ $(s,t \in F - \{0,1\})$ *in* (4.2) *are related to the sums* S', S'_s *and* $S'(s,t)$ *as follows.*

$$S_{2^{-1}} = S' \qquad \textit{and hence} \qquad S = 2\varepsilon'[u(1)wu(2^{-1})] + S'. \tag{5.9}$$

$$S_s + S_{1-s} = 4\varepsilon'[u(1)wu(2^{-1})] + 2S'_s \qquad \textit{for } s \in F'. \tag{5.10}$$

$$S(s,t) + S(1-s,t) = 4|K_{s,t}|\varepsilon'[u(1)wu(2^{-1})] + 2S'(s,t) \qquad \textit{for } s,t \in F - \{0,1\}. \tag{5.11}$$

Proof. Since $S_{2^{-1}} = \sum_{x \in F'} \varepsilon[u(1)wu(x)] = \sum_{x \in F'} \varepsilon[u(1)wu(1-x)]$, it follows that

$$S_{2^{-1}} = \frac{1}{2}\left(\sum_{x \in F'} \varepsilon[u(1)wu(x)] + \sum_{x \in F'} \varepsilon[u(1)wu(1-x)]\right).$$

Using (3.7), we have $S_{2^{-1}} = S'$. Since $S = \varepsilon[u(1)wu(2^{-1})] + S_{2^{-1}}$, $S = 2\varepsilon'[u(1)wu(2^{-1})] + S'$ is obvious. Let $s \in F'$. Then we can write

$$S_s = \varepsilon[u(1)wu(2^{-1})] + \sum_{x \in F' - \{s\}} \varepsilon[u(1)wu(x)]$$

and

$$S_{1-s} = \varepsilon[u(1)wu(2^{-1})] + \sum_{x \in F' - \{1-s\}} \varepsilon[u(1)wu(x)].$$

Replacing x by $1 - x$, we obtain

$$S_{1-s} = \varepsilon[u(1)wu(2^{-1})] + \sum_{x \in F' - \{s\}} \varepsilon[u(1)wu(1-x)].$$

Therefore we have

$$S_s + S_{1-s} = 2\varepsilon[u(1)wu(2^{-1})] + \sum_{x \in F' - \{s\}} (\varepsilon[u(1)wu(x)] + \varepsilon[u(1)wu(1-x)]).$$

By (3.6) and (3.7), we conclude that

$$S_s + S_{1-s} = 4\varepsilon'[u(1)wu(2^{-1})] + 2S_s'.$$

It follows from (5.4) that

$$S(s,t) + S(1-s,t) = \sum_{x \in F - J_{s,t}} (\varepsilon[u(1)wu(\psi_{s,t}(x))] + \varepsilon[u(1)wu(1 - \psi_{s,t}(x))]),$$

which is transformed into

$$S(s,t) + S(1-s,t) = 2|K_{s,t}|\varepsilon[u(1)wu(2^{-1})] + \sum_{x \in F - J_{s,t} \cup K_{s,t}} (\varepsilon[u(1)wu(\psi_{s,t}(x))] + \varepsilon[u(1)wu(1 - \psi_{s,t}(x))]).$$

By (3.6) and (3.7), we get

$$S(s,t) + S(1-s,t) = 4|K_{s,t}|\varepsilon'[u(1)wu(2^{-1})] + 2S'(s,t).$$

Now we are ready to describe the multiplication table of $\mathscr{H}(G,H)$. In the table below, we omit the contribution of ε' because it is the identity element of $\mathscr{H}(G,H)$ and we also omit the upper half part because $\mathscr{H}(G,H)$ is commutative.

