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Existence and stability of solutions for a class of partial neutral functional differential equations

Mostafa Adimy and Khalil Ezzinbi*

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Abstract. In this work, we consider a class of nonlinear partial neutral functional differential equations with a nondensely defined Hille-Yosida operator. We first prove the local existence, uniqueness and regularity of solutions. Second, we study the global existence and stability. In the end, we extend in the autonomous case, results of Hale ([21], [23]) concerning dissipativeness and existence of a global attractor to our situation.

1. Introduction

In [45], Wu and Xia considered a system of partial neutral functional differential-difference equations, defined on the unit circle S^1, of the form

$$(1)\quad \frac{\partial}{\partial t}[x(.\,,t) - qx(.\,,t-\tau)] = K\frac{\partial^2}{\partial \xi^2}[x(.\,,t) - qx(.\,,t-\tau)] + f(x_t), \qquad t \geq 0,$$

where $\xi \in S^1$, K a positive constant and $0 \leq q < 1$. The space of initial data was chosen to be $\mathscr{C}([-\tau,0],H^1(S^1))$. This system is a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions. In ([22], [23]) Hale presented the basic theory of existence and uniqueness, and properties of the solution operator, as well as Hopf bifurcation and conditions for the stability and instability of periodic orbits for a more general class of PNFDE on the unit circle S^1. Let us briefly restate the equations considered by Hale in ([22], [23]). Let $E = H^1(S^1)$. If $\varphi \in \mathscr{C}_E := \mathscr{C}([-\tau,0],E)$, we write it as $\varphi(\theta,\xi)$, for $\theta \in [-\tau,0]$ and $\xi \in S^1$. For any function $\tilde{f} \in \mathscr{C}^{k+1}(\mathscr{C}([-\tau,0],\mathbf{R});\mathbf{R})$, $k \geq 1$, we let $f \in \mathscr{C}^{k+1}(\mathscr{C}_E, L^2(S^1))$ be defined by $f(\varphi)(\xi) = \tilde{f}(\varphi(.\,,\xi))$, $\xi \in S^1$. Let $\tilde{\mathscr{D}} \in \mathscr{L}(\mathscr{C}([-\tau,0],\mathbf{R});\mathbf{R})$ be defined by

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$$\begin{cases} \tilde{\mathscr{D}}\psi = \psi(0) - \tilde{g}(\psi), \\ \tilde{g}(\psi) = \int_{-\tau}^{0} [d_\theta \eta(\theta)]\psi(\theta), \end{cases}$$

where η is of bounded variation and non-atomic at 0; that is, there exists a continuous nondecreasing function $\delta : [0, \tau] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\left| \int_{-s}^{0} [d_\theta \eta(\theta)]\psi(\theta) \right| \le \delta(s)\|\psi\|, \qquad s \in [0, \tau],\ \psi \in \mathscr{C}([-\tau, 0], \mathbf{R}).$$

We define $\mathscr{D} \in \mathscr{L}(\mathscr{C}_E, E)$ as

$$\mathscr{D}(\varphi)(\xi) = \tilde{\mathscr{D}}(\varphi(\,.\,, \xi)), \qquad \xi \in S^1. \tag{2}$$

Hale considered in ([22], [23]), PNFDE of the form

$$\frac{\partial}{\partial t}\mathscr{D}x_t = K\frac{\partial^2}{\partial \xi^2}\mathscr{D}x_t + f(x_t), \qquad t \ge 0, \tag{3}$$

with $\mathscr{C}_E$ as a space of initial data. He considered the Laplace operator $A_0 = K\dfrac{\partial^2}{\partial \xi^2}$ with domain $H^2(S^1)$, which is a generator of a C_0-semigroup.

Motivated by the works discussed by Xia and Wu [45], and Hale ([22], [23]), we consider the following partial neutral functional differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}[u(t,x) - Bu(t-r,x)] \\ \qquad = A_0[u(t,x) - Bu(t-r,x)] + f(t,x,u_t(\cdot\,,x)), & t \ge 0,\ x \in \Omega, \\ u(t,x) = 0, & t \ge 0,\ x \in \partial\Omega, \\ u(\theta,x) = \varphi(\theta,x), & \theta \in [-r,0],\ x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set with regular boundary $\partial\Omega$, $B \in \mathscr{L}(E)$ and $A_0 = \Delta$ is the Laplace operator in the sense of distributions on Ω, which is a generator of a C_0-semigroup in $E = H_0^1(\Omega)$. If instead of $H_0^1(\Omega)$, one considers the space of continuous functions $\mathscr{C}(\bar{\Omega}, \mathbf{R})$, the domain of the operator A_0 is

$$D(A_0) = \{u \in \mathscr{C}(\bar{\Omega}, \mathbf{R}) : \Delta u \in \mathscr{C}(\bar{\Omega}, \mathbf{R}) \text{ and } u = 0 \text{ on } \partial\Omega\}$$

and the density property is not satisfied in $\mathscr{C}(\bar{\Omega}, \mathbf{R})$.

The idea of studying partial neutral functional differential equations with nondensely defined Hille-Yosida operator begins with [3], for a class of partial neutral functional differential-difference equations of the type

$$\frac{\partial}{\partial t}[x(t) - Bx(t-\tau)] = A_0[x(t) - Bx(t-\tau)] + Cx(t-\tau) + L(x_t), \qquad t \ge 0. \tag{4}$$

It was shown in particular that the solutions generate a locally Lipschitz continuous integrated semigroup. In [5], we considered a class of nonlinear partial neutral functional differential equations of the type

$$(5)\qquad \begin{cases} \dfrac{\partial}{\partial t}[x(t) - Lx_t] = A_0 x(t) + F(x_t), & t \geq 0, \\ x_0 = \varphi \in C_E, \end{cases}$$

where $A_0 : D(A_0) \subset E \to E$ is a Hille-Yosida operator, L is a continuous linear functional from $\mathscr{C}_E$ into E such that $Range(L) \subseteq D(A_0)$ and $\|L\| < 1$, and F is a globally Lipshitz continuous mapping from $\mathscr{C}_E$ into E. Basic existence and uniqueness results were given. In [4], we considered the following class of nonlinear partial neutral functional differential equations

$$(6)\qquad \begin{cases} \dfrac{\partial}{\partial t}[\mathscr{D}x_t - G(t, x_t)] = A_0[\mathscr{D}x_t - G(t, x_t)] + F(t, x_t), & t \geq 0, \\ x_0 = \varphi \in C_E. \end{cases}$$

In particular, we used a principle of linearized stability for strongly continuous semigroups given by Desh and Schappacher [14] to study, in the nonlinear autonomous case, the stability of solutions.

In this paper, we consider a class of nonlinear partial neutral functional differential equations of the type

$$(7)\qquad \begin{cases} \dfrac{\partial}{\partial t}\mathscr{D}u_t = A_0 \mathscr{D}u_t + F(t, u_t), & t \geq 0, \\ u_0 = \varphi \in \mathscr{C}_E, \end{cases}$$

where $A_0 : D(A_0) \subset E \to E$ is a linear operator on a Banach space $(E, |\cdot|)$, $\mathscr{C}_E$ is the space of all continuous functions on $[-\tau, 0]$ with values in E endowed with the uniform convergence topology, $\mathscr{D}$ is a bounded linear operator from $\mathscr{C}_E$ into E defined by

$$\mathscr{D}\varphi = \varphi(0) - P\varphi, \qquad \varphi \in \mathscr{C}_E,$$

F is an E-valued nonlinear continuous mapping on $\mathbf{R}_+ \times \mathscr{C}_E$ and for every $t \geq 0$, the function $u_t \in \mathscr{C}_E$ is defined by

$$(8)\qquad u_t(\theta) = u(t + \theta), \qquad \theta \in [-\tau, 0].$$

One can consider the following more general system

$$(9)\qquad \begin{cases} \dfrac{\partial}{\partial t}\mathscr{D}_1 u_t = A_0 \mathscr{D}_2 u_t + G(t, u_t), & t \geq 0, \\ u_0 = \varphi \in \mathscr{C}_E, \end{cases}$$

with two distinct bounded linear operators $\mathscr{D}_1$ and $\mathscr{D}_2$. But, for Problem (9) to be well posed, we need the following assumption

$$Range(\mathscr{D}_2 - \mathscr{D}_1) \subseteq D(A_0). \tag{10}$$

This assumption permits to write Equation (9) as an equation (7), with $F = A_0(\mathscr{D}_2 - \mathscr{D}_1) + G$ and $\mathscr{D} = \mathscr{D}_1$. Thanks to the closed graph theorem, (10) implies that

$$A_0(\mathscr{D}_2 - \mathscr{D}_1) \in \mathscr{L}(\mathscr{C}_E, E).$$

Then, Equation (9) can be solved along the same line as Equation (7) and is, indeed, covered by our study.

We assume, in this work, that A_0 is a Hille-Yosida operator on E. This means that A_0 satisfies the usual assumptions of the Hille-Yosida theorem characterizing the generators of strongly continuous semigroups except the density of $D(A_0)$ in E, i.e.,

–(**H1**) there exist $M_0 \geq 0$ and $\omega_0 \in \mathbf{R}$ such that $(\omega_0, +\infty) \subset \rho(A_0)$ and

$$\sup\{(\lambda - \omega_0)^n \|(\lambda I - A_0)^{-n}\| : n \in \mathbf{N}, \lambda > \omega_0\} \leq M_0.$$

There are many examples where A_0 is not densely defined. One can refer for this to [12] for more details.

The following assumption implies that Problem (7) is well posed and will be assumed throughout this paper.

–(**H2**) There exists a continuous nondecreasing function $\delta : [0, \tau] \to [0, +\infty)$, $\delta(0) = 0$ and a family of continuous linear operators $W_\varepsilon : \mathscr{C}_E \to E$, $\varepsilon \in [0, \tau]$, such that

$$|P\varphi - P_\varepsilon\varphi| \leq \delta(\varepsilon)\|\varphi\|, \qquad \varepsilon \in [0, \tau],\ \varphi \in \mathscr{C}_E,$$

where the linear operator $P_\varepsilon : \mathscr{C}_E \to E$, is defined, for $\varepsilon \in [0, \tau]$, by

$$\begin{cases} P_\varepsilon = W_\varepsilon \circ \tau_\varepsilon, \\ \tau_\varepsilon(\varphi)(\theta) = \varphi\left(\dfrac{\tau - \varepsilon}{\tau}\theta - \varepsilon\right), \quad \varphi \in \mathscr{C}_E,\ \theta \in [-\tau, 0]. \end{cases}$$

The idea of the assumption (**H2**) is very simple, it means that the operator P does not depend very strongly upon $\varphi(0)$. In particular, if P depends only upon of $\varphi(\theta)$ for $-\tau \leq \theta \leq -\varepsilon < 0$, then P satisfies (**H2**). It is not difficult to show that the operators $\mathscr{D}$ considered by Xia and Wu in [45] and Hale in ([22], [23]), satisfy the assumption (**H2**). This condition (**H2**) was introduced in [4], for the general problem (6).

In this paper, we first prove the local existence, uniqueness and regularity of solutions under a local condition on the nonlinear part. Second, we study the global existence and stability of the trivial solution and we give some simple examples. In the end, we extend in the autonomous case, results of Hale ([21], [23]) concerning dissipativeness and existence of a global attractor, to our situation. For global existence and stability, techniques employed in [27] were generalized to our equations. The method used in this work is based on the integrated semigroups theory.

2. Preliminaries

The following definitions are due to Arendt [6].

DEFINITION 2.1 [6]. Let E be a real Banach space. An integrated semigroup $(S(t))_{t\geq 0}$ is a family of bounded linear operators $S(t)$ on E, with the following properties

(i) $S(0)=0$;

(ii) for any $y\in E$, $t\to S(t)y$ is strongly continuous with values in E;

(iii) $S(s)S(t)=\int_0^s (S(t+r)-S(r))dr$ for $t,s\geq 0$.

DEFINITION 2.2 [6]. An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist constants $M\geq 0$ and $\omega\in\mathbf{R}$ such that $\|S(t)\|\leq Me^{\omega t}$ for all $t\geq 0$. Moreover, $(S(t))_{t\geq 0}$ is called non-degenerate, if $S(t)x=0$ for all $t\geq 0$ implies that $x=0$.

If $(S(t))_{t\geq 0}$ is an integrated semigroup, exponentially bounded, then the Laplace transform $R(\lambda):=\lambda\int_0^{+\infty} e^{-\lambda t}S(t)dt$ exists for λ with $\mathcal{R}e(\lambda)>\omega$, but $R(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is non-degenerate. $R(\lambda)$ satisfies the following equation

$$R(\lambda)-R(\mu)=(\mu-\lambda)R(\lambda)R(\mu),$$

and if $R(\lambda)$ is injective, there exists a unique operator A satisfying $(\omega,+\infty)\subset\rho(A)$ and

$$R(\lambda)=(\lambda I-A)^{-1},\qquad \lambda>\omega,$$

where $\rho(A)$ is the resolvent set of A. The operator A is called the generator of $(S(t))_{t\geq 0}$.

