Hiroshima Math. J.
34 (2004), 307–343

Selberg zeta functions for cofinite lattices acting on line bundles over complex hyperbolic spaces

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(Received May 23, 2003)
(Revised May 4, 2004)

Abstract. For a line bundle over a finite volume quotient of the complex hyperbolic space, we write down an explicit trace formula for an admissible function lying in the Harish-Chandra p-Schwartz space $\mathscr{C}^p(G)$, $0 < p < 1$, we apply it to a suitable admissible function in order to discuss the analytic continuation of the associated Selberg zeta function.

1. Introduction

Let Y be a finite volume non compact locally symmetric space of negative curvature, that is $Y = \Gamma\backslash G/K$ where G is a real semi-simple Lie group of $\mathbf{R}$-rank one, K is a maximal compact subgroup of G, $\Gamma \subset G$ a cofinite discrete subgroup of G.

In 1956, for $G = SL(2,\mathbf{R})$, $G/K = H$ the upper half plane and Γ a discrete subgroup of G, Atle Selberg in his famous paper [10] introduced a function $Z_\Gamma(s)$ of one complex variable, so called Selberg zeta function and showed that the location and the order of the zeros of this function gives information on the topology of the manifold $Y = \Gamma\backslash H$ as well as on the spectrum of the associated Laplace-Beltrami operator.

In 1977, R. Gangolli [7] extended the result of Selberg to a general G of rank one and $Y = \Gamma\backslash G/K$ compact by constructing Selberg type Zeta function for this general case. Two years after, the same author jointly with G. Warner [6] treated analogously the case where $\Gamma\backslash G$ is not compact but of finite volume for a general G of rank one. However, for technical reasons, they avoided the case where $G = SU(2n,1)$. Their work was based on the explicit Selberg trace formula written down by G. Warner for $G = SU(2n+1,1)$ in his survey paper [15]. This zeta function provides some topological data on the manifold $\Gamma\backslash G/K$ as well as some spectral information. That is, the class one spectrum induced from the trivial representation of K contained in $L^2_{disc}(\Gamma\backslash G)$.

2000 *Mathematics subject Classification*. Primary 11M36; Secondary 33C60.
Key words and phrases. Lattice, complex hyperbolic space, trace formula, zeta function, Harish-Chandra Schwartz space, Abel transform.

For τ an irreducible non trivial representation of M $(M = U(1))$, D. Scott [11] constructed for $GL(2, \mathbf{C})$ and Γ cocompact, a zeta function $Z_{\tau,\Gamma}$ associated to the data (τ, G, Γ) which gives information about the representations induced from τ and appearing in $L^2(\Gamma\backslash G)$.

Later, in 1984 in the same context, for τ non trivial representation of M, M. Wakayama [13] considered the case where $G = SU(n,1)$ and Γ a discrete cocompact subgroup of G and studied the Zeta function $Z_{\tau,\Gamma}$ associated with τ, a one dimensional representation of $K = U(n+1) \cap G$.

In [3], more generally, for locally homogeneous vector bundles over compact locally symmetric spaces, U. Bunke and M. Olbrich have developed a new approach in the theory of Theta and Zeta functions which is different from the approach of Gangolli and uses operator theory and index theory.

For τ an irreducible one dimensional representation of $U(1)$ and $G = SU(n,1)$, the main purpose of the present paper is the extension of the result of M. Wakayama to the finite volume case, i.e. the study of the associated zeta function of Selberg type Z_τ^Γ for Γ a cofinite discrete subgroup of $SU(n,1)$. This will be accomplished firstly by writing an explicit trace formula for this case and secondly by applying it to some suitably chosen test function as developed by R. Gangolli [7]. For $\tau = 1$, we recover the result in [7] and at the same time we treat the case of $G = SU(2n,1)$ omitted there.

This function Z_τ^Γ will allow us to give some information about the $[\tau]$-class spectrum induced from the representation τ contained in $L^2_{dis}(\Gamma\backslash G)$ as well as some topological information.

This paper is organized as follows: in the section 1, for Γ a non uniform lattice in $SU(n,1)$, we recall the general setting of the Selberg trace formula at its second stage as exposed by G. Warner in [15] for a K-finite function. This in order to explicit it further for the special case of a τ-function (τ an irreducible representation of $U(1)$) that we will use later. In section 2, we expose some general facts about the spherical Fourier analysis on homogeneous vector bundles associated with τ over G/K and write it explicitly in the form we will use later.

In Theorem 3.2 in section 3, we write down an explicit form for both sides of the Selberg trace formula for a τ-function belonging to the functional space $\mathscr{C}^p(G)$, $0 < p < 1$, $\tau \in \hat{U}(1)$, where $\mathscr{C}^p(G)$ stands for the Harish-Chandra L^p-Schwartz space. While in the forth section we apply the explicit trace formula to the study of the analytic continuation of the attached zeta function to the whole s-plane, we give for it a functional equation and a product representation taken over the set of primitive elements of conjugacy class of Γ, plus information on the location and orders of zeros and poles (see Theorem 4.2).

Acknowledgments

The author express his gratitude to Martin Olbrich for suggesting the subject of this paper, for his encouragements and helpful comments. He also thanks Werner Hoffmann for his concern on this paper as well as reading and correcting the manuscript.

This work is done during the author stay in Mathematische Institute of Göttingen. He wishes to express his thanks to "Forschernachwuchsgruppe des Landes Niedersachsen (Clausthal/Göttingen)" for financial support.