Theorem 5.4. *The multiplication table of $\mathscr{H}(G,H)$ with respect to the standard basis is given as follows.*

Table II

	$\varepsilon'[u(1)]$	$\varepsilon'[u(1)wu(2^{-1})]$	$\varepsilon'[u(1)wu(t)]$ $(t \in F')$
$\varepsilon'[u(1)]$	$2(q-1)\varepsilon' + 2S' + (q-1)\varepsilon'[u(1)] + 4\varepsilon'[u(1)wu(2^{-1})]$		
$\varepsilon'[u(1)wu(2^{-1})]$	$\varepsilon'[u(1)] + S'$	$E'(2^{-1}, 2^{-1})$	
$\varepsilon'[u(1)wu(s)]$ $(s \in F')$	$2\varepsilon'[u(1)] + 4\varepsilon'[u(1)wu(2^{-1})] + 2S'_s$	$E'(s, 2^{-1})$	$E'(s,t)$

where $F' = F - \{0, 1, 2^{-1}\}$ *and*

$$S' = \sum_{x \in F'} \varepsilon'[u(1)wu(x)], \qquad S'_s = \sum_{x \in F' - \{s\}} \varepsilon'[u(1)wu(x)] \qquad \textit{for } s \in F'.$$

Furthermore

$$E'(2^{-1}, 2^{-1}) = 2^{-1}(q-1)\varepsilon'[e] + 2^{-1}|K_{2^{-1},2^{-1}}|\varepsilon'[u(1)wu(2^{-1})] + 4^{-1}S'(2^{-1}, 2^{-1}),$$

$$E'(s, 2^{-1}) = \varepsilon'[u(1)] + |K_{s,2^{-1}}|\varepsilon'[u(1)wu(2^{-1})] + 2^{-1}S'(s, 2^{-1}),$$

$$E'(s,t) = \begin{cases} (q-1)\varepsilon' + \varepsilon'[u(1)] \\ \qquad + 2|K_{s,s}|\varepsilon'[u(1)wu(2^{-1})] + S'(s,s) & \textit{for } t = s, \textit{ or } 1-s, \\ 2\varepsilon'[u(1)] + 2|K_{s,t}|\varepsilon'[u(1)wu(2^{-1})] + S'(s,t) & \textit{for } t \neq s, 1-s. \end{cases}$$

where

$$S'(s,t) = \sum_{x \in F - J_{s,t} \cup K_{s,t}} \varepsilon'[u(1)wu(\psi_{s,t}(x))].$$

PROOF. By (3.5), we have

$$\varepsilon'[u(1)]^2 = 4^{-1}(\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)])^2.$$

We can derive from Table I that the right-side is given by

$$(q-1)(\varepsilon + \varepsilon[w]) + 2^{-1}(q-1)(\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)]) + 2S,$$

which can be written, by (3.4), (3.5) and (5.9), as

$$2(q-1)\varepsilon' + (q-1)\varepsilon'[u(1)] + 4\varepsilon'[u(1)wu(2^{-1})] + 2S'.$$

By (3.5) and (3.6), we have

$$\varepsilon'[u(1)wu(2^{-1})]\varepsilon'[u(1)] = 4^{-1}\varepsilon[u(1)wu(2^{-1})] \times (\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)]).$$

It follows from Table I that the right-side is equal to

$$2^{-1}(\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)]) + S_{2^{-1}},$$

which can be written, by (3.5) and (5.9), as $\varepsilon'[u(1)] + S'$. By (3.5) and (3.7), we have for $s \in F'$

$$\varepsilon'[u(1)wu(s)]\varepsilon'[u(1)] = 4^{-1}(\varepsilon[u(1)wu(s)] + \varepsilon[u(1)wu(1-s)]) \\ \times (\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)]).$$

We can derive from Table I that the right-side is equal to

$$\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)] + S_s + S_{1-s},$$

which can be written, by (3.5) and (5.10), as

$$2\varepsilon'[u(1)] + 4\varepsilon'[u(1)wu(2^{-1})] + 2S'_s.$$

By (3.6) and Table I, we obtain

$$E'(2^{-1}, 2^{-1}) = \varepsilon'[u(1)wu(2^{-1})]^2 = 4^{-1}\{(q-1)(\varepsilon + \varepsilon[w]) + S(2^{-1}, 2^{-1})\},$$

which can be written, by (3.4) and (5.11), as

$$2^{-1}(q-1)\varepsilon' + 2^{-1}|K_{2^{-1},2^{-1}}|\varepsilon'[u(1)wu(2^{-1})] + 4^{-1}S'(2^{-1}, 2^{-1}).$$