DEFINITION 2.3 [6]. An operator A is called a generator of an integrated semigroup, if there exists $\omega\in\mathbf{R}$ such that $(\omega,+\infty)\subset\rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded linear operators such that $S(0)=0$ and $(\lambda I-A)^{-1}=\lambda\int_0^{+\infty} e^{-\lambda t}S(t)dt$ for all $\lambda>\omega$.

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time).

Definition 2.4 [6]. An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous, if for all $h>0$ there exists a constant $k(h)>0$ such that

$$\|S(t)-S(s)\|\leq k(h)|t-s| \qquad \text{for all } t,s\in[0,h].$$

In this case, we have the following result.

Proposition 2.5 [6]. *Every locally Lipschitz continuous integrated semigroup is exponentially bounded.*

Definition 2.6 [6]. We say that a linear operator A satisfies the Hille-Yosida condition (HY) if there exist $M\geq 0$ and $\omega\in\mathbf{R}$ such that $(\omega,+\infty)\subset\rho(A)$ and

$$(HY) \qquad \sup\{(\lambda-\omega)^n\|(\lambda I-A)^{-n}\| : n\in\mathbf{N}, \lambda>\omega\}\leq M.$$

The following theorem shows that the Hille-Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroups.

Theorem 2.7 [34]. *The following assertions are equivalent*

(i) *A is the generator of a locally Lipschitz continuous integrated semigroup,*

(ii) *A satisfies the condition (HY).*

This result and the assumption (**H1**) show that the operator A_0 in Equation (7) is the generator of a locally Lipschitz continuous integrated semigroup $(S_0(t))_{t\geq 0}$ on E.

Proposition 2.8 [6]. *Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then, for all $x\in E$ and $t\geq 0$*

$$\int_0^t S(s)x\,ds\in D(A) \qquad \textit{and} \qquad S(t)x=A\int_0^t S(s)x\,ds+tx,$$

and the part A_Y of A in $Y:=\overline{D(A)}$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on Y and we have, for $x\in Y$, $t\geq 0$, $S(t)x=\int_0^t T(s)x\,ds$.

We now turn to our problem and remark that Proposition 2.8 implies that $S_0'(t):\overline{D(A_0)}\to\overline{D(A_0)}$ is a C_0-semigroup, where $(S_0(t))_{t\geq 0}$ is the integrated semigroup generated by the operator A_0 on E. To prove our results on global existence, stability and existence of global attractor we will need to assume that

–(**H3**) there exist constants $K_0\geq 1$ and $\omega_0>0$, such that

$$|S_0'(t)y|\leq K_0e^{-\omega_0 t}|y| \qquad \text{for all } t\geq 0 \text{ and } y\in\overline{D(A_0)},$$

and

–(**H4**) the operator $\mathscr{D}$ is stable, i.e., there exist positive constants α,β such that the solution of the homogeneous functional equation

$$\begin{cases} \mathscr{D}u_t = 0, & t \geq 0, \\ u_0 = \varphi, \end{cases}$$

with $\varphi \in \mathscr{C}_E$ and $\mathscr{D}\varphi = 0$, satisfies the inequality

$$\|u_t\| \leq \beta e^{-\alpha t}\|\varphi\|,$$

where u_t, $t \geq 0$, is defined by (8).

We remark that the operator $\mathscr{D}\varphi = \varphi(0) - q\varphi(-\tau)$ considered by Xia and Wu in [45] is stable if $0 \leq q < 1$ and it is not stable if $q \geq 1$.

Consider the following Cauchy problem, for $\varphi \in \mathscr{C}_E$,

$$\begin{cases} \dfrac{\partial}{\partial t}\mathscr{D}u_t = A_0\mathscr{D}u_t & \text{if } t \geq 0, \\ u(t) = \varphi(t) & \text{if } t \in [-\tau, 0]. \end{cases} \tag{11}$$

This system is a strong version of an integrated once given by

$$\begin{cases} \mathscr{D}u_t = S_0'(t)\mathscr{D}\varphi, & t \geq 0, \\ u_0 = \varphi, \end{cases} \tag{12}$$

where

$$\varphi \in Y := \{\varphi \in \mathscr{C}_E : \mathscr{D}\varphi \in \overline{D(A_0)}\}. \tag{13}$$

Consider the general nonhomogeneous equation

$$\mathscr{D}v_t = h(t), \qquad t \geq 0, \tag{14}$$

with the initial condition

$$v_0 = \varphi \in \mathscr{C}_E.$$

We have the following preliminary result.

Lemma 2.9. *Assume that the conditions* (**H2**) *and* (**H4**) *are satisfied. Then, there are positive constants a, b, c, d such that, for any $\varepsilon \in (0, \tau]$ sufficiently small and any continuous function h from $[0, +\infty)$ into E, the solution v of the equation* (14) *satisfies the inequality*

$$\|v_t\| \leq e^{-a(t-\varepsilon)}\left[b\|v_0\| + c \sup_{0 \leq s \leq \varepsilon} |h(s)|\right] + d \sup_{\max(\varepsilon, t-\tau) \leq s \leq t} |h(s)|, \qquad t \geq \varepsilon. \tag{15}$$

Proof. The idea of the proof comes from [11], in which there is the same estimate (15) but in finite dimensional case. First, it is clear that there exist positive constants K_1, K_2 and K_3 such that

$$\|v_t\| \leq \left[K_1\|v_0\| + K_2 \sup_{0 \leq s \leq t} |h(s)|\right] e^{K_3 t}, \qquad t \geq 0. \tag{16}$$

To prove (15) we will make a transformation of variables in Equation (14) such that $h(\varepsilon) = 0$.

Consider the mapping $\Lambda : E \to \mathscr{C}_E$ be defined, for $c \in E$ and $\theta \in [-\tau, 0]$ by

$$\Lambda(c)(\theta) = \begin{cases} 0, & -\tau \le \theta \le -\varepsilon, \\ \left(1 + \dfrac{\theta}{\varepsilon}\right)c, & -\varepsilon < \theta \le 0. \end{cases} \tag{17}$$

Since

$$-\tau \le \frac{\tau - \varepsilon}{\tau}\theta - \varepsilon \le -\varepsilon \qquad \text{for all } \theta \in [-\tau, 0],$$

(17) implies, for all $c \in E$ and $\theta \in [-\tau, 0]$, that

$$\tau_\varepsilon(\Lambda(c))(\theta) = 0,$$

where τ_ε is given by (**H2**). Hence

$$P_\varepsilon(\Lambda(c)) = 0 \qquad \text{for all } c \in E.$$

Consequently

$$\|P(\Lambda(c))\| \le \delta(\varepsilon)|c|.$$

We can choose $\varepsilon \in (0, \tau]$ sufficiently small such that

$$\delta(\varepsilon) < 1.$$

We conclude that the linear operator $\mathscr{D}(\Lambda) : E \to E$ defined by

$$\mathscr{D}(\Lambda)(c) = \mathscr{D}(\Lambda(c)),$$

is invertible.

We make now the following transformation of variables in Equation (14)

$$z(t) = v(t) - y(t) \qquad \text{for } t \ge \varepsilon - \tau.$$

where $y : [\varepsilon - \tau, +\infty) \to E$ is defined by

$$y(t) = \begin{cases} \Lambda([\mathscr{D}(\Lambda)]^{-1}(h(\varepsilon)))(t - \varepsilon), & \varepsilon - \tau \le t \le \varepsilon, \\ [\mathscr{D}(\Lambda)]^{-1}(h(t)), & t > \varepsilon. \end{cases}$$

Thus, we can rewrite Equation (14) as

$$\mathscr{D}(z_t) = h^*(t), \qquad t \ge \varepsilon, \tag{18}$$

where

$$h^*(t) = h(t) - \mathscr{D}(y_t), \qquad t \geq \varepsilon.$$

Note that

$$y_\varepsilon(\theta) = y(\varepsilon + \theta) = \Lambda([\mathscr{D}(\Lambda)]^{-1}(h(\varepsilon)))(\theta) \qquad \text{for } \theta \in [-\tau, 0].$$

This gives

$$\mathscr{D}(y_\varepsilon) = \mathscr{D}(\Lambda([\mathscr{D}(\Lambda)]^{-1}(h(\varepsilon)))) = h(\varepsilon).$$

Hence

$$h^*(\varepsilon) = 0.$$

We can now start the proof of the estimate (15).

It is immediate that

$$|h^*(t)| \leq |h(t)| + K_4\|y_t\|, \qquad t \geq \varepsilon,$$

and

$$\|y_t\| = \sup_{t-\tau \leq s \leq t} |y(s)|, \qquad t \geq \varepsilon.$$

Let $s \in [t-\tau, t]$. If $t - \tau \geq \varepsilon$

$$y(s) = [\mathscr{D}(\Lambda)]^{-1}(h(s)),$$

and if $t - \tau < \varepsilon$

$$y(s) = \begin{cases} \Lambda([\mathscr{D}(\Lambda)]^{-1}(h(\varepsilon)))(s - \varepsilon), & t - \tau \leq s \leq \varepsilon, \\ [\mathscr{D}(\Lambda)]^{-1}(h(s)), & \varepsilon < s \leq t. \end{cases}$$

Then, we can assert that

(19)
$$\|y_t\| \leq K_5 \sup_{\max(\varepsilon, t-\tau) \leq s \leq t} |h(s)|, \qquad t \geq \varepsilon,$$

and

(20)
$$|h^*(t)| \leq K_6 \sup_{\max(\varepsilon, t-\tau) \leq s \leq t} |h(s)|, \qquad t \geq \varepsilon.$$

Our next objective is to estimate $\|z_t\|$, for $t \geq \varepsilon$.

By the superposition principle of solutions of linear systems, we have

$$z(t) = z^1(t) + z^2(t), \qquad t \geq \varepsilon - \tau,$$

where

$$\begin{cases} \mathscr{D}(z_t^1) = 0, & t \geq \varepsilon, \\ z_\varepsilon^1 = z_\varepsilon, \end{cases}$$

and

$$\begin{cases} \mathscr{D}(z_t^2) = h^*(t), & t \geq \varepsilon, \\ z_\varepsilon^1 = 0. \end{cases}$$

Since $\mathscr{D}$ is stable, it follows that

$$\|z_t^1\| \leq \beta e^{-\alpha(t-\varepsilon)}\|z_\varepsilon\|.$$

As $z_\varepsilon = v_\varepsilon - y_\varepsilon$, we obtain

$$\|z_\varepsilon\| \leq \|v_\varepsilon\| + K_5|h(\varepsilon)|.$$

Applying (16), we conclude that

$$\|v_\varepsilon\| \leq K_7\|v_0\| + K_8 \sup_{0 \leq s \leq \varepsilon} |h(s)|. \tag{21}$$

This gives

$$\|z_t^1\| \leq \beta e^{-\alpha(t-\varepsilon)}\left(K_7\|v_0\| + (K_5 + K_8) \sup_{0 \leq s \leq \varepsilon} |h(s)|\right).$$

We also have

$$\|z_t^2\| \leq K_9 \sup_{\varepsilon \leq s \leq t} |h^*(s)|,$$

(see, for example, Theorem 2.1 in [17] which is easy to extend to infinite dimensional case). Then, (20) implies

$$\|z_t^2\| \leq K_6 K_9 \sup_{\varepsilon \leq s \leq t}\left(\sup_{\max(\varepsilon, s-\tau) \leq \sigma \leq s} |h(\sigma)|\right) \leq K_{10} \sup_{\max(\varepsilon, t-\tau) \leq s \leq t} |h(s)|.$$

Consequently, for $t \geq \varepsilon$

$$\|z_t\| \leq e^{-\alpha(t-\varepsilon)}\left(\beta K_7\|v_0\| + \beta(K_5 + K_8) \sup_{0 \leq s \leq \varepsilon} |h(s)|\right) + K_{10} \sup_{\max(\varepsilon, t-\tau) \leq s \leq t} |h(s)|.$$

Finally, using (19) we obtain

$$\|v_t\| \leq e^{-\alpha(t-\varepsilon)}\left(\beta K_7\|v_0\| + \beta(K_5 + K_8) \sup_{0 \leq s \leq \varepsilon} |h(s)|\right) + (K_5 + K_{10}) \sup_{\max(\varepsilon, t-\tau) \leq s \leq t} |h(s)|.$$

As the interval $(0, \tau]$ is bounded, the constants K_i can be chosen independent of ε. This completes the proof. □

The estimate (15) is very interesting because, if $|h(s)|$ is bounded on $[0, +\infty)$, then the ultimate bound on v_t as $t \to +\infty$ is determined by the bound on $|h(s)|$ for s in the delay interval $[t-\tau, t]$ as $t \to +\infty$.

Proposition 2.10 [4]. *Assume that the conditions* (**H1**) *and* (**H2**) *are satisfied. Then, for given* $\varphi \in Y$ *there exists a unique function* $u \in \mathscr{C}([-\tau, +\infty), E)$ *which solves Equation* (12) *and, the family of operators* $(T(t))_{t \geq 0}$*, defined on* Y *by* $T(t)\varphi = u_t(.\,, \varphi)$ *is a* C_0*-semigroup on* Y.

The estimate (15) for $h(t) = S_0'(t)\mathscr{D}\varphi$ and $\varepsilon \to 0$, and the condition (**H3**) prove the following result.