2. Preliminaries and Selberg trace formula

Let $G = SU(n,1)$ be the non compact connected semi-simple Lie group with finite center preserving the complex quadratic form $\sum_{i=1}^{n} |Z_i|^2 - |Z_{n+1}|^2 = 1$. Let $K = S(U(n) \times U(1))$ be its maximal compact subgroup. Then the symmetric space of rank one G/K is the complex hyperbolic space $H^n(\mathbf{C})$.

Also, let Γ be a discrete subgroup of G such that $vol(\Gamma\backslash G) < \infty$ and $\Gamma\backslash G$ is not compact, let $P = NAM$ be the Langlands decomposition of a minimal parabolic subgroup P of G.

r denotes the number of Γ inequivalent cusps, then there exists $\{k_i\}_{i=1}^{r} \in K$ such that $P_i = {}^{k_i}P = k_i P k_i^{-1}$ form a complete set of representatives of Γ-cuspidal parabolic subgroups of $G \bmod \Gamma$. Further, throughout this paper, we make the following assumptions on Γ:

- $\Gamma \cap P^i = Z_\Gamma \cdot (\Gamma \cap N^i)$, $1 \leq i \leq r$, where Z_Γ is the center of Γ.
- Γ has no finite order element other than those in Z_Γ.

It is known that for Γ cofinite, the Hilbert space $L^2(\Gamma\backslash G)$ has the following decomposition (the continuous part and the discrete part) with respect to the action of the left regular representation L of G:

$$L^2(\Gamma\backslash G) = L^2_{disc}(\Gamma\backslash G) \oplus L^2_{cont}(\Gamma\backslash G).$$

Now, for $0 < p < 1$ let us consider the following Schwartz space $\mathscr{C}^p(G)$ which generalizes the well known Harish-Chandra space $\mathscr{C}(G) = \mathscr{C}^2(G)$. The space $\mathscr{C}^p(G)$ consists of smooth functions f of G such that

$$\nu^p_{D,n}(f) = \sup_{x \in G}\{(1+\sigma(x))^n \Theta^{-2/p}(x)|(D_1 f D_2)(x)|\} < \infty$$

$$\text{for every } n \in \mathbf{Z} \text{ and } D_1, D_2 \in \mathscr{U}(\mathfrak{g}),$$

where $\sigma(x)$ is the hyperbolic distance between K and $x.K$, $\Theta(x) = \int_K e^{-\rho \log(a(xk))}\, dk$ the elementary spherical function and $\mathscr{U}(\mathfrak{g})$ is the universal enveloping algebra of G. $\mathscr{C}^p(G)$ is a Frechet space with $\nu^p_{D,n}$ as semi-norms and for $0 < p < 1$ we have the following inclusions with dense ranges $C_0^\infty(G) \subset \mathscr{C}^p(G) \subset \mathscr{C}^1(G) \subset C^\infty(G)$.

Then, for $\alpha \in \mathscr{C}^p(G)$ we denote by $\pi_\Gamma(\alpha)$ the convolution operator associated to α acting on $L^2(\Gamma\backslash G)$ and defined as an integral operator as follows. For $f \in L^2(\Gamma\backslash G)$,

$$[\pi_\Gamma(\alpha)f](x) = \int_{\Gamma\backslash G} K_\alpha^\Gamma(x,y)f(y)d_G(y),$$

where $d_G(y)$ denotes the Haar measure on G and the kernel $K_\alpha^\Gamma(x,y)$ has the following expression

$$K_\alpha^\Gamma(x,y) = \sum_{\gamma\in\Gamma} \alpha(x^{-1}\gamma y), \qquad x, y \in G \text{ and } \gamma \in \Gamma.$$

The kernel $K_\alpha^\Gamma(x,y)$ converges uniformly on compact subsets of $G \times G$.

Because of the continuous part $L^2_{cont}(\Gamma\backslash G)$, the operator $\pi_\Gamma(\alpha)$ need not be compact but for α right K-finite its restriction to $L^2_{dis}(\Gamma\backslash G)$ that we will denote by π_Γ^d is of trace class.

A function $\alpha \in \mathscr{C}^\infty(G)$ is said to be right K-finite if there exists a finite set F in the unitary dual $\hat{K}$ of K such that $\alpha * \chi_F = \alpha$, $\chi_F = \sum_{\tau\in F}\chi_\tau$.

Then for α right K-finite in $\mathscr{C}^p(G)$, the operator $\pi_\Gamma^d(\alpha)$ is of trace class and we have the following theorem (cf [15, page 85]) giving its Selberg trace formula

Theorem 2.1.

$$\begin{aligned}\operatorname{Tr}\pi_\Gamma^d(\alpha) &= \operatorname{vol}(\Gamma\backslash G)\left(\sum_{z\in Z_\Gamma}\alpha(z)\right) + \sum_{\{\gamma\}\in\Gamma_s}\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G}\alpha(x^{-1}\gamma x)d(G_\gamma\backslash G)(x)\\ &\quad + \lim_{s\mapsto 0}\frac{d}{ds}(s\varphi_\alpha(s)) + \frac{1}{4\pi}\sum_{v\in\hat{F}}\left(\int_{\Re(S)=0} tr\left(M_v(-s)\left(\frac{d}{ds}M_v(s)\right).U^{v,s}(\alpha)\right)ds\right.\\ &\quad\left. - \frac{1}{4}\sum_v tr(M_v(0)U^{v,0}(\alpha))\right).\end{aligned}$$

Here, $\Gamma_s \subset \Gamma$ is the set of semi simple elements, G_γ the centralizer of γ in G, $M_v(s)$ is the intertwining operator of Γ and $U^{v,s}$ is the principal series representation of G induced from $(s,v) \in \mathbf{C} \times F$.