By (3.6) and (3.7), we have

$$E'(s, 2^{-1}) = \varepsilon'[u(1)wu(s)]\varepsilon'[u(1)wu(2^{-1})] \\ = 4^{-1}(\varepsilon[u(1)wu(s)] + \varepsilon[u(1)wu(1-s)])\varepsilon[u(1)wu(2^{-1})].$$

It follows from Table I that the right-side is given by

$$2^{-1}(\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)]) + 4^{-1}(S(s, 2^{-1}) + S(1-s, 2^{-1})),$$

which can be written, by (3.5) and (5.11), as

$$\varepsilon'[u(1)] + |K_{s,2^{-1}}|\varepsilon'[u(1)wu(2^{-1})] + 2^{-1}S'(s, 2^{-1}).$$

Finally we consider the product $E'(s,t) = \varepsilon'[u(1)wu(s)]\varepsilon'[u(1)wu(t)]$ for $s, t \in F'$. By (3.7) and the definition of $E(s,t)$, we have

$$E'(s,t) = 4^{-1}(E(s,t) + E(s, 1-t) + E(1-s, t) + E(1-s, 1-t)).$$

This implies $E'(s,s) = E'(s, 1-s)$. We can deduce from Table I

$$E'(s,s) = 2^{-1}(q-1)(\varepsilon + \varepsilon[w]) + 2^{-1}(\varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)]) \\ + 4^{-1}(S(s,s) + S(s, 1-s) + S(1-s, s) + S(1-s, 1-s)).$$

By (3.4), (3.5), (5.3) and (5.4), we obtain

$$E'(s,s) = (q-1)\varepsilon' + \varepsilon'[u(1)] + 2^{-1}(S(s,s) + S(1-s,s)).$$

Applying (5.11), we get

$$E'(s,s) = (q-1)\varepsilon' + \varepsilon'[u(1)] + 2|K_{s,s}|\varepsilon'[u(1)wu(2^{-1})] + S'(s,s).$$

For $s, t \in F'$ and $t \neq s, 1-s$, we can derive from Table I that

$$E'(s,t) = \varepsilon[u(1)] + \varepsilon[wu(1)] + \varepsilon[u(1)w] + \varepsilon[u(1)wu(1)]$$
$$+ 4^{-1}(S(s,t) + S(1-s,t) + S(s,1-t) + S(1-s,1-t)).$$

By (3.5), (5.3) and (5.4), we have

$$E'(s,t) = 2\varepsilon'[u(1)] + 2^{-1}(S(s,t) + S(1-s,t)).$$

Using (5.11), we get

$$E'(s,t) = 2\varepsilon'[u(1)] + 2|K_{s,t}|\varepsilon'[u(1)wu(2^{-1})] + S'(s,t).$$

References

[1] C. W. Curtis and T. V. Fossum, On Centralizer Rings and Characters of Representations of Finite Groups, Math. Zeitschr. **107** (1968), 402–406.
[2] W. Fulton and J. Harris, Representation Theory: A First Course, Springer-Verlag, N.Y., 1991.
[3] M. Hashizume and Y. Mori, Spectra of Vertex-Transitive Graphs and Hecke Algebras of Finite Groups, The Bulletin of Okayama Univ. Sci., **31** (1996), 7–15.
[4] M. Hashizume and Y. Mori, The Character Table of the Hecke Algebra $\mathscr{H}(GL_2(F_q), A)$, Preprint.
[5] N. Iwahori, On the structure of Hecke ring of a Chevalley group over a finite field, J. Fac. Sci. Univ. Tokyo, Sect. I, **10** (1964), 215–236.
[6] A. Krieg, Hecke Algebras, Mem. Amer. Math. Soc., **87** (1990) (#435).

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