Proposition 2.11. *Assume that the conditions* (**H1**), (**H2**), (**H3**) *and* (**H4**) *are satisfied. Then, there exist constants* $K \geq 1$ *and* $\omega > 0$ *such that*

$$\|T(t)\varphi\| \leq Ke^{-\omega t}\|\varphi\| \qquad \text{for } t \geq 0 \text{ and } \varphi \in Y. \tag{22}$$

As in [4], we will define a fundamental integral solution $Z(t)$, associated to Equation (7). Consider, for $c \in E$ given, the following equation

$$\begin{cases} \mathscr{D}z_t = S_0(t)c & \text{if } t \geq 0, \\ z(t) = 0 & \text{if } t \in [-\tau, 0]. \end{cases} \tag{23}$$

Proposition 2.12. *Assume that the conditions* (**H1**) *and* (**H2**) *are satisfied. Then, Problem* (23) *has a unique solution* $z := z(.)c$ *which is a continuous mapping from* $[-\tau, +\infty)$ *into* E. *Moreover, the operator* $Z(t) : E \to \mathscr{C}_E$ *defined by*

$$Z(t)c = z_t(.)c$$

satisfies the following properties

(i) *there exist* $\alpha \geq 0$ *and* $\beta \in \mathbf{R}$ *such that* $\|Z(t)\| \leq \alpha e^{t\beta}$, *for* $t \geq 0$;

(ii) $Z(t)(E) \subseteq Y$, *for* $t \geq 0$;

(iii) *for all* $H > 0$ *there exists a constant* $k(H) > 0$ *such that*

$$\|Z(t)c - Z(s)c\| \leq k(H)|t-s|\,|c| \qquad \text{for } t, s \in [0, H] \text{ and } c \in E;$$

(iv) *for any continuous function* $f : [0, +\infty) \to E$, *the functions*

$$t \mapsto \int_0^t Z(t-s)f(s)ds \qquad \text{and} \qquad t \mapsto \int_0^t S_0(t-s)f(s)ds$$

are continuously differentiable for $t \geq 0$ *and satisfy*

$$\frac{d}{dt}\int_0^t Z(t-s)f(s)ds = \lim_{h \to 0^+} \frac{1}{h}\int_0^t T(t-s)Z(h)f(s)ds$$

and

$$\mathscr{D}\left(\frac{d}{dt}\int_0^t Z(t-s)f(s)ds\right) = \lim_{h\to 0^+}\frac{1}{h}\int_0^t S_0'(t-s)S_0(h)f(s)ds,$$
$$= \frac{d}{dt}\int_0^t S_0(t-s)f(s)ds.$$

Proof. The proof of the existence, uniqueness and continuity of the solutions is easy and it is omitted.

The property (i) comes from (16) with $h(t) = S_0(t)c$ and $v_0 = 0$.

If $c \in E$, then $S_0(t)c \in \overline{D(A_0)}$, for $t \geq 0$. This implies that $Z(t)c \in Y$. Then (ii) is true.

Since $S_0(.)$ is locally Lipschitz continuous, then (iii) holds.

We will prove now the assertion (iv). It is clear that the functions

$$t \mapsto \int_0^t Z(t-s)f(s)ds \qquad \text{and} \qquad t \mapsto \int_0^t S_0(t-s)f(s)ds$$

are continuously differentiable (see for example the proof of Theorem 2.5 in [34]). We know by the definition of an integrated semigroup that

$$S_0'(t)S_0(h)c = S_0(t+h)c - S_0(t)c,$$

for $t, h \geq 0$ and $c \in E$. On the other hand, we have

$$\frac{d}{dt}\int_0^t S_0(t-s)f(s)ds = \lim_{h\to 0^+}\left(\frac{1}{h}\int_0^t (S_0(t+h-s) - S_0(t-s))f(s)ds + \frac{1}{h}\int_t^{t+h} S_0(t+h-s)f(s)ds\right).$$

If we put, in the second integral of the right-hand side, $r = \frac{1}{h}(s-t)$, we obtain

$$\frac{1}{h}\int_t^{t+h} S_0(t+h-s)f(s)ds = \int_0^1 S_0(h(1-r))f(t+hr)dr.$$

Since $S_0(0) = 0$, we obtain

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h} S_0(t+h-s)f(s)ds = 0.$$

Hence

$$\frac{d}{dt}\int_0^t S_0(t-s)f(s)ds = \lim_{h\to 0^+}\frac{1}{h}\int_0^t (S_0(t+h-s) - S_0(t-s))f(s)ds.$$

It follows that, for $t \geq 0$

$$\frac{d}{dt}\int_0^t S_0(t-s)f(s)ds = \lim_{h\to 0^+} \frac{1}{h}\int_0^t S_0'(t-s)S_0(h)f(s)ds.$$

On the other hand, by the definition of $T(t)$ and $Z(t)$

$$T(t)Z(h)c = Z(t+h)c - Z(t)c.$$

Then, we can use the same argument as above to prove that

$$\frac{d}{dt}\int_0^t Z(t-s)f(s)ds = \lim_{h\to 0^+} \frac{1}{h}\int_0^t T(t-s)Z(h)f(s)ds.$$

Consequently

$$\mathscr{D}\left(\frac{d}{dt}\int_0^t Z(t-s)f(s)ds\right) = \lim_{h\to 0^+} \frac{1}{h}\int_0^t \mathscr{D}(T(t-s)Z(h)f(s))ds.$$

Since

$$\mathscr{D}(T(s)Z(h)c) = S_0'(s)\mathscr{D}(Z(h)c) = S_0'(s)S_0(h)c,$$

it follows that

$$\begin{aligned}\mathscr{D}\left(\frac{d}{dt}\int_0^t Z(t-s)f(s)ds\right) &= \lim_{h\to 0^+} \frac{1}{h}\int_0^t S_0'(t-s)S_0(h)f(s)ds,\\ &= \frac{d}{dt}\int_0^t S_0(t-s)f(s)ds.\end{aligned}$$

This proves (iv). □

Let $T > 0$ and $\varphi \in \mathscr{C}_E$ such that $\mathscr{D}\varphi \in \overline{D(A_0)}$. We consider the following definitions.

Definition 2.13. We say that a function $u \in \mathscr{C}([-\tau, T], E)$ is a mild solution of Equation (7) if

$$\begin{cases} \mathscr{D}u_t = S_0'(t)\mathscr{D}\varphi + \dfrac{d}{dt}\displaystyle\int_0^t S_0(t-s)F(s,u_s)ds, & 0 \le t \le T,\\ u_0 = \varphi. \end{cases}$$

Note that the property (iv) of Proposition 2.12 says that the function $t \mapsto \int_0^t S_0(t-s)F(s,u_s)ds$ is continuously differentiable. Then, it gives a sense to Definition 2.13.

Definition 2.14. We say that a function $u \in \mathscr{C}([-\tau, T], E)$ is a strict solution of Equation (7) if

(i) $t \to \mathscr{D}u_t \in \mathscr{C}^1([0,T],E) \cap \mathscr{C}([0,T],D(A_0))$,
(ii) $t \mapsto u_t$ satisfies Equation (7) on $[0,T]$,
(iii) $u_0 = \varphi$.

A sufficient condition for the mild solution of Equation (7) to be a strict solution is given by the following lemma.

Lemma 2.15. *If u is a mild solution of Equation* (7) *such that $t \to \mathscr{D}u_t \in \mathscr{C}^1([0,T],E)$, then u is a strict solution.*

Proof. We know (see [34]) that if u is a mild solution of Equation (7) then

$$\int_0^t S_0(t-s)F(s,u_s)ds \in D(A_0)$$

and

$$\mathscr{D}u_t = S_0'(t)\mathscr{D}\varphi + A_0\int_0^t S_0(t-s)F(s,u_s)ds + \int_0^t F(s,u_s)ds. \tag{24}$$

On the other hand, since $\mathscr{D}\varphi \in \overline{D(A_0)}$, we have

$$S_0(t)\mathscr{D}\varphi \in D(A_0) \qquad \text{and} \qquad S_0'(t)\mathscr{D}\varphi = \mathscr{D}\varphi + A_0 S_0(t)\mathscr{D}\varphi.$$

So, we deduce that

$$S_0(t)\mathscr{D}\varphi + \int_0^t S_0(t-s)F(s,u_s)ds \in D(A_0)$$

and

$$\mathscr{D}u_t = \mathscr{D}\varphi + A_0\left(S_0(t)\mathscr{D}\varphi + \int_0^t S_0(t-s)F(s,u_s)ds\right) + \int_0^t F(s,u_s)ds.$$

It follows that

$$\int_0^t \mathscr{D}u_s\, ds \in D(A_0)$$

and

$$\mathscr{D}u_t = \mathscr{D}\varphi + A_0\left(\int_0^t \mathscr{D}u_s\, ds\right) + \int_0^t F(s,u_s)ds. \tag{25}$$

If we assume that $t \to \mathscr{D}u_t \in \mathscr{C}^1([0,T],E)$, then we have the existence of the limit

$$\lim_{h\to 0} A_0\left(\frac{1}{h}\int_t^{t+h} \mathscr{D}u_s\, ds\right) = \frac{d}{dt}\mathscr{D}u_t - F(t,u_t).$$

Since the operator A_0 is closed, it follows that

$$\lim_{h\to 0}\left(\frac{1}{h}\int_t^{t+h} \mathscr{D}u_s\, ds\right) = \mathscr{D}u_t \in D(A_0)$$

and

$$A_0\mathscr{D}u_t = \frac{d}{dt}\mathscr{D}u_t - F(t,u_t).$$

Consequently, u is a strict solution of Equation (7). □

Proposition 2.16. *If there exists a mild solution* $u := u(.,\varphi) \in \mathscr{C}([-\tau,T],E)$ *of Equation* (7)*, then the function* $t \in [0,T] \to u_t \in \mathscr{C}_E$ *satisfies*

$$u_t = T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s,u_s)ds, \tag{26}$$

$$= T(t)\varphi + \lim_{h\to 0^+}\frac{1}{h}\int_0^t T(t-s)Z(h)F(s,u_s)ds.$$

Conversely, if there exists a function $v \in \mathscr{C}([0,T],\mathscr{C}_E)$ *such that*

$$v(t) = T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s,v(s))ds, \qquad t \in [0,T], \tag{27}$$

then $v(t) = u_t$, $t \in [0,T]$, *where*

$$u(t) = \begin{cases} v(t)(0), & \text{if } t \in [0,T], \\ \varphi(t), & \text{if } t \in [-\tau,0], \end{cases} \tag{28}$$

and $u(t)$ *is a mild solution of Equation* (7).

Proof. In the beginning we have, from Proposition 2.12, the following observation. If $f : [0,T] \to E$ is a continuous function, then

$$W(t) := \int_0^t Z(t-s)f(s)ds$$

is continuously differentiable and $W'(0) = 0$. Set

$$w(t) = \begin{cases} (W(t))(0), & \text{if } t \geq 0, \\ 0, & \text{if } t \in [-\tau,0]. \end{cases}$$

Then $w(t)$ is continuously differentiable, and it is given by

$$w(t) = \left(\int_0^t Z(t-s)f(s)ds\right)(0) = \int_0^t (Z(t-s)f(s))(0)ds,$$
$$= \int_0^t z(t-s)f(s)ds.$$

Furthermore, for $t \geq 0$ and $\theta \in [-\tau, 0]$

$$(W(t))(\theta) = \left(\int_0^t Z(t-s)f(s)ds\right)(\theta) = \int_0^t (Z(t-s)f(s))(\theta)ds,$$
$$= \int_0^t z(t+\theta-s)f(s)ds.$$

Since $z(s) = 0$ for $s \in [-\tau, 0]$, we have

$$\int_0^t z(t+\theta-s)f(s)ds = \int_0^{t+\theta} z(t+\theta-s)f(s)ds.$$

Thus it follows that $(W(t))(\theta) = w(t+\theta)$, that is $W(t) = w_t$. Furthermore, since

$$\lim_{h\to 0^+} \frac{1}{h}(W(t+h) - W(t)) = W'(t)$$

in the sense of supremum norm, it follows that

$$(W'(t))(\theta) = \lim_{h\to 0^+} \left(\frac{1}{h}(W(t+h) - W(t))\right)(\theta),$$
$$= \lim_{h\to 0^+} \frac{1}{h}(W(t+h)(\theta) - W(t)(\theta)),$$
$$= \lim_{h\to 0^+} \frac{1}{h}(w(t+h+\theta) - w(t+\theta)) = w'(t+\theta),$$

that is, $W'(t) = (w')_t$.

We first prove the latter half of the proposition. Suppose that $v(t)$ is a solution of Equation (27). The function $T(t)\varphi$ is given by $T(t)\varphi = x_t$, where $x \in \mathscr{C}([-\tau, +\infty), E)$ is the solution of $\mathscr{D}(x_t) = S_0'(t)\mathscr{D}\varphi$ such that $x_0 = \varphi$. From the above observation, we set

$$w(t) = \int_0^t z(t-s)F(s, v(s))ds.$$

Then it follows that

$$v(t) = x_t + (w')_t = (x + w')_t.$$

Thus, if we set $u(t) = x(t) + w'(t)$, then $v(t) = u_t$ and

$$u_t = T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, v(s))ds,$$
$$= T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds.$$

Since

$$\mathscr{D}(T(t)\varphi) = S_0'(t)\mathscr{D}\varphi,$$

and since

$$\mathscr{D}\left(\frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds\right) = \frac{d}{dt}\int_0^t S_0(t-s)F(s, u_s)ds,$$

it follows that $u(t)$ is a mild solution.