All the integrals in the above formula are absolutely convergent and

$$\begin{cases}\varphi_\alpha(s) = \int_{-\infty}^{+\infty}\int_{N\cap\Gamma\backslash N}(\sum_{\eta\in\Gamma\cap N;\eta\neq 1}\alpha_K(a(t)n^{-1}\eta n a(t)^{-1}))e^{2|\rho|t(1+s)}d_N(n)dt,\\ \alpha_K = \int_K \alpha(kxk^{-1})dk,\end{cases}$$

where there is a strip $\mathscr{S}_P$ of $\mathbf{C}$ containing the imaginary axis such that the integral defining φ_α is absolutely convergent for $s \in \mathscr{S}_P$ as a meromorphic function whose only possible singularity is a simple pole at $s = 0$.

In particular, $\lim_{s\to 0} \frac{d}{ds}(s\varphi_\alpha(s))$ exists and it is just the constant term in the Laurent expansion of $\varphi_\alpha(s)$. More precisely (cf G. Warner [15]), it is the sum of some tempered distributions, i.e. we have

$$\lim_{s\to 0} \frac{d}{ds}(s\varphi_\alpha(s)) = \frac{1}{|\lambda|} \operatorname{vol}(N \cap \Gamma \backslash N)[c_{\Gamma_\lambda} T_\lambda(\alpha) + R_{\Gamma_\lambda} T'_\lambda(\alpha) + c_{\Gamma_{2\lambda}} T_{2\lambda}(\alpha)],$$

where

- $T_\lambda(\alpha) = c_1 \int_N \alpha_K(n)dn.$
- $T'_\lambda(\alpha) = c_2 \int_{N_\lambda} \int_{N_{2\lambda}} \alpha_K(n_\lambda n_{2\lambda}) \log(\|n_\lambda\|)d_{N_\lambda}(n_\lambda)d_{N_{2\lambda}}(n_{2\lambda});\ N = N_\lambda N_{2\lambda}.$
- $T_{2\lambda}(\alpha) = \frac{1}{2}|\lambda|[\int_{G/G_{n_0}} \alpha_K(xn_0x^{-1})dG/G_{n_0}(x)]$
$$+\frac{1}{2}|\lambda|[\int_{G/G_{n_0}^{-1}} \alpha_K(xn_0^{-1}x^{-1})dG/G_{n_0^{-1}}(x)],$$

where n_0 is a fixed element in $N_{2\lambda}$ such that $\|\exp^{-1}(n_0)\| = 1$.

For some interesting applications of the trace formula, for instance the investigation of the attached Selberg zeta function, it is essential to go further in the computations, i.e. to give the Fourier transform in the sense of Harish-Chandra of the distributions $\alpha \to T_\lambda(\alpha)$, $\alpha \to T'_\lambda(\alpha)$ and $\alpha \to T_{2\lambda}(\alpha)$ involved in the expression of $\lim_{s\to 0} \frac{d}{ds}(s\varphi_\alpha(s))$.

In their investigation of the meromorphic continuation of the logarithmic derivative of the Selberg Zeta function for $G = SU(2n+1, 1)$, R. Gangolli and G. Warner ([6]) studied separately each term figuring in the trace formula when applied to a certain function $h_s(t)$ and prove that it can be continued meromorphically to the complex line $\mathbf{C}$ with simple poles and integer residues. For technical reasons they avoid the case when $G = SU(2n, 1)$ because for that case the function $J(v)$, appearing in the expression of a certain weighted integral, is no longer polynomial ($J(v)$ is a polynomial for $G = SU(2n+1, 1)$). Therefore the meromorphic extension of the corresponding term needs more detailed analysis.

Let $G = SU(n, 1)$. One of the objects of this work is to write down an explicit formula of the Selberg trace formula for the convolution operator associated to a τ_l-radial function f_{τ_l} where τ_l is a one dimensional irreducible representation of $K = S(U(n) \times U(1))$.

In this case the functions f_{τ_l} generalize to the line bundles E_{τ_l} associated to τ_l the notion of radial functions.

So, in the next section, we will discuss harmonic analysis on such bundles (see [4]).

3. Spherical Fourier transform on the vector bundle E_τ

Let (τ, V_τ) be a unitary finite dimensional irreducible representation of K of degree d_τ and χ_τ its character. Let E_τ be the homogeneous vector bundle over G/K associated to τ. Then, a cross section of E_τ may be identified with a vector valued function $f : G \to V_\tau$ which is right K-equivariant of type τ, i.e.

$$f(gk) = \tau(k^{-1})f(g), \qquad \forall g \in G \text{ and } k \in K$$

We denote by $\mathscr{C}^p(G,\tau)$ and $L^2(G,\tau)$ the following spaces of cross-sections of E_τ.

$$\mathscr{C}^p(G,\tau) = \{f \in \mathscr{C}^p(G) \otimes V_\tau \text{ and the components } f_i \text{ of } f \text{ are right } K\text{-equivariant of type } \tau\},$$

$$L^2(G,\tau) = \left\{ f : G \to V_\tau \,|\, \text{the components } f_i \text{ are right } K\text{-invariant of type } \tau \text{ and } \int_G |f_i|^2 < \infty \right\}.$$