If $u(t)$ is a mild solution, then from the definition of $T(t)$ we have that

$$\mathscr{D}(u_t) = S_0'(t)\mathscr{D}\varphi + \frac{d}{dt}\int_0^t S_0(t-s)F(s, u_s)ds,$$
$$= \mathscr{D}(T(t)\varphi) + \mathscr{D}\left(\frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds\right),$$
$$= \mathscr{D}\left(T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds\right),$$
$$= \mathscr{D}(x_t + (w')_t),$$

where $x(t)$ is the solution of $\mathscr{D}(x_t) = S_0'(t)\mathscr{D}\varphi$, and $w(t)$ is defined as

$$w(t) = \int_0^t z(t-s)F(s, u_s)ds.$$

Hence $\mathscr{D}((u-(x+w'))_t) = 0$; consequently $u-(x+w') = 0$. Therefore,

$$u_t = x_t + (w')_t$$
$$= T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, u_s)ds.$$

This completes the proof of Proposition 2.16. □

3. Local existence and regularity of solutions

In this section, we will prove the local existence, uniqueness and regularity of solutions of Equation (7), under the assumptions (**H1**), (**H2**) and the following additional condition.

–(**H5**) $F:[0,+\infty)\times\mathscr{C}_E\to E$ is continuous and locally Lipschitz continuous with respect to φ, i.e., for each $r>0$ there exists a constant $C_0(r)>0$ such that if $t\ge 0$, $\varphi_1,\varphi_2\in\mathscr{C}_E$ and $\|\varphi_1\|,\|\varphi_2\|\le r$ then

$$|F(t,\varphi_1)-F(t,\varphi_2)|\le C_0(r)\|\varphi_1-\varphi_2\|.$$

Theorem 3.1. *Suppose that the assumptions* (**H1**), (**H2**) *and* (**H5**) *are satisfied and $\varphi\in\mathscr{C}_E$ such that $\mathscr{D}\varphi\in\overline{D(A_0)}$. Then, there exists a maximal interval of existence $[-\tau,t_\varphi)$, $t_\varphi>0$, and a unique mild solution $u(.,\varphi)$ of Equation* (7)*, defined on $[-\tau,t_\varphi)$ and either $t_\varphi=+\infty$ or*

$$\limsup_{t\to t_\varphi^-}\|u_t(.,\varphi)\|=+\infty.$$

Moreover, $u_t(.,\varphi)$ is a continuous function of φ, in the sense that if $\varphi\in\mathscr{C}_E$, $\mathscr{D}\varphi\in\overline{D(A_0)}$ and $t\in[0,t_\varphi)$, then there exist positive constants L and α such that, for $\psi\in\mathscr{C}_E$, $\mathscr{D}\psi\in\overline{D(A_0)}$ and $\|\varphi-\psi\|<\alpha$, we have $t\in[0,t_\psi)$ and

$$\|u_s(.,\varphi)-u_s(.,\psi)\|\le L\|\varphi-\psi\|\qquad \text{for all } s\in[0,t].$$

Proof. Note that (**H5**) implies that, for each $r>0$ there exists $C_0(r)>0$ such that

$$|F(t,\varphi)|\le rC_0(r)+|F(t,0)|\qquad \text{for } t\ge 0,\ \varphi\in\mathscr{C}_E \text{ and } \|\varphi\|\le r.$$

Let $T_1>0$. Suppose that $\varphi\in\mathscr{C}_E$, $\mathscr{D}\varphi\in\overline{D(A_0)}$, $r:=\|\varphi\|+1$ and

$$c_1:=rC_0(r)+\sup_{t\in[0,T_1]}|F(t,0)|.$$

Consider the following set

$$\Omega_\varphi:=\left\{v\in\mathscr{C}([0,T_1],\mathscr{C}_E):\sup_{0\le s\le T_1}\|v(s)-\varphi\|\le 1\right\},$$

where $\mathscr{C}([0,T_1],\mathscr{C}_E)$ is endowed with the uniform convergence topology. It is clear that Ω_φ is a closed set of $\mathscr{C}([0,T_1],\mathscr{C}_E)$. Consider the mapping

$$H:\Omega_\varphi\to\mathscr{C}([0,T_1],\mathscr{C}_E),$$

defined, for $v\in\Omega_\varphi$ and $t\in[0,T_1]$, by

$$\begin{aligned}H(v)(t)&:=T(t)\varphi+\frac{d}{dt}\int_0^t Z(t-s)F(s,v(s))ds,\\&=T(t)\varphi+\lim_{h\to 0^+}\frac{1}{h}\int_0^t T(t-s)Z(h)F(s,v(s))ds.\end{aligned}$$

We will show that

$$H(\Omega_\varphi) \subseteq \Omega_\varphi.$$

On can remark, as in the proof of Proposition 2.2 of [34], that

$$\limsup_{h\to 0^+} \frac{1}{h}\|Z(h)\| < +\infty.$$

Then we can set

$$k := \limsup_{h\to 0^+} \frac{1}{h}\|Z(h)\|. \tag{29}$$

We have for suitable constants M and ω

$$\|H(v)(t) - \varphi\| \le \|T(t)\varphi - \varphi\| + Me^{\omega t}\int_0^t e^{-\omega s}\frac{1}{h}\|Z(h)\|\,|F(s, v(s))|ds.$$

We can assume here without loss of generality that $\omega > 0$. Thus we obtain the estimate

$$\|H(v)(t) - \varphi\| \le \|T(t)\varphi - \varphi\| + Mke^{\omega t}\int_0^t |F(s, v(s))|ds.$$

Since $\|v(s) - \varphi\| \le 1$, for $s \in [0, T_1]$ and $r = \|\varphi\| + 1$, we obtain that $\|v(s)\| \le r$, for $s \in [0, T_1]$. Then

$$|F(s, v(s))| \le C_0(r)\|v(s)\| + |F(s, 0)| \le c_1.$$

Consider $T_1 > 0$ sufficiently small for

$$\sup_{0\le s\le T_1}\{\|T(s)\varphi - \varphi\| + Mke^{\omega s}c_1 s\} < 1.$$

Then, we deduce, for $t \in [0, T_1]$

$$\|H(v)(t) - \varphi\| \le \|T(t)\varphi - \varphi\| + Mke^{\omega t}c_1 t < 1.$$

Hence

$$H(\Omega_\varphi) \subseteq \Omega_\varphi.$$

On the other hand, let $u, v \in \Omega_\varphi$ and $t \in [0, T_1]$. We have

$$\|H(u)(t) - H(v)(t)\| \le Mke^{\omega T_1}C_0(r)T_1\|u - v\|_{\mathscr{C}([0,T_1],\mathscr{C}_E)}.$$

Note that $r \ge 1$. Then, by definition of c_1, $C_0(r) \le c_1$. Consequently

$$Mke^{\omega T_1}C_0(r)T_1 \le \sup_{0\le s\le T_1}\{\|T(s)\varphi - \varphi\| + Mke^{\omega s}c_1 s\} < 1.$$

We conclude that there exists a unique function $v \in \Omega_\varphi$ such that $H(v) = v$. Then, Equation (7) has one and only one mild solution $u : [-\tau, T_1] \to E$ defined by

$$u(t) = \begin{cases} v(t)(0) & \text{if } t \geq 0, \\ \varphi(t) & \text{if } t \in [-\tau, 0]. \end{cases}$$

Let $[-\tau, t_\varphi)$ be the maximal interval of existence of u. Assume that $t_\varphi < +\infty$ and

$$\limsup_{t \to t_\varphi^-} \|u_t(\,.\,, \varphi)\| < +\infty.$$

Then, there exists a constant $r > 0$ such that $\|u_t(\,.\,, \varphi)\| \leq r$, for all $t \in [0, t_\varphi)$. Let $t, t+h \in [0, t_\varphi)$, $h > 0$. We obtain

$$\begin{aligned} u_{t+h} &= T(t+h)\varphi + \lim_{d \to 0^+} \frac{1}{d} \int_0^{t+h} T(t+h-s) Z(d) F(s, u_s)\,ds, \\ &= T(t+h)\varphi + \lim_{d \to 0^+} \frac{1}{d} \left(\int_0^h T(t)T(h-s) Z(d) F(s, u_s)\,ds \right. \\ &\qquad \left. + \int_h^{t+h} T(t+h-s) Z(d) F(s, u_s)\,ds \right) \end{aligned}$$

and

$$u_t = T(t)\varphi + \lim_{d \to 0^+} \frac{1}{d} \int_0^t T(t-s) Z(d) F(s, u_s)\,ds.$$

Since

$$\int_h^{t+h} T(t+h-s) Z(d) F(s, u_s)\,ds = \int_0^t T(t-s) Z(d) F(s+h, u_{s+h})\,ds,$$

we have

$$\begin{aligned} (30) \quad u_{t+h} - u_t &= T(t+h)\varphi - T(t)\varphi + \lim_{d \to 0^+} \frac{1}{d} \left(T(t) \int_0^h T(h-s) Z(d) F(s, u_s)\,ds \right. \\ &\qquad \left. + \int_0^t T(t-s) Z(d) (F(s+h, u_{s+h}) - F(s, u_s))\,ds \right). \end{aligned}$$

From Proposition 2.12 (iv), we have that

$$\lim_{d \to 0^+} \frac{1}{d} \int_0^h T(h-s) Z(d) F(s, u_s)\,ds = \frac{d}{dh} \int_0^h Z(h-s) F(s, u_s)\,ds.$$

This means that the limit in (30) can be separated in two parts. Hence we have

$$u_{t+h} - u_t = T(t+h)\varphi - T(t)\varphi + \lim_{d\to 0^+} \frac{1}{d} T(t) \int_0^h T(h-s)Z(d)F(s,u_s)ds$$
$$+ \lim_{d\to 0^+} \frac{1}{d} \int_0^t T(t-s)Z(d)(F(s+h,u_{s+h}) - F(s,u_s))ds.$$

We estimate each limit in the right-side. In the beginning,

$$\left\| T(t) \int_0^h T(h-s)Z(d)F(s,u_s)ds \right\| \le Me^{\omega(t+h)} \int_0^h \|Z(d)\| \, |F(s,u_s)| ds,$$
$$\le Me^{\omega(t+h)} \|Z(d)\| h \left(rC_0(r) + \sup_{s\in[0,t_\varphi]} |F(s,0)| \right).$$

Notice that $\sup_{s\in[0,t_\varphi]} |F(s,0)| < \infty$ since $t_\varphi < \infty$. Hence we have that

$$\left\| \lim_{d\to 0^+} \frac{1}{d} T(t) \int_0^h T(h-s)Z(d)F(s,u_s)ds \right\| \le Me^{\omega t_\varphi} khc_2,$$

where

$$c_2 := rC_0(r) + \sup_{s\in[0,t_\varphi]} |F(s,0)|.$$

In the next step, we decompose as

$$\text{(31)} \qquad \int_0^t T(t-s)Z(d)(F(s+h,u_{s+h}) - F(s,u_s))ds$$
$$= \int_0^t T(t-s)Z(d)(F(s+h,u_{s+h}) - F(s+h,u_s))ds$$
$$+ \int_0^t T(t-s)Z(d)(F(s+h,u_s) - F(s,u_s))ds.$$

The first integral is estimated as

$$\text{(32)} \qquad \left\| \int_0^t T(t-s)Z(d)(F(s+h,u_{s+h}) - F(s+h,u_s))ds \right\|$$
$$\le \int_0^t Me^{\omega(t-s)} \|Z(d)\| C_0(r) \|u_{s+h} - u_s\| ds,$$
$$\le Me^{\omega t_\varphi} \|Z(d)\| C_0(r) \int_0^t \|u_{s+h} - u_s\| ds.$$

The second integral is estimated as

$$\left\|\int_0^t T(t-s)Z(d)(F(s+h,u_s)-F(s,u_s))ds\right\|$$
$$\leq \int_0^t Me^{\omega(t-s)}\|Z(d)\|\,|F(s+h,u_s)-F(s,u_s)|ds,$$
$$\leq Me^{\omega t_\varphi}\|Z(d)\|\int_0^t |F(s+h,u_s)-F(s,u_s)|ds.$$

We set

$$f(t,h) := \int_0^t |F(s+h,u_s)-F(s,u_s)|ds.$$

Then

$$\left\|\lim_{d\to 0^+}\frac{1}{d}\int_0^t T(t-s)Z(d)(F(s+h,u_{s+h})-F(s,u_s))ds\right\|$$
$$\leq Me^{\omega t_\varphi}k\left(C_0(r)\int_0^t \|u_{s+h}-u_s\|ds + f(t,h)\right).$$

Thus we obtain that

$$\|u_{t+h}-u_t\| \leq Me^{\omega t_\varphi}\|T(h)\varphi-\varphi\| + Me^{\omega t_\varphi}khc_2$$
$$+ Me^{\omega t_\varphi}kC_0(r)\int_0^t \|u_{s+h}-u_s\|ds + Me^{\omega t_\varphi}kf(t,h).$$

By Gronwall's lemma, it follows that

$$\|u_{t+h}-u_t\| \leq Me^{\omega t_\varphi}(\|T(h)\varphi-\varphi\| + khc_2 + kf(t,h))\exp(Me^{\omega t_\varphi}kC_0(r)t),$$
$$\leq Me^{\omega t_\varphi}(\|T(h)\varphi-\varphi\| + khc_2 + kf(t_\varphi,h))\exp(Me^{\omega t_\varphi}kC_0(r)t_\varphi).$$

The bounded convergence theorem by Lebesgue implies that

$$\lim_{h\to 0} f(t_\varphi,h) = 0.$$

Therefore,

$$\lim_{h\to 0}\|u_{t+h}(\,.\,,\varphi)-u_t(\,.\,,\varphi)\| = 0$$

uniformly for $t\in[0,t_\varphi)$, and $\lim_{t\to t_\varphi^-}\|u_t(\,.\,,\varphi)\|$ exists; the solution can be continued to the right of t_φ, which contradicts the maximality of $[-\tau,t_\varphi)$.

We will now prove that the solution depends continuously on the initial data. Let $\varphi \in \mathscr{C}_E$, $\mathscr{D}\varphi \in \overline{D(A_0)}$ and $t \in [0, t_\varphi)$ fixed. We put

$$\begin{cases} r = 1 + \sup\limits_{0 \le s \le t} \|u_s(\,.\,,\varphi)\|, \\ c(t) = Me^{\omega t} \exp(Me^{\omega t} C_0(r)kt). \end{cases}$$

Let $\alpha \in (0,1)$ such that $c(t)\alpha < 1$ and $\psi \in \mathscr{C}_E$, $\mathscr{D}\psi \in \overline{D(A_0)}$ such that $\|\varphi - \psi\| < \alpha$. We have

$$\|\psi\| \le \|\varphi\| + \alpha < r.$$

Let

$$T_0 = \sup\{s \in (0, t_\psi) : \|u_\sigma(\,.\,,\psi)\| \le r \text{ for all } \sigma \in [0,s]\}.$$

If we suppose that $T_0 < t$, we obtain for $s \in [0, T_0]$, as in (32)

$$\|u_s(\,.\,,\varphi) - u_s(\,.\,,\psi)\| \le Me^{\omega t}\left(\|\varphi - \psi\| + C_0(r)k \int_0^s \|u_\sigma(\,.\,,\varphi) - u_\sigma(\,.\,,\psi)\| d\sigma\right).$$

By Gronwall's Lemma, we deduce that

$$\|u_s(\,.\,,\varphi) - u_s(\,.\,,\psi)\| \le c(t)\|\varphi - \psi\|.$$

This implies that

$$\|u_s(\,.\,,\psi)\| \le c(t)\alpha + r - 1 < r \text{ for all } s \in [0, T_0].$$

It follows that T_0 cannot be the largest number $s > 0$ such that $\|u_\sigma(\,.\,,\psi)\| \le r$, for all $\sigma \in [0,s]$. Thus, $T_0 \ge t$ and $t < t_\psi$. Furthermore, $\|u_s(\,.\,,\psi)\| \le r$, for $s \in [0,t]$. Then, we deduce the continuous dependence on the initial data. □

Under more restrictive conditions on F and φ, we obtain strict solutions of Equation (7).

Theorem 3.2. *Assume that the hypotheses of Theorem* 3.1 *hold. Furthermore, assume that* $F : [0, +\infty) \times \mathscr{C}_E \to E$ *is continuously differentiable and* $D_tF, D_\varphi F$ *satisfy the locally Lipschitz condition* (**H5**), *i.e., for each* $r > 0$ *there exist constants* $C_1(r), C_2(r) > 0$ *such that if* $t \ge 0$, $\varphi, \psi \in \mathscr{C}_E$ *and* $\|\varphi\|, \|\psi\| \le r$ *then*

$$\begin{cases} |D_tF(t,\varphi) - D_tF(t,\psi)| \le C_1(r)\|\varphi - \psi\|, \\ \|D_\varphi F(t,\varphi) - D_\varphi F(t,\psi)\| \le C_2(r)\|\varphi - \psi\|, \end{cases}$$

where D_tF *and* $D_\varphi F$ *denote the derivatives. Let* $\varphi \in \mathscr{C}_E$ *such that* $\varphi \in \mathscr{C}_E^1 := \mathscr{C}^1([-\tau, 0], E)$, $\mathscr{D}\varphi \in D(A_0)$, $\mathscr{D}\varphi' \in \overline{D(A_0)}$ *and* $\mathscr{D}\varphi' = A_0\mathscr{D}\varphi + F(0,\varphi)$.

Then, the unique mild solution $u(.,\varphi):[-\tau,t_\varphi)\to E$ of Equation (7) is continuously differentiable on $[-\tau,t_\varphi)$ and it is a strict solution of Equation (7).

Proof. Let $\varphi\in\mathscr{C}_E^1$ such that $\mathscr{D}\varphi\in D(A_0)$, $\mathscr{D}\varphi'\in\overline{D(A_0)}$ and $\mathscr{D}\varphi'=A_0\mathscr{D}\varphi+F(0,\varphi)$. Let $u:=u(.,\varphi)$ be the unique mild solution of Equation (7) on $[-\tau,t_\varphi)$. Consider the linear equation

$$\begin{cases}\mathscr{D}v_t=S_0'(t)\mathscr{D}\varphi'+\dfrac{d}{dt}\displaystyle\int_0^t S_0(t-s)(D_tF(s,u_s)+D_\varphi F(s,u_s)v_s)ds,\\ v_0=\varphi'.\end{cases}$$

It is clear that there is a unique solution v on $[-\tau,t_\varphi)$ of this equation. Define, for $t\geq 0$, the function

$$g:t\to\varphi+\int_0^t v_s\,ds.$$

Then, for $\theta\in[-\tau,0]$

$$\begin{aligned}g(t)(\theta)&=\varphi(\theta)+\int_0^t v(s+\theta)ds,\\ &=\varphi(\theta)+\int_\theta^{t+\theta}v(\xi)d\xi.\end{aligned}$$

If $t+\theta\geq 0$, then

$$\begin{aligned}g(t)(\theta)&=\varphi(\theta)+\int_\theta^0\varphi'(\xi)d\xi+\int_0^{t+\theta}v(\xi)d\xi,\\ &=\varphi(\theta)+\varphi(0)-\varphi(\theta)+\int_0^{t+\theta}v(\xi)d\xi,\\ &=\varphi(0)+\int_0^{t+\theta}v(\xi)d\xi,\\ &=g(t+\theta)(0).\end{aligned}$$

If $t+\theta<0$, then

$$\begin{aligned}g(t)(\theta)&=\varphi(\theta)+\int_\theta^{t+\theta}\varphi'(\xi)d\xi,\\ &=\varphi(\theta)+\varphi(t+\theta)-\varphi(\theta),\\ &=\varphi(t+\theta).\end{aligned}$$

Thus if we define

$$w(t) = \begin{cases} g(t)(0), & \text{for } t \geq 0, \\ \varphi(t), & \text{for } -\tau \leq t \leq 0, \end{cases}$$

then $w : [-\tau, t_\varphi) \to E$ is a continuous function, and

$$g(t) = w_t, \qquad t \geq 0.$$

Integrating the equation of v_t, we have that

$$\int_0^t \mathscr{D}v_\xi \, d\xi = S_0(t)\mathscr{D}\varphi' + \int_0^t S_0(t-s)(D_tF(s,u_s) + D_\varphi F(s,u_s)v_s)ds.$$

Since the left-hand side becomes

$$\int_0^t \mathscr{D}v_\xi \, d\xi = \mathscr{D}\left(\int_0^t v_\xi \, d\xi\right) = \mathscr{D}(g(t) - \varphi) = \mathscr{D}w_t - \mathscr{D}\varphi,$$

we have that

$$\mathscr{D}w_t = \mathscr{D}\varphi + S_0(t)\mathscr{D}\varphi' + \int_0^t S_0(t-s)(D_tF(s,u_s) + D_\varphi F(s,u_s)v_s)ds.$$

On the other hand, by assumption

$$\mathscr{D}\varphi' = A_0\mathscr{D}\varphi + F(0,\varphi).$$

Then

$$S_0(t)\mathscr{D}\varphi' = S_0(t)A_0\mathscr{D}\varphi + S_0(t)F(0,\varphi).$$

Since $\mathscr{D}\varphi \in D(A_0)$, it follows that

$$S_0(t)A_0\mathscr{D}\varphi = S_0'(t)\mathscr{D}\varphi - \mathscr{D}\varphi.$$

Hence

$$S_0(t)\mathscr{D}\varphi' = S_0'(t)\mathscr{D}\varphi - \mathscr{D}\varphi + S_0(t)F(0,\varphi).$$

Thus w_t satisfies

$$\mathscr{D}w_t = S_0'(t)\mathscr{D}\varphi + S_0(t)F(0,\varphi) + \int_0^t S_0(t-s)(D_tF(s,u_s) + D_\varphi F(s,u_s)v_s)ds. \tag{33}$$

Now we compute $S_0(t)F(0,\varphi)$. Notice that

$$\int_0^t S_0(t-s)F(s,w_s)ds = \int_0^t S_0(s)F(t-s,w_{t-s})ds.$$

Since $w_t = g(t)$ is continuously differentiable, and since $F(t-s,\varphi)$ is also continuously differentiable, it follows that $F(t-s,w_{t-s})$ is continuously differentiable with respect to t. Thus we have

$$\begin{aligned}\frac{d}{dt}\int_0^t & S_0(t-s)F(s,w_s)ds\\ &= S_0(t)F(0,\varphi)+\int_0^t S_0(s)\left(D_tF(t-s,w_{t-s})+D_\varphi F(t-s,w_{t-s})\frac{d}{dt}w_{t-s}\right)ds,\\ &= S_0(t)F(0,\varphi)+\int_0^t S_0(t-s)(D_tF(s,w_s)+D_\varphi F(s,w_s)v_s)ds.\end{aligned}$$

So, we deduce that

$$\begin{aligned}S_0(t)F(0,\varphi) = {} & \frac{d}{dt}\int_0^t S_0(t-s)F(s,w_s)ds\\ & -\int_0^t S_0(t-s)(D_tF(s,w_s)+D_\varphi F(s,w_s)v_s)ds.\end{aligned}$$

Therefore, Equation (33) becomes

$$\begin{aligned}\mathscr{D}w_t = S_0'(t)\mathscr{D}\varphi & + \frac{d}{dt}\int_0^t S_0(t-s)F(s,w_s)ds\\ & -\int_0^t S_0(t-s)(D_tF(s,w_s)+D_\varphi F(s,w_s)v_s)ds\\ & +\int_0^t S_0(t-s)(D_tF(s,u_s)+D_\varphi F(s,u_s)v_s)ds.\end{aligned}$$

Since the mild solution u satisfies

$$\mathscr{D}u_t = S_0'(t)\mathscr{D}\varphi+\frac{d}{dt}\int_0^t S_0(t-s)F(s,u_s)ds,$$

we obtain

$$\begin{aligned}\mathscr{D}(u_t-w_t) = {} & \frac{d}{dt}\int_0^t S_0(t-s)(F(s,u_s)-F(s,w_s))ds\\ & -\int_0^t S_0(t-s)(D_tF(s,u_s)-D_tF(s,w_s))ds\\ & -\int_0^t S_0(t-s)(D_\varphi F(s,u_s)-D_\varphi F(s,w_s))v_s ds.\end{aligned}$$

If we choose $T_1 := \min\{\varepsilon, t_\varphi - t_\varphi/2\}$ and $\varepsilon\in(0,\tau]$, we obtain for $t\in(0,T_1)$ and $\theta\in[-\tau,0]$

$$-\tau < t-\tau \le t+\frac{\tau-\varepsilon}{\tau}\theta-\varepsilon \le t-\varepsilon < 0.$$

Since $u(\theta) = w(\theta) = \varphi(\theta)$, it follows that

$$\tau_\varepsilon(u_t)(\theta) = u_t\left(\frac{\tau-\varepsilon}{\tau}\theta - \varepsilon\right) = u\left(t + \frac{\tau-\varepsilon}{\tau}\theta - \varepsilon\right) = \varphi\left(t + \frac{\tau-\varepsilon}{\tau}\theta - \varepsilon\right),$$

$$\tau_\varepsilon(w_t)(\theta) = w_t\left(\frac{\tau-\varepsilon}{\tau}\theta - \varepsilon\right) = w\left(t + \frac{\tau-\varepsilon}{\tau}\theta - \varepsilon\right) = \varphi\left(t + \frac{\tau-\varepsilon}{\tau}\theta - \varepsilon\right).$$

Thus we have that

$$P_\varepsilon(u_t - w_t) = W_\varepsilon \circ \tau_\varepsilon(u_t - w_t) = 0,$$

and that

$$\begin{aligned}\mathscr{D}(u_t - w_t) &= u(t) - w(t) - P(u_t - w_t),\\ &= u(t) - w(t) - (P(u_t - w_t) - P_\varepsilon(u_t - w_t)).\end{aligned}$$