Also, let denote by $\mathscr{C}^p(G,\tau,\tau)$ the related convolution algebra of radial systems of sections of E_τ defined as follows

$$\mathscr{C}^p(G,\tau,\tau) = \left\{ F : G \to \mathrm{End}(V_\tau) \,\middle|\, \begin{array}{l} F(k_1 g k_2) = \tau(k_1)F(g)\tau(k_2) \ \ \forall k_1,k_2 \in K; g \in G \\ \text{and } \ tr\, F \in \mathscr{C}^p(G) \end{array} \right\}.$$

Remark 3.1. *The algebra above generalizes to the bundle case E_τ the convolution algebra $\mathscr{C}^p(K\backslash G/K)$ of K bi-invariant functions on G.*

An interesting feature of this algebra is that $\tau|_M$ is multiplicity free (every $\sigma \in \hat{M}$ occurs at most once in $\tau|_M$), it is commutative and can be identified with a certain subalgebra $I_{p,\tau}(G)$ of $\mathscr{C}^p(G)$ defined by

$$I_{p,\tau}(G) = \left\{ f \in \mathscr{C}^p(G) \text{ such that } \begin{array}{l} \text{i) } f(kxk^{-1}) = f(x) \ \ \forall x \in G, k \in K \ \ (\text{i.e. } f \text{ is } K\text{-central}) \\ \text{ii) } d_\tau \bar{\chi}_\tau * f = f = (f * d_\tau \bar{\chi}_\tau), \ \text{ the convolution is over } K \end{array} \right\}.$$

The following map

$$\mathscr{C}^p(G,\tau,\tau) \to I_{p,\tau}(G)$$
$$F \mapsto f_F(x) = d_\tau \ tr\, F(x),$$

where d_τ is the formal degree of τ gives a linear bijection between the two convolution algebras and its inverse is given by

$$I_{p,\tau}(G) \to \mathscr{C}^p(G,\tau,\tau)$$

$$f \mapsto F_f(x) = \frac{1}{d_\tau}\int_K f(kx)\tau(k)^{-1}dk,$$

also

$$F_{f_1 * f_2} = F_{f_1} * F_{f_2} \qquad \text{for every } f_1; f_2 \in I_{p,\tau}(G).$$

Now, let $P = MAN$ be the Langlands decomposition of the parabolic subgroup P of G. Before defining the Spherical Fourier transform acting on the algebra $\mathscr{C}^p(G,\tau,\tau)$, we assume that $\tau|_M$ is still irreducible and we keep denoting it by τ. So, for $\lambda \in i\mathbf{R}$, let $U^{\tau,\lambda}$ be the representation of G induced from the following representation of P: $man \mapsto \tau(m)a^{\rho_P-\lambda}1_N$ $(A \simeq \mathbf{R}^+)$ and we denote by $H^{\tau,\lambda}$ the space of the representation $U^{\tau,\lambda}$ defined by

$$H^{\tau,\lambda} = \{f : G \mapsto V_\tau; f(manx) = a^{\lambda-\rho_P}\tau^{-1}(m)f(x) \text{ and } f|_K \in L^2(K)\}.$$

For $f \in H^{\tau,\lambda}$, we have $[U^{\tau,\lambda}(g)f](x) = f(g^{-1}.x)$, $g, x \in G$.

Then take $F \in \mathscr{C}^p(G,\tau,\tau)$, $\hat{F}(U^{\tau,\lambda}) := U^{\tau,\lambda}(F)$ is called the spherical Fourier transform of F (Gelfand Fourier transform) and it is defined as follows.

$$\hat{F}(U^{\tau,\lambda}) = \frac{1}{d_\tau}\int_G Tr[u^{\tau,\lambda}(x)F(x)]dx,$$

while for $f \in I_{p,\tau}(G)$, $\hat{f}(U^{\tau,\lambda}) := U^{\tau,\lambda}(f)$ is called the spherical trace Fourier transform of f and it is defined by

$$\hat{f}(U^{\tau,\lambda}) = \frac{1}{d_\tau}\int_G f(x)\ Tr(u^{\tau,\lambda}(x))dx,$$

where $Tr(u^{\tau,\lambda})$ is called the spherical trace function of type τ and $u^{\tau,\lambda}(g) = P_\tau U^{\tau,\lambda}(g)P_\tau$ is the operator valued spherical function. P_τ is the projection operator from $H^{\tau,\lambda}$ onto V_τ given by $P_\tau = d_\tau \int_K U^{\tau,\lambda}(k)\chi_\tau(k^{-1})dk$.

Furthermore, we have the following important relation $\hat{F}(U^{\tau,\lambda}) = \hat{f}_F(U^{\tau,\lambda})$.

In the case we are concerned with in this paper, that is $G = SU(n,1)$, we suppose that

$$\tau\left(\begin{pmatrix} B & 0 \\ 0 & z \end{pmatrix}\right) = \tau_l\left(\begin{pmatrix} B & 0 \\ 0 & z \end{pmatrix}\right) = z^l,$$

where $B \in U(n)$, $z \in U(1)$, $\det\begin{pmatrix} B & 0 \\ 0 & z \end{pmatrix} = z \det B = 1$ and $l \in \mathbf{Z}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G.

If $H_0 = \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathfrak{p}$, then $\mathbf{R}H_0 =: \mathfrak{a}$ is a maximal Abelian subalgebra of $\mathfrak{p}$.