Consequently

$$\begin{aligned} u(t) - w(t) = P(u_t - w_t) - P_\varepsilon(u_t - w_t) &+ \frac{d}{dt}\int_0^t S_0(t-s)(F(s,u_s) - F(s,w_s))ds\\ &- \int_0^t S_0(t-s)(D_tF(s,u_s) - D_tF(s,w_s))ds\\ &- \int_0^t S_0(t-s)(D_\varphi F(s,u_s) - D_\varphi F(s,w_s))v_s\, ds.\end{aligned}$$

By using Proposition 2.12, we have that

$$\begin{aligned}&\frac{d}{dt}\int_0^t S_0(t-s)(F(s,u_s) - F(s,w_s))ds\\ &\quad = \lim_{h\to 0^+}\frac{1}{h}\int_0^t T(t-s)Z(h)(F(s,u_s) - F(s,w_s))ds.\end{aligned}$$

Hence we obtain, for suitable constants M, $\omega > 0$ and for all $t \in [0, T_1)$,

$$\left|\frac{d}{dt}\int_0^t S_0(t-s)(F(s,u_s) - F(s,w_s))ds\right| \le Me^{\omega T_1}k\int_0^t |F(s,u_s) - F(s,w_s)|ds.$$

Since $S_0(t)$ is assumed to be exponentially bounded, we have also for suitable positive constants which we label the same

$$\left|\int_0^t S_0(t-s)(D_tF(s,u_s) - D_tF(s,w_s))ds\right| \le Me^{\omega T_1}\int_0^t |D_tF(s,u_s) - D_tF(s,w_s)|ds,$$

and

$$\left|\int_0^t S_0(t-s)(D_\varphi F(s,u_s) - D_\varphi F(s,w_s))v_s\, ds\right|$$
$$\le Me^{\omega T_1}\int_0^t \|D_\varphi F(s,u_s) - D_\varphi F(s,w_s)\|\,\|v_s\| ds.$$

Thus for all $t \in [0, T_1)$

$$|u(t) - w(t)| \le \delta(\varepsilon)\|u_t - w_t\| + Me^{\omega T_1}k\int_0^t |F(s,u_s) - F(s,w_s)| ds$$
$$+ Me^{\omega T_1}\int_0^t |D_t F(s,u_s) - D_t F(s,w_s)| ds$$
$$+ Me^{\omega T_1}\int_0^t \|D_\varphi F(s,u_s) - D_\varphi F(s,w_s)\|\,\|v_s\| ds.$$

Since $t < T_1 \le \varepsilon \le \tau$, and $u_0 = w_0 = \varphi$, the function $\|u_t - w_t\|$ is nondecreasing with respect to t. Thus we can replace the left-hand side of this last inequality by $\|u_t - w_t\|$. As a result,

$$(1-\delta(\varepsilon))\|u_t - w_t\| \le Me^{\omega T_1}k\int_0^t |F(s,u_s) - F(s,w_s)| ds$$
$$+ Me^{\omega T_1}\int_0^t |D_t F(s,u_s) - D_t F(s,w_s)| ds$$
$$+ Me^{\omega T_1}\int_0^t \|D_\varphi F(s,u_s) - D_\varphi F(s,w_s)\|\,\|v_s\| ds.$$

We choose $\varepsilon > 0$ such that $\delta(\varepsilon) < 1$. Set

$$r := \max\left(\sup_{-\tau \le s < T_1} |u(s)|, \sup_{-\tau \le s < T_1} |v(s)|, \sup_{-\tau \le s < T_1} |w(s)|\right),$$

which is finite since $T_1 < t_\varphi$. There exist $C_0(r), C_1(r), C_2(r) > 0$ such that, for $s \in [0, T_1)$

$$\begin{cases} |F(s,u_s) - F(s,w_s)| \le C_0(r)\|u_s - w_s\|, \\ |D_t F(t,u_s) - D_t F(t,w_s)| \le C_1(r)\|u_s - w_s\|, \\ \|D_\varphi F(t,u_s) - D_\varphi F(t,w_s)\| \le C_2(r)\|u_s - w_s\|. \end{cases}$$

This implies that

$$\|u_t - w_t\| \le \frac{M}{1-\delta(\varepsilon)} e^{\omega T_1}(kC_0(r) + C_1(r) + rC_2(r))\int_0^t \|u_s - w_s\| ds.$$

By the Gronwall's Lemma, $u_t = w_t$ for $t \in [0, T_1)$.

We can repeat the previous argument on $[0, T_2)$, where $T_2 := \min\{2\varepsilon, t_\varphi - t_\varphi/2^2\}$ and $\varepsilon \in (0, \tau]$, $\delta(\varepsilon) < 1$, with the initial condition $u(t)$ for $t \in [-\tau, T_1]$. In this case, we have also, for $t \in [0, T_2)$,

$$\|u_t - w_t\| \leq \frac{M}{1 - \delta(\varepsilon)} e^{\omega T_2}(kC_0(r) + C_1(r) + rC_2(r)) \int_0^t \|u_s - w_s\| ds.$$

Then, $u_t = w_t$ in $[0, T_2)$. Proceeding inductively we obtain $u_t = w_t$ in $[0, t_\varphi)$. Finally, since

$$t \to \mathscr{D}w_t = \mathscr{D}g(t) = \mathscr{D}\varphi + \mathscr{D}\left(\int_0^t v_s \, ds\right) = \mathscr{D}\varphi + \int_0^t \mathscr{D}v_s \, ds$$

is continuously differentiable, then the function $t \to \mathscr{D}u_t$ is continuously differentiable. This ends the proof of Theorem 3.2. □

4. Global existence and stability of solutions

In this section, simple results on global existence and stability of solutions will be given. We add the following assumption

–(**H6**) there exist $r > 0$ and $\rho \in \left(0, \dfrac{\omega}{kK}\right)$ such that $\varphi \in \mathscr{C}_E$, $\|\varphi\| \leq r$ and $t \geq 0$ implies

$$|F(t, \varphi)| \leq \rho\|\varphi\|,$$

where $k > 0$ is given by (29), and $K \geq 1$, $\omega > 0$ are given in Proposition 2.11. A spacial case of this assumption is $|F(t, \varphi)| = o(\|\varphi\|)$ uniformly in $t \geq 0$.

Theorem 4.1. *If all the assumptions* (**H1**)–(**H6**) *are satisfied, then the trivial solution* $u = 0$ *of Equation* (7) *is exponentially stable.*

Proof. Let $\varphi \in \mathscr{C}_E$ such that $\mathscr{D}\varphi \in \overline{D(A_0)}$. Consider the problem

$$u_t(.\,, \varphi) = T(t)\varphi + \frac{d}{dt}\int_0^t Z(t - s)F(s, u_s(.\,, \varphi))ds. \tag{34}$$

We have proved in Theorem 3.1, that under the assumptions (**H1**), (**H2**) and (**H5**) there exist a maximal interval of existence $[-\tau, t_\varphi)$, $t_\varphi > 0$, and a unique solution $u(.\,, \varphi)$ of Equation (34), defined on $[-\tau, t_\varphi)$ and either $t_\varphi = +\infty$ or $\limsup_{t \to t_\varphi^-} \|u_t(.\,, \varphi)\| = +\infty$.

Assume that $t_\varphi < +\infty$ and let $\varepsilon = \dfrac{r}{K}$, where $r > 0$ and $K \geq 1$ are given by the condition (**H6**). Suppose that $\|\varphi\| < \varepsilon$ and consider the positive number

$$t_1 = \sup\{t \in [0, t_\varphi) : \|u_s(.\,,\varphi)\| \le r \text{ for all } s \in [0, t]\}. \tag{35}$$

By the continuity of the function u, we have $\|u_s(.\,,\varphi)\| \le r$ for all $s \in [0, t_1]$. Then, for $t \in [0, t_1]$,

$$\|u_t(.\,,\varphi)\| \le Ke^{-\omega t}\|\varphi\| + Kk\int_0^t e^{-\omega(t-s)}|F(s, u_s(.\,,\varphi))|ds,$$

and therefore,

$$e^{\omega t}\|u_t(.\,,\varphi)\| \le K\varepsilon + Kk\rho\int_0^t e^{\omega s}\|u_s(.\,,\varphi)\|ds.$$

Applying the Gronwall's lemma to this inequality, we obtain

$$\|u_t(.\,,\varphi)\| \le K\varepsilon e^{(Kk\rho-\omega)t} = re^{(Kk\rho-\omega)t} < r, \qquad t \in [0, t_1].$$

Consequently, there exists $\delta > 0$ such that

$$\|u_t(.\,,\varphi)\| < r, \qquad t \in [0, t_1 + \delta].$$

This contradicts (35). We conclude that the solution u is global on $[-\tau, +\infty)$ and satisfies

$$\|u_t(.\,,\varphi)\| \le re^{(Kk\rho-\omega)t} \qquad \text{for } t \ge 0 \text{ and } \|\varphi\| < \varepsilon,$$

with $Kk\rho - \omega < 0$. □

Our next objective is to give other sufficient conditions for the global existence and stability of the trivial solution of Problem (7). We keep the assumptions (**H1**)–(**H4**) and instead of the hypotheses (**H5**) and (**H6**), we make the following conditions.

–(**H7**) $F : \mathbf{R}_+ \times \mathscr{C}_E \to E$ is continuous, $F(t, 0) = 0$ and F satisfies the following local Lipschitz condition

$$|F(t, \varphi_1) - F(t, \varphi_2)| \le \gamma(t, \|\varphi_1\|, \|\varphi_2\|)\|\varphi_1 - \varphi_2\|$$

for $\varphi_1, \varphi_2 \in \mathscr{C}_E$, where $\gamma(t, y_1, y_2)$ is continuous with respect to $t \in \mathbf{R}_+$, $y_1, y_2 \in \mathbf{R}_+$, and is monotonically nondecreasing with respect to y_1 and y_2.

Since $F(t, 0) = 0$, (**H7**) implies that

$$|F(t, \varphi)| \le \gamma(t, \|\varphi\|, 0)\|\varphi\|.$$

Set

$$G(t, x) = \gamma(t, x, 0)x \qquad \text{for } t \ge 0 \text{ and } x \ge 0.$$

Then, $G(t,x)$ is monotonically nondecreasing with respect to its second argument.

–(**H8**) There exist $r > 0$ and $g \in \mathscr{C}(\mathbf{R}_+ \times [0,r), \mathbf{R}_+)$ such that for each $p \in [0,r)$, (r can take the value $+\infty$)

$$g(t,p) \geq Kpe^{-\omega t} + \int_0^t Kke^{-\omega(t-s)} G(s, g(s,p))ds, \qquad t \geq 0,$$

where, $\omega > 0$ and $K \geq 1$ are given in Proposition 2.11.

Theorem 4.2. *Let $\varphi \in \mathscr{C}_E$ such that $\mathscr{D}\varphi \in \overline{D(A_0)}$ and $\|\varphi\| < r$. If all the assumptions* (**H1**), (**H2**), (**H3**), (**H4**), (**H7**) *and* (**H8**) *are satisfied, then Problem* (7) *has a unique global mild solution $u(.,\varphi) : [-\tau, +\infty) \to \mathscr{C}_E$, and the following inequality holds*

$$\|u_t(.,\varphi)\| \leq g(t, \|\varphi\|), \qquad t \geq 0.$$

In addition, the following properties hold.