Let $A_{\mathfrak{p}}$ denote the Lie algebra of $\mathfrak{a}$, then $A_{\mathfrak{p}}$ is identified to $\mathbf{R}$ as follows.

$$A_{\mathfrak{p}} := \exp a_{\mathfrak{p}} = \left\{ \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}; t \in \mathbf{R} \right\}.$$

Also, the centralizer M of $A_{\mathfrak{p}}$ in K is given by

$$M = \left\{ \begin{pmatrix} U & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}; U \in U(n-1) \text{ and } e^{2i\theta} \det U = 1 \right\}.$$

In this case, the spherical trace function of type τ_l defined by $\xi_{l,\lambda}(t) = \int_K \tau_l(\kappa(x))\, exp(-(i\lambda + n) \log a(xk))dk$ is given in terms of the Jacobi functions, more precisely we have

$$\xi_{l,\lambda}(t) := \xi_{l,\lambda}(a_t) = Tr(u^{\tau_l,\lambda}) = (2 \cosh t)^l \varphi_{i\lambda}^{n-1,l}(t),$$

where

$$\varphi_\lambda^{\alpha,\beta} = {}_2F_1\left(\frac{\alpha+\beta+1-i\lambda}{2}, \frac{\alpha+\beta+1+i\lambda}{2}; \alpha+1, -\sinh^2 t\right).$$

Hence for $f \in I_{p,l}(G)$, its spherical trace Fourier transform is

$$\hat{f}(U^{\tau_l,\lambda}) = \hat{f}_l(\lambda) = \int_0^{+\infty} f(a_t)\xi_{l,\lambda}(t)\Delta(t)dt,$$

where the density Δ is given by

$$\Delta(t) = (2 \sinh t)^{2(n-1)}(2 \cosh 2t).$$

***Discrete series representations.**

As $Rank(G) = Rank(K)$, according to Harish Chandra, in addition to the principal series, there are also discrete series representations that figure, in the spectral decomposition of $L^2(G)$ and that we will describe below.

Let T be the subgroup of K formed by diagonal matrices (i.e. a maximal Abelian subgroup of K) and we denote its Lie algebra by $\mathfrak{t}$. The unitary dual $\hat{T}$ of T can be identified with a lattice L_T in $i\mathfrak{t}^*$, the set of regular elements is denoted by L'_T. The Weyl group of G/T acts on L'_T. Let L_T^+ denote its fundamental domain. Then it is known that L_T^+ uniquely parameterizes the discrete series representations $\hat{G}^2$ of G.

For every $v \in L_T^+$, we denote by $w(v)$ its corresponding representation in $\hat{G}^2$. Also let us denote by $\hat{G}^2(\tau_l)$ the set given by $\hat{G}^2(\tau_l) = \{v \in L_T^+ \mid [w(v) : \tau_l] \neq 0\}$. Then for $l \leq n$ the set $\hat{G}^2(\tau_l)$ is empty.

For $l > n$, $\hat{G}^2(\tau_l) = \{U^{\tau,v_j}/v_j \in D_l\}$, where

$$D_l := \{v \in \mathbf{C} \mid \text{some } w \in \hat{G}^2(\tau_l) \text{ can be embedded in } U^{\tau,v}\}$$
$$= \left\{v_j = i(2j + n - |l|); 0 \le j \le \frac{(l-n)}{2}\right\}.$$

The associated Harish-Chandra c-function $c_l(v)$ has the expression (cf. [13])

$$c_l(v) = \frac{(n-1)!\Gamma\left(\frac{iv}{2}\right)\Gamma\left(\frac{iv}{2}+\frac{1}{2}\right)}{\pi^{1/2}2^{1-n}\Gamma\left(\frac{n+iv+l}{2}\right)\Gamma\left(\frac{n+iv-l}{2}\right)}$$

and zeros which lie in the set D_l.

4. Selberg trace formula for non uniform lattices acting on $E_{\tau_l}^{\Gamma}$ over $\Gamma\backslash G/K$

4.1 τ_l-Eisenstein series

Let Γ be a cofinite discrete subgroup of $G = SU(n,1)$ which is subject to the same assumptions as in the introduction and let r be the number of Γ-inequivalent cusps, $P_i = k_i P k_i^{-1}$ $(k_i \in K)$ a complete set of representatives of Γ-cuspidal parabolic subgroups of $G \bmod \Gamma$.

The τ_l-Eisenstein series (with parameter ϕ_i) corresponding to $P^i = k_i MAN k_i^{-1}$ is defined by the series (cf. [8])

$$E_l^i(\lambda, \phi_i, x) = \sum_{\gamma \in \Gamma \cap N_i \backslash \Gamma} \tau_l(\kappa(\gamma x))^{-1} \phi_i(m(\gamma x) e^{(\lambda - n)(\log a(\gamma . x))},$$

where $\lambda \in \mathfrak{a}_{\mathbf{C}}^* \simeq \mathbf{C}$, $x \in G$ a, $\phi_i \in L^2(\Gamma_{M_i}\backslash M_i, \tau_l)$, $\Gamma_{M_i} = \Gamma \cap M_i$ and

$$L^2(\Gamma_M\backslash M, \tau_l) \cong \left\{ f : M \to V_{\tau_l} \left| \begin{array}{r} f(m_1 m_2) = \tau_l^{-1}(m_2) f(m_1); f(\gamma m) = f(m) \\ \text{for every } m_1, m_2 \in M; \gamma \in \Gamma_M \end{array} \right. \right\}.$$

For example, one can take $\phi_i|_{M_i} = cste = 1_{V_\tau}$ such that

$$E_l^i(\lambda, \phi_i, x) = \sum_{\gamma \in \Gamma \cap N_i \backslash \Gamma} \tau_l(\kappa(\gamma x))^{-1} e^{(\lambda - n)(\log a(\gamma . x))}.$$

$E_l^i(\lambda, \phi_i, x)$ are τ_l functions $\mathscr{C}^\infty$ on $\{\lambda, \Re(\lambda) < -n\} \times G$ and left invariant under Γ. Also $E_l^i(\lambda, \phi, x)$ is an eigenfunction of $\mathscr{Z}(\mathfrak{g})$

$$zE_l^i(\lambda, \phi_i, x) = E_l^i(\lambda, \phi_i, x)\chi_M(\mu_\lambda(z)) \qquad \text{for } z \in \mathscr{Z}(\mathfrak{g}).$$

Here $\mathscr{Z}(\mathfrak{g})$ denotes the center of the enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$, χ_M is a representation of $\mathscr{Z}(\mathfrak{m})$ (the center of the enveloping algebra of $\mathfrak{m}_{\mathbf{C}}$) on the space V_l commuting with τ_l, then there exist an injective homomorphism μ (cf. [8]); $\mu : \mathscr{Z}(\mathfrak{g}) \to \mathscr{Z}(\mathfrak{m})$ such that for $z \in \mathscr{Z}(\mathfrak{g})$ we have $\mu(z) = \sum \xi_i q_i$, $q_i \in S(\mathfrak{a}_{\mathbf{C}})$; $\xi_i \in \mathscr{Z}(\mathfrak{m})$ and $\mu_\lambda(z) = \sum \xi_i q_i(\lambda)$.

Remark 4.1. *In our case $V_l = \mathbf{C}$, χ_M is the infinitesimal character of M determined by τ_l, so if we take $\phi = 1$ then for $C \in \mathscr{Z}(\mathfrak{g})$ the Casimir operator we have $\chi_M(\mu_\lambda(C)) = (\lambda^2 + l^2 - n^2)$.*

However, $E_l^i(\lambda, \phi, x)$ is not square integrable with respect to x, $E_l^i(\lambda, \phi_i, x) \notin L^2(G, \tau_l)$, we denote by $E_l^{P_j}(P_i, \lambda, \phi, x)$ its constant term along $P^j = M^j A^j N^j$; i.e.

$$E_l^{P_j}(P_i, \lambda, \phi_i, x) = \int_{N_j \cap \Gamma \backslash N_j} E_l^i(\lambda, \phi_i, n_j x) dn_j.$$

Then, it is known from the theory of Eisenstein series that there exists a linear transformation $M_l(s, \lambda)$ from $\bigoplus_{i=1}^r L^2(\Gamma_M \backslash M, \tau_l)$ to itself called the intertwining operator of Γ such that we have

$$E_l^{P_j}(P_i, \lambda, \phi, man) = \sum_{s \in W(A_i, A_j)} [M_l^{ij}(s, \lambda)\phi_i](m) e^{(s\lambda - n)\log a}.$$

$W(A_i, A_j)$ is the set of bijections $s : A_i \mapsto A_j$, where for w the non trivial element of W (the Weyl group), the linear transformation $M_l(w, \lambda) = (M_l^{ij}(k_j w k_i^{-1}, k_i \lambda))_{i,j}$ that we will denote from now on simply by $M_l(\lambda)$ as a function of λ is holomorphic in the region $\Re(\lambda) < -n$ and it can be meromorphically continued to the whole complex λ-plane, its poles in $\Re(\lambda) < 0$ lies in the set $\{s \in \mathbf{R}, -n \leq s < 0\}$ and are all simple and of finite number.

Furthermore, $M_l(\lambda)$ satisfies the functional equation $M_l(-\lambda)M_l(\lambda) = Id$.

Let $E_l(\lambda, x)$ be the column vector with entries $E_i(\lambda, 1, x)$ hence, as a function of λ it can be meromorphically continued to the whole λ-plane, moreover its poles are the poles of $M_l(\lambda)$ and we have the following functional relation for $E_l(\lambda, x)$:

$$E_l(\lambda, x) = M_l(\lambda) E_l(-\lambda, x).$$

Now, let π denote the left regular representation of G.

Let $\psi \in I_{p, \tau_l}(G)$ and $\pi(\psi)$ the associated operator acting on $L^2(\Gamma \backslash G, \tau_l)$ defined by:

For $f \in L^2(\Gamma \backslash G, \tau_l)$, $$[\pi^\Gamma(\psi) f](x) = \int_G \psi(x^{-1} y) f(y) dy = (\psi * f)(x).$$

The operator $\pi^\Gamma(\psi)$ acts on the L^2-sections of the homogeneous vector bundle E_{τ_l} associated with τ_l, i.e. we have the following mapping

$$\pi^{\Gamma}(\psi) : L^2(\Gamma\backslash G, \tau_l) \mapsto L^2(\Gamma\backslash G, \tau_l),$$

where for $f \in L^2(\Gamma\backslash G, \tau_l)$ we have

$$[\pi^{\Gamma}(\psi)(f)](x) = \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash G} \psi(x^{-1}\gamma y) f(y)dy = \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} \psi(x^{-1}\gamma y) \right) f(y)dy.$$

Let us denote by $K_{\Gamma}^{\psi}(x, y) = \sum_{\gamma\in\Gamma} \psi(x\gamma y^{-1})$ the integral kernel of the integral operator $\pi^{\Gamma}(\psi)$ when acting on $L^2(\Gamma\backslash G, \tau_l)$.