(i) *If, for any $\varepsilon > 0$, there is an $\eta = \eta(\varepsilon) > 0$ such that $0 \leq p < \eta$ implies $g(t,p) < \varepsilon$ for $t \geq 0$, then, the trivial solution of Problem* (7) *is stable. This means that for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\|\varphi\| < \delta$ implies $\|u_t(.,\varphi)\| < \varepsilon$ for $t \geq 0$.*

(ii) *If*

$$\lim_{t \to +\infty} g(t,p) = 0 \qquad \text{for small } p > 0,$$

then the trivial solution of Problem (7) *is asymptotically stable.*

Proof. Let $\varphi \in \mathscr{C}_E$ such that $\mathscr{D}\varphi \in \overline{D(A_0)}$ and $\|\varphi\| < r$. By virtue of Proposition 2.16, it suffices to prove Theorem 4.2 for the following equation

$$u_t = T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s,u_s)ds.$$

Let $(v^n)_{n\in\mathbf{N}}$ be a sequence of continuous functions defined for $t \geq 0$ by

$$\begin{cases} v^0(t) = T(t)\varphi, \\ v^n(t) = v^0(t) + \dfrac{d}{dt}\displaystyle\int_0^t Z(t-s)F(s, v^{n-1}(s))ds, \quad n \geq 1. \end{cases}$$

We have, from Proposition 2.11 and Condition (**H8**), that

$$\|v^0(t)\| \leq Ke^{-\omega t}\|\varphi\| \leq g(t, \|\varphi\|).$$

Proposition 2.12 and Condition (**H7**) yield

$$\|v^1(t)\| \le Ke^{-\omega t}\|\varphi\| + \int_0^t Kke^{-\omega(t-s)}\gamma(s, \|v^0(s)\|, 0)\|v^0(s)\|ds,$$
$$\le Ke^{-\omega t}\|\varphi\| + \int_0^t Kke^{-\omega(t-s)}G(s, \|v^0(s)\|)ds.$$

Since

$$\|v^0(s)\| \le g(s, \|\varphi\|)$$

and $G(s, x)$ is nondecreasing with respect to x, we obtain

$$G(s, \|v^0(s)\|) \le G(s, g(s, \|\varphi\|)).$$

Hence (**H8**) implies that

$$\|v^1(t)\| \le Ke^{-\omega t}\|\varphi\| + \int_0^t Kke^{-\omega(t-s)}G(s, g(s, \|\varphi\|))ds,$$
$$\le g(t, \|\varphi\|).$$

By using induction, we prove

$$\|v^n(t)\| \le g(t, \|\varphi\|) \qquad \text{for all } n \in \mathbf{N}.$$

On the other hand, we have

$$\|v^1(t) - v^0(t)\| \le \int_0^t Kke^{-\omega(t-s)}\gamma(s, \|v^0(s)\|, 0)\|v^0(s)\|ds.$$

Then

$$\|v^1(t) - v^0(t)\| \le \int_0^t Kke^{-\omega(t-s)}G(s, \|v^0(s)\|)ds,$$
$$\le \int_0^t Kke^{-\omega(t-s)}G(s, g(s, \|\varphi\|))ds.$$

Let $T > 0$. Since $g(\,.\,, \|\varphi\|)$ and γ are both continuous, we can set

$$\begin{cases} \alpha = \sup_{s\in[0,T]} g(s, \|\varphi\|), \\ \beta = \sup_{s\in[0,T]} \gamma(s, g(s, \|\varphi\|), g(s, \|\varphi\|)). \end{cases}$$

In particular

$$\sup_{s\in[0,T]} G(s, g(s, \|\varphi\|)) \le \alpha\beta.$$

Hence

$$\|v^1(t) - v^0(t)\| \le Kk\alpha\beta t.$$

In general case, we have

$$\begin{aligned}\|v^n(t) - v^{n-1}(t)\| &\le \int_0^t Kke^{-\omega(t-s)}\gamma(s, \|v^{n-1}(s)\|, \|v^{n-2}(s)\|)\|v^{n-1}(s) - v^{n-2}(s)\|ds,\\ &\le \int_0^t Kke^{-\omega(t-s)}\gamma(s, g(s, \|\varphi\|), g(s, \|\varphi\|))\|v^{n-1}(s) - v^{n-2}(s)\|ds,\\ &\le Kk\beta\int_0^t \|v^{n-1}(s) - v^{n-2}(s)\|ds.\end{aligned}$$

So, by induction

$$\|v^n(t) - v^{n-1}(t)\| \le \alpha\frac{(Kk\beta t)^n}{n!}.$$

Consequently, the limit $v := \lim_{n\to\infty} v^n$ exists uniformly on $[0, T]$, it is continuous on $[0, T]$ and it satisfies

$$\|v(t)\| \le g(t, \|\varphi\|) \qquad \text{for } t \in [0, T].$$

On the other hand, we have from Proposition 2.12 and (**H7**)

$$\begin{aligned}&\left\|\frac{d}{dt}\int_0^t Z(t-s)F(s, v^{n-1}(s))ds - \frac{d}{dt}\int_0^t Z(t-s)F(s, v(s))ds\right\|\\ &\quad\le kK\int_0^t e^{-\omega(t-s)}\gamma(s, \|v^{n-1}(s)\|, \|v(s)\|)\|v^{n-1}(s) - v(s)\|ds,\\ &\quad\le kK\int_0^t \gamma(s, g(s, \|\varphi\|), g(s, \|\varphi\|))\|v^{n-1}(s) - v(s)\|ds,\\ &\quad\le kK\beta T\sup_{s\in[0,T]}\|v^{n-1}(s) - v(s)\|.\end{aligned}$$

Then

$$\frac{d}{dt}\int_0^t Z(t-s)F(s, v^{n-1}(s))ds \xrightarrow[n\to+\infty]{} \frac{d}{dt}\int_0^t Z(t-s)F(s, v(s))ds$$

uniformly on $[0, T]$. Consequently, v satisfies

$$v(t) = T(t)\varphi + \frac{d}{dt}\int_0^t Z(t-s)F(s, v(s))ds, \qquad t \in [0, T].$$

Then, if we consider the function $u : [-\tau, T] \to E$ defined by

$$u(t) = \begin{cases} v(t)(0) & \text{if } t \geq 0, \\ \varphi(t) & \text{if } t \leq 0, \end{cases}$$

we obtain

$$u_t = v(t), \qquad t \geq 0.$$

To show uniqueness, suppose that $w(t)$ is also a mild solution of Equation (7) with the initial condition φ. Then,

$$\|v(t) - w(t)\| \leq kK\bar{\beta} \int_0^t \|v(s) - w(s)\| ds,$$

where $\bar{\beta} = \sup_{s \in [0, T]} \gamma(s, \|v(s)\|, \|w(s)\|)$. By Gronwall's inequality, $w = v$ on $[0, T]$. This proves the theorem. □

We end this part with simple examples.

Examples

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$, Δ the Laplace operator on Ω, $\mathscr{C}_E := \mathscr{C}([-\tau, 0], E)$ with $E = \mathscr{C}(\bar{\Omega}, \mathbf{R})$ and $\mathscr{D} : \mathscr{C}_E \to E$ the operator defined by $(\mathscr{D}\varphi)(x) = \varphi(0)(x) - q\varphi(-\tau)(x)$ for $\varphi \in \mathscr{C}_E$, $x \in \bar{\Omega}$ and $q \in [0, 1)$. We consider the problem

$$(36) \qquad \begin{cases} \dfrac{\partial}{\partial t} \mathscr{D}u_t(.\,, x) = \Delta \mathscr{D}u_t(.\,, x) - u^3(t - \tau, x), & t \geq 0,\ x \in \Omega, \\ u(t, x) = 0, & t \geq 0,\ x \in \partial\Omega, \\ u(\theta, x) = \varphi(\theta, x), & \theta \in [-\tau, 0],\ x \in \Omega. \end{cases}$$

Problem (36) can be reformulated as an abstract semilinear neutral functional differential equations

$$\begin{cases} \dfrac{\partial}{\partial t} \mathscr{D}U_t = A_0 \mathscr{D}U_t + F(U_t), & t \geq 0, \\ U(0) = \varphi, \end{cases}$$

with

$$\begin{cases} D(A_0) = \{u \in \mathscr{C}(\bar{\Omega}, \mathbf{R}) : \Delta u \in \mathscr{C}(\bar{\Omega}, \mathbf{R}) \text{ and } u = 0 \text{ on } \partial\Omega\}, \\ A_0 u = \Delta u, \end{cases}$$

and $F : \mathscr{C}_E \to E$ the nonlinear mapping defined by

$$F(\varphi)(x) = -\varphi^3(-\tau)(x) \qquad \text{for } \varphi \in \mathscr{C}_E \text{ and } x \in \bar{\Omega}.$$

We have $\overline{D(A_0)} = \{u \in \mathscr{C}(\overline{\Omega}, \mathbf{R}); u = 0 \text{ on } \partial\Omega\} \neq E$.

Moreover

$$\begin{cases} \rho(A_0) \subset (0, +\infty) \\ \|(\lambda I - A_0)^{-1}\| \leq \dfrac{1}{\lambda} \quad \text{for } \lambda > 0. \end{cases}$$

This implies that A_0 satisfies (**H1**) on E. Then, A_0 is the generator of a locally Lipschitz continuous integrated semigroup $(S_0(t))_{t \geq 0}$ on E.

We know that there are positive constants $K_0 \geq 1$ and $\omega_0 > 0$ such that

$$|S_0'(t)y| \leq K_0 e^{-\omega_0 t}|y|, \qquad t \geq 0, \ y \in \overline{D(A_0)}.$$

Then, (**H3**) is satisfied. The operator $\mathscr{D}$ satisfies the assumption (**H2**) and for $q \in [0, 1)$ the assumption (**H4**). Furthermore, we have

$$|F(\varphi)| = |\varphi(-\tau)|^3 \leq \|\varphi\|^3, \qquad \varphi \in \mathscr{C}_E,$$

and

$$|F(\varphi_1) - F(\varphi_2)| \leq (\|\varphi_1\|^2 + \|\varphi_1\|\,\|\varphi_2\| + \|\varphi_2\|^2)\|\varphi_1 - \varphi_2\|.$$

We choose

$$\begin{cases} \gamma(p, q) = p^2 + pq + q^2, \\ g(t, p) = 2Kpe^{-\omega t}. \end{cases}$$

Then (**H5**), (**H6**) and (**H7**) are satisfied. Let $p > 0$ and

$$I(t, p) := g(t, p) - Kpe^{-\omega t} - \int_0^t kKe^{-\omega(t-s)} G(g(s, p))ds, \qquad t \geq 0,$$

where $G(p) := \gamma(p, 0)p = p^3$. Then,

$$\begin{aligned} I(t, p) &= Kpe^{-\omega t} - 8kK^4p^3e^{-\omega t} \int_0^t e^{-2\omega s}\, ds, \\ &= Kpe^{-\omega t}\left(1 - \frac{4kK^3p^2}{\omega}(1 - e^{-2\omega t})\right), \\ &\geq Kpe^{-\omega t}\left(1 - \frac{4kK^3p^2}{\omega}\right). \end{aligned}$$

Consequently, for $0 < p \leq \dfrac{1}{2K}\sqrt{\dfrac{\omega}{kK}}$, we have $I(t, p) \geq 0$. Hence, the condition (**H8**) holds. So, we have from Theorem 4.1 or 4.2, the following result.

THEOREM 4.3. *Suppose that* $\varphi \in \mathscr{C}_E$, $\varphi(0) - q\varphi(-\tau) \in \overline{D(A_0)}$, $0 \le q < 1$ *and* $\|\varphi\|$ *sufficiently small. Then, Problem* (36) *has a unique global mild solution* $u \in \mathscr{C}([-\tau, +\infty), E)$. *Furthermore, the inequality*

$$\|u_t(\cdot, \varphi)\| \le 2K\|\varphi\|e^{-\omega t}, \qquad t \ge 0,$$

holds, and hence its trivial solution is exponentially asymptotically stable.

The same results can be obtained for the following examples
a) $F(\varphi)(x) = -\varphi^2(-\tau)(x)\varphi(0)(x)$,
b) $F(\varphi)(x) = -\varphi(-\tau)(x)\varphi^2(0)(x)$,
c) $F(\varphi)(x) = -(\varphi(0)(x) - q\varphi(-\tau)(x))^3$.

5. Dissipativeness and existence of global attractor

In this section, we investigate the dissipativeness and the existence of global attractor of the solution operator in the autonomous case, that is, the system

$$\begin{cases} \dfrac{\partial}{\partial t}\mathscr{D}u_t = A_0\mathscr{D}u_t + F(u_t), & t \ge 0, \\ u_0 = \varphi. \end{cases} \tag{37}$$

We assume the following.

–(**H9**) $F : \mathscr{C}_E \to E$ satisfies enough smoothness conditions to ensure that

(a) for each initial condition $\varphi \in Y := \{\varphi \in \mathscr{C}_E : \mathscr{D}\varphi \in \overline{D(A_0)}\}$, Problem (37) has a unique global mild solution defined on $[-\tau, +\infty)$ and this solution is continuous in φ;

(b) F maps bounded subsets of Y into bounded subsets of E.

For example, if F is locally Lipschitz continuous and satisfies $|F(\varphi)| \le a\|\varphi\| + b$, then F satisfies the assumption (**H9**).

Define, for each $t \ge 0$, the nonlinear operator $U(t) : Y \to \mathscr{C}_E$ by

$$U(t)(\varphi) = u_t(\cdot, \varphi),$$

where $u(\cdot, \varphi)$ is the unique mild solution of Equation (37). We know from [4], that

$$U(t)(Y) \subseteq Y \qquad \text{for all } t \ge 0.$$

Furthermore, we have the following result.

PROPOSITION 5.1 [4]. *Assume that the conditions* (**H1**), (**H2**) *and* (**H9**) *hold. Then, the family of operators* $(U(t))_{t \ge 0}$ *is a nonlinear strongly continuous semigroup of continuous operators on* Y, *that is*

(i) $U(0) = I$,

(ii) $U(t+s) = U(t)U(s)$ *for all* $t, s \geq 0$,

(iii) *for all* $\varphi \in Y$, $U(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* Y,

(iv) *for all* $t \geq 0$, $U(t)$ *is continuous from* Y *into* Y.

Moreover,

(v) $(U(t))_{t \geq 0}$ *satisfies, for* $\varphi \in Y$, $t \geq 0$ *and* $\theta \in [-\tau, 0]$, *the translation property*

$$(U(t)(\varphi))(\theta) = \begin{cases} (U(t+\theta)(\varphi))(0) & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0. \end{cases}$$

Let us consider, in addition, the following assumption.

–(**H10**) The C_0-semigroup $S_0'(t) : Y \to Y$ is compact for each $t > 0$.