We have the following decomposition for $L^2(\Gamma\backslash G, \tau_l)$ with respect to the action of the left regular representation of G:

$$L^2(\Gamma\backslash G, \tau_l) = L^2_{dis}(\Gamma\backslash G, \tau_l) \oplus L^2_{cont}(\Gamma\backslash G, \tau_l);$$
$$L^2_{dis}(\Gamma\backslash G, \tau_l) = L^2_{cusp}(\Gamma\backslash G, \tau_l) \oplus L^2_{res}(\Gamma\backslash G, \tau_l).$$

Here $L^2_{cont}(\Gamma\backslash G, \tau_l), L^2_{cusp}(\Gamma\backslash G, \tau_l)$ and $L^2_{res}(\Gamma\backslash G, \tau_l)$ are respectively the closure of the subspaces spanned by wave packets formed with Eisenstein Series of type τ_l, the cusp forms and the residues of the Eisenstein Series.

Then, for $\psi \in I_{p,\tau_l}(G)$ $(0 < p < 1)$ the operator $\pi^{\Gamma}(\psi)$ is of trace class on $L^2_{dis}(\Gamma\backslash G, \tau)$ and as an integral operator its trace can be written as follows:

$$\begin{aligned} \mathrm{Tr}\, \pi^{\Gamma}(\psi) &= \int_{\Gamma\backslash G} \{\mathrm{tr}\, K_{\Gamma}^{\psi}(x, x) - \mathrm{tr}\, K_{\Gamma}^{\psi, c}(x, x)\} dx \\ &= \int_{\Gamma\backslash G} \left\{ \sum_{\gamma\in\Gamma} \psi(x\gamma x^{-1}) - K_{\Gamma}^{\psi, c}(x, x) \right\} dx, \end{aligned}$$

where

$$K_{\Gamma}^{\psi}(x, y) = \int_{\Im s=0} [E_l(s, x)E_l(s, y)^*] u^{l,s}(\psi) ds = \sum_{i=1}^{r} \int_{\Im s=0} [E_l^i(s, x)\overline{E_l^i(s, y)}] u^{l,s}(\psi) ds.$$

Therefore, as ψ is in the space $\mathscr{C}^p(G)$ and K-finite $(\psi * \chi_{\tau_l} = \psi)$, we can apply the trace formula given in [6] to the operator $\pi^{\Gamma}(\psi)$.

On the other hand, $tr\ \pi^{\Gamma}(\psi)$ can be written also as $\sum_j n_{\Gamma}(U^{l,\lambda_j})\ tr\ U^{l,\lambda_j}(\psi)$, where $\{U^{l,\lambda_j}\}_j \in \hat{G}(\tau_l)$ denotes the set of τ_l-spherical representations occurring in $L^2_{dis}(\Gamma\backslash G, \tau_l)$ and n_j^l their multiplicities. Then, we have

$$tr\ \pi_d^{\Gamma}(\psi) = tr\ \pi_d^{\Gamma}(\psi) = \sum_j n_j^l \hat{\psi}(U^{l,\lambda_j}).$$

Also, we should mention that every function $\psi \in I_{p,\tau_l}(G)$ can be determined from its values in the Abelian part A of G, where $G = KAK$ is the Cartan decomposition of G, more precisely we have

$$\psi \in I_{p,\tau_l}(G); \qquad \psi(g) = \frac{1}{d_l}\chi_l(\kappa_1(g))\psi(a_t(g)),$$

where

$$\kappa(g) = \begin{pmatrix} \kappa_1(g) & 0 \\ 0 & \kappa_2(g) \end{pmatrix}, \qquad \kappa_1(g) \in U(n), \ \ \kappa_2(g) \in U(n)$$

$$\text{and } \det \kappa_1(g) \times \kappa_2(g) = 1.$$

Hence, the trace formula of theorem 1.1 applied to $\pi^{\Gamma}(\psi)$ at its first stage for $\psi \in I_{p,\tau_l}(G)$ leads to the following:

THEOREM 4.1. *For $0 < p < 1$, let $\psi \in I_{p,\tau_l}(G)$ be a τ_l-radial function. Then, the Selberg trace formula for the corresponding operator $\pi^{\Gamma}(\psi)$ on $L^2_{dis}(\Gamma\backslash G, \tau_l)$ reads as follows.*

$$\sum_{j\geq 0} n_j^l \hat{\psi}(\lambda_j) = vol(\Gamma\backslash G) \sum_{z\in Z_\Gamma} \psi(z) + \sum_{\{\gamma\}\in C\Gamma_s} vol(G_\gamma\backslash\Gamma_\gamma) \int_{G_\gamma\backslash G} \psi(x\gamma x^{-1}) d(G_\gamma\backslash G)(x)$$

$$+ k_1 c_{\Gamma_\lambda} T_\lambda(\psi) + k_2 r_{\Gamma_\lambda} T'_\lambda(\psi) + k_3 c_{\Gamma_{2\lambda}} T_{2\lambda}(\psi)$$

$$+ \frac{1}{4\pi}\int_{\Re s=0} tr\left(M_l(-s) \times \frac{d}{ds} M_l(s)\right).\hat{\psi}_l(s)ds - \frac{1}{4}\ tr(M_l(0))\hat{\psi}_l(0).$$

where $k_1 c_{\Gamma_\lambda}; k_2 r_{\Gamma_\lambda}; k_3 c_{\Gamma_{2\lambda}}$ are constants depending on G and Γ (for their explicit expression see [15]*) and the distributions T_λ, T'_λ and $T_{2\lambda}$ have respectively the following integral representations*

$$T_\lambda(\psi) = \frac{1}{d_l}\int_N \chi_l(\kappa_1(n))\psi(a_t(n))dn,$$

$$T'_\lambda(\psi) = \int_{N_\lambda}\left[\int_{N_{2\lambda}} \chi_l(\kappa_1(n_\lambda n_{2\lambda}))\psi(a_t(n_\lambda n_{2\lambda}))d_{N_{2\lambda}}\right] log\|n_\lambda\| d_{N_\lambda},$$

$$T_{2\lambda}(\psi) = \int_{G_{n_0}\backslash G} \chi_l(\kappa_1(xn_0x^{-1}))\psi(a_t(xn_0x^{-1}))d_{G_{n_0}\backslash G}(x)$$

$$+ \int_{G_{n_0^{-1}}\backslash G} \chi_l(\kappa_1(xn_0^{-1}x^{-1}))\psi(a_t(xn_0^{-1}x^{-1}))d_{G_{n_0^{-1}}\backslash G}(x).$$