Theorem 5.2. *Assume that the assumptions* (**H1**)–(**H4**), (**H9**) *and* (**H10**) *hold and for all* $t \geq 0$, $U(t)$ *maps bounded subsets of* Y *into bounded subsets. Then, the semigroup* $(U(t))_{t \geq 0}$ *is an* α*-contraction on* Y *in the sense that*

$$U(t)(\varphi) = T(t)\varphi + V(t)(\varphi), \qquad \varphi \in Y,$$

where $V(t)$ *is a compact operator for each* $t > 0$ *and* $T(t)$ *is the* C_0*-semigroup given by Proposition* 2.10.

Proof. We will use the same arguments as in [23] in the paper. In the beginning, Proposition 2.10 implies that

$$\mathscr{D}(T(t)\varphi) = S_0'(t)\mathscr{D}\varphi.$$

On the other hand, by Proposition 2.11

$$\|T(t)\varphi\| \leq Ke^{-\omega t}\|\varphi\|,$$

and from definition of mild solutions

$$\mathscr{D}(U(t)(\varphi)) = S_0'(t)\mathscr{D}\varphi + \frac{d}{dt}\int_0^t S_0(t-s)F(U(s)(\varphi))ds.$$

Let $V(t)(\varphi) = v_t = U(t)(\varphi) - T(t)\varphi$. Then

$$\begin{cases} \mathscr{D}v_t = \mathscr{D}(U(t)(\varphi)) - S_0'(t)\mathscr{D}\varphi = \dfrac{d}{dt}\displaystyle\int_0^t S_0(t-s)F(U(s)(\varphi))ds, \\ v_0 = 0. \end{cases} \tag{38}$$

Consequently, from Proposition 2.12 we obtain

$$\mathscr{D}v_t = f(t,\varphi) := \lim_{h \to 0^+} \frac{1}{h}\int_0^t S_0'(t-s)S_0(h)F(U(s)(\varphi))ds.$$

Let B be a bounded subset of Y, we will show that for each $t \geq 0$, $\mathscr{D}V(t)(B)$ is totally bounded in E. By assumption $U(t)(B)$ is bounded in Y and $F(U(t)(B))$ is bounded in E. Since $S_0(.)$ is locally Lipschitz continuous and $S_0(0) = 0$, it is not difficult to see that the following set

$$\left\{\frac{1}{h}S_0(h)F(U(t)(\varphi)) : h \in (0, H], t \in [0, \sigma] \text{ and } \varphi \in B\right\}$$

is contained in a bounded subset of E; that is, there exists a constant c such that

$$\left|\frac{1}{h}S_0(h)F(U(t)(\varphi))\right| \leq c, \qquad \text{for } h \in (0, H],\ t \in [0, \sigma] \text{ and } \varphi \in B.$$

We prove that, for $t \in [0, \sigma]$,

$$L(t) := \left\{\frac{1}{h}\int_0^t S_0'(s)S_0(h)F(U(t-s)(\varphi))ds : h \in (0, H] \text{ and } \varphi \in B\right\}$$

is totally bounded in E. If $0 < \delta < t \leq \sigma$, we can write

$$\frac{1}{h}\int_0^t S_0'(s)S_0(h)F(U(t-s)(\varphi))ds = \int_0^\delta S_0'(s)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds + \int_\delta^t S_0'(s)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds.$$

At first we obtain, from (**H3**)

$$\left|\int_0^\delta S_0'(s)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds\right| \leq \int_0^\delta K_0 e^{-\omega_0 s}c\, ds, \leq \frac{cK_0}{\omega_0}(1 - e^{-\omega_0\delta}).$$

Let $\varepsilon > 0$. Then we can take a $\delta > 0$ such that

$$\left|\int_0^\delta S_0'(s)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds\right| \leq \varepsilon.$$

Next we rewrite

$$\int_\delta^t S_0'(s)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds = S_0'(\delta)\int_\delta^t S_0'(s-\delta)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds.$$

Then

$$\left|\int_\delta^t S_0'(s-\delta)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds\right| \le \int_\delta^t K_0 e^{-\omega_0(s-\delta)}c\,ds,$$

$$\le \frac{cK_0}{\omega_0}(1-e^{-\omega_0(t-\delta)}).$$

Since $S_0'(\delta)$ is compact, the set

$$\left\{S_0'(\delta)\int_\delta^t S_0'(s-\delta)\frac{1}{h}S_0(h)F(U(t-s)(\varphi))ds : h \in (0,H] \text{ and } \varphi \in B\right\}$$

is totally bounded. Hence the set $L(t)$ is totally bounded, that is, it is contained in a compact subset of E. Then the following set is contained in the same compact subset:

$$\left\{\lim_{h\to 0^+}\frac{1}{h}\int_0^t S_0'(s)S_0(h)F(U(t-s)(\varphi))ds : \varphi \in B\right\}.$$

Next we will show that the family $\{t \mapsto f(t,\varphi)\}$, $\varphi \in B$, is equicontinuous in t. Suppose that $0 \le t < t'$. Then

$$(39)\qquad \left|\frac{1}{h}\left(\int_0^{t'} S_0'(t'-s)S_0(h)F(U(s)(\varphi))ds - \int_0^t S_0'(t-s)S_0(h)F(U(s)(\varphi))ds\right)\right|$$

$$\le \int_t^{t'}\|S_0'(t'-s)\|\frac{1}{h}|S_0(h)F(U(s)(\varphi))|ds$$

$$+\int_0^t\left|(S_0'(t'-t)-I)S_0'(t-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|ds.$$

The first term in the right side of (39) is estimated as

$$\int_t^{t'}\|S_0'(t'-s)\|\frac{1}{h}|S_0(h)F(U(s)(\varphi))|ds \le \int_t^{t'}K_0e^{-\omega_0(t'-s)}\frac{1}{h}\|S_0(h)\|\,|F(U(s)(\varphi))|ds,$$

$$\le (t'-t)K_0\sup\left\{\frac{1}{h}\|S_0(h)\|\,|F(U(s)(\varphi))| : 0 < h \le H, t \le s \le t', \varphi \in B\right\}.$$

Let $0 < \delta < t$. The second term in the right side of (39) is separated as

$$(40)\qquad \int_0^t\left|(S_0'(t'-t)-I)S_0'(t-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|ds$$

$$=\int_0^{t-\delta}\left|(S_0'(t'-t)-I)S_0'(t-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|ds$$

$$+\int_{t-\delta}^t\left|(S_0'(t'-t)-I)S_0'(t-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|ds.$$

The second term in the right side of (40) is estimated as

$$\begin{aligned}&\int_{t-\delta}^{t}\left|(S_0'(t'-t)-I)S_0'(t-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|ds\\&\quad\le\int_{t-\delta}^{t}(K_0+1)K_0\frac{1}{h}\|S_0(h)\|\,|F(U(s)(\varphi))|ds,\\&\quad\le\delta(K_0+1)K_0\sup\left\{\frac{1}{h}\|S_0(h)\|\,|F(U(s)(\varphi))|:0<h\le H,0\le s\le t,\varphi\in B\right\}.\end{aligned}$$

Let $\varepsilon>0$. Then we can take a $\delta>0$ such that the last term in this inequality is less than ε. Fix such a $\delta>0$. The first term in the right side of (40) is then estimated as

$$\begin{aligned}&\int_0^{t-\delta}\left|(S_0'(t'-t)-I)S_0'(t-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|ds\\&\quad=\int_0^{t-\delta}\left|(S_0'(t'-t)-I)S_0'(\delta)S_0'(t-\delta-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|ds.\end{aligned}$$

Since

$$\begin{aligned}&\left|S_0'(t-\delta-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|\le K_0e^{-\omega_0(t-\delta-s)}\frac{1}{h}\|S_0(h)\|\,|F(U(s)(\varphi))|\\&\quad\le K_0\sup\left\{\frac{1}{h}\|S_0(h)\|\,|F(U(s)(\varphi))|:0<h\le H,0\le s\le t-\delta,\varphi\in B\right\},\end{aligned}$$

and $S_0'(\delta)$ is a compact operator, there exists a $\delta_1>0$ such that if $t<t'<t+\delta_1$ then

$$\left|(S_0'(t'-t)-I)S_0'(\delta)S_0'(t-\delta-s)\frac{1}{h}S_0(h)F(U(s)(\varphi))\right|<\frac{\varepsilon}{t-\delta},$$

for $0<h\le H$, $0\le s\le t-\delta$ and $\varphi\in B$. This implies that if $t<t'<t+\delta_1$, $0<h\le H$ and $\varphi\in B$ the first term in the right side of (40) is less than ε. Therefore, if $|t'-t|$ is small enough, $0<h\le H$ and $\varphi\in B$, then

$$\left|\frac{1}{h}\int_0^{t'}S_0'(t'-s)S_0(h)F(U(s)(\varphi))ds-\frac{1}{h}\int_0^{t}S_0'(t-s)S_0(h)F(U(s)(\varphi))ds\right|<\varepsilon.$$

Let $(\varphi_k)_{k\ge0}$ be a bounded sequence in Y. Then there exists a subsequence, which we label the same, such that the sequence $(f(t,\varphi_k))_{k\ge0}$ converges in E as $k\to+\infty$ uniformly on $[0,\sigma]$ to some function $f(t)\in E$. Let v^k be the solution of Equation (38) in the paper with $\varphi=\varphi_k$. Then,

$$\begin{cases} \mathscr{D}(v_t^k - v_t^j) = f(t, \varphi_k) - f(t, \varphi_j), \\ v_0^k - v_0^j = 0. \end{cases}$$

From Lemma 2.9 in the paper, we deduce that there exists a positive constant c such that

$$\|v_t^k - v_t^j\| \leq c \sup_{0 \leq s \leq t} |f(s, \varphi_k) - f(s, \varphi_j)|.$$

This implies that the sequence $(v_t^k)_{k \geq 0}$ is a Cauchy sequence, which proves that $V(t)$ is a completely continuous operator on Y. This completes the proof of the theorem. □

Let $(X, |\cdot|)$ be a Banach space. We recall the following definitions.

Definition 5.3 [21]. A family of mappings $W(t) : X \to X$, $t \geq 0$, is said to be a C^r-semigroup, $r \geq 0$, provided that

(i) $W(0) = I$,

(ii) $W(t + s) = W(t)W(s)$ for all $t, s \geq 0$,

(iii) $W(t)x$ is continuous in t, x together with Fréchet derivatives in x up through order r for $(t, x) \in \mathbf{R}_+ \times X$.

Definition 5.4 [21]. Let $W(t) : X \to X$ be a C^r-semigroup for some $r \geq 0$.

(i) A set $B \subset X$ is said to attract a set $C \subset X$ under $W(t)$ if $dist(W(t)C, B) \to 0$ as $t \to +\infty$.

(ii) A set $S \subset X$ is said to be invariant if, for any $x \in S$, there is a complete orbit $\gamma(x)$ through x such that $\gamma(x) \subset S$.

(iii) $W(.)$ is asymptotically smooth if, for any nonempty, closed, bounded set $B \subset X$ for which $W(t)B \subset B$, there is a compact set $J \subset B$ such that J attracts B.

(iv) A compact invariant set $\mathscr{A}$ is said to be a maximal compact invariant set if every compact invariant set of the semigroup belongs to $\mathscr{A}$.

(v) An invariant set $\mathscr{A}$ is said to be a global attractor if $\mathscr{A}$ is maximal compact invariant set which attracts each bounded set $B \subset X$.

As a consequence of Theorem 5.2, we obtain the following result.

Proposition 5.5. *Under the same assumptions as in Theorem* 5.2, *the semigroup* $U(.)$ *is asymptotically smooth on* Y.

The proof is based on the following lemma.

Lemma 5.6 [21]. *For each* $t \geq 0$, *suppose that* $W(t) = W_1(t) + W_2(t) : X \to X$ *has the property that* $W_1(t)$ *is completely continuous and there is a continuous function* $k : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $k(t, r) \to 0$ *as* $t \to +\infty$ *and*

$|W_2(t)x| \leq k(t,r)$ *if* $|x| \leq r$. *Then, the semigroup* $W(t)$ *is asymptotically smooth on* X.

DEFINITION 5.7. The semigroup $(W(t))_{t\geq 0}$ on X is said to be point dissipative (compact dissipative) if there is a bounded set $B \subset X$ that attracts each point of X (each compact set of X) under $W(t)$.

It follows from Hale [21] and Theorem 5.2 that the following result is true.

THEOREM 5.8. *Assume that the assumptions of Theorem* 5.2 *are satisfied.*
(i) *If the semigroup* $U(.)$ *is compact dissipative, then there exists a global attractor* $\mathscr{A}$ *for* $U(.)$.
(ii) *If the semigroup* $U(.)$ *is point dissipative and orbits of bounded sets are bounded, then there exists a global attractor* $\mathscr{A}$ *for* $U(.)$.

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Mostafa Adimy
Laboratoire de Mathématiques Appliquées, I.P.R.A., FRE 2570
Université de Pau, Avenue de l'université, 64000 *Pau, France*
E-mail: mostafa.adimy@univ-pau.fr

Khalil Ezzinbi
Département de Mathématiques, Faculté des Sciences Semlalia
Université Cadi Ayyad, B.P. 2390, *Marrakesh, Morocco*
E-mail: ezzinbi@ucam.ac.ma