In order to investigate the meromorphic extension of the Selberg zeta function for the vector bundle E_l associated with τ_l, we need to give an explicit formula for the integrals $T_\lambda(\psi), T'_\lambda(\psi)$ and $T_{2\lambda}(\psi)$ in terms of the τ_l-spherical Fourier transform of ψ.

For this, we will need first to give an explicit inverse for the τ_l-Abel transform defined on the algebra $I_{p,\tau_l}(G)$.

4.2 τ_l-Abel transform acting on $I_{p,\tau_l}(G)$

For every function f in $I_{p,\tau_l}(G)$, its τ_l-Abel transform is given by the following integral transform

$$[A_l f](t) = e^{nt} \int_N dn f(a_t n),$$

and we have the following well-known formula

$$\hat{f}_l(\lambda) = \int_{-\infty}^{+\infty} dt e^{+i\lambda t} [A_l f](t) = \widehat{A_l f}(t),$$

which means that the τ_l-spherical Fourier transform of f is the Euclidean Fourier transform on $\mathbf{R}$ of its τ_l-Abel transform.

Hence

$$\hat{f}_l(0) = \int_{-\infty}^{+\infty} dt [A_l f](t),$$

i.e.

$$A_l f(t) = \int_{-\infty}^{+\infty} e^{-i\lambda t} \hat{f}_l(\lambda) dt.$$

Then, we have the proposition.

Proposition 4.1. *Let A_l denotes τ_l-Abel transform defined on the algebra $I_{p,\tau_l}(G)$, then its inverse is given by the following integral transform*

$$[[A_l^{-1}]f](t) = 2^{-2l-3n-5/2} \sqrt{\pi} \frac{1}{(n-1)!} (\cosh t)^{-l} \frac{1}{\Gamma\left(l - \frac{1}{2}\right)} \times \int_t^{\infty} d(\cosh 2s)(\cosh 2s - \cosh 2t)^{l-3/2} \left(\frac{-d}{d(\cosh s)}\right)^{l+n-1} f(s) ds.$$

Proof. Our method of giving the inverse of the Abel transform is torelate it to the Abel-Jacobi transform $A_{\alpha,\beta}$ investigated by T. H. Koornwinder ([9]), by choosing suitable parameters α, β.

For $f \in \mathscr{C}_0^{\infty}(\mathbf{R})$, its Jacobi transform is defined as follows

$$[J_{\alpha,\beta} f](x) = \int_0^{\infty} f(t) \varphi_{i\lambda}^{\alpha,\beta} \Delta_{\alpha,\beta}(t) dt; \qquad \Delta_{\alpha,\beta}(t) = (2 \sinh t)^{2\alpha+1} (2 \cosh t)^{2\beta+1},$$

where $\varphi_{i\lambda}^{\alpha,\beta}$ is the usual Jacobi function.

The Abel-Jacobi transform considered in [9] is defined as follows

$$A_{\alpha,\beta}(f)(s) := \int_s^{\infty} f(t) A_{\alpha,\beta}(s,t) dt \qquad \text{for } f \in \mathscr{C}^{\infty}(\mathbf{R}),$$

where

$$A_{\alpha,\beta}(s,t) = \frac{2^{3\alpha+2\beta+1}\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\,\sinh(2t)(\cosh t)^{\beta-1/2}(\cosh t - \cosh s)^{\alpha-1/2}$$
$$\times F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; \alpha+\frac{1}{2}; \frac{\cosh t - \cosh s}{2\cosh t}\right).$$

We have also

$$J_{\alpha,\beta}f = \widehat{A_{\alpha,\beta}f}.$$

Hence one can write

$$\hat{f}_l(\lambda) = J_{n-1,-l}\{2^{2l}\cosh t^l f(t)\} = \int_{\mathbf{R}} e^{i\lambda t}(A_{n-1,l}g)(t)dt,$$

where $g(t) = 2^{2l}\cosh t^l f(t)$. Then we have

$$f(t) = A_l^{-1}A_l f(t) = 2^{-2l}(\cosh t)^{-l}g(t) = 2^{-2l}(\cosh t)^{-l}A_{n-1,-l}^{-1}A_{n-1,-l}g(t),$$

and as $A_{n-1,-l}g(t) = A_l f(t)$ we write

$$f(t) = 2^{-2l}(\cosh t)^{-l}A_{n-1,-l}^{-1}[A_l f(t)].$$

The inverse of the Abel-Jacobi transform was given explicitly in [9] in terms of fractional Weyl's transform as $A_{n-1,-l}^{-1}(g) = 2^{-3(n-1)-1/2}\pi^{1/2}\frac{1}{\Gamma(n)}\mathscr{W}_{l-1/2}^{2} \circ \mathscr{W}_{-l-n+1}^{1}(g)$ where for $\mu > 0$, the fractional Weyl's transform $\mathscr{W}_{\mu}^{\tau}$ is defined as follows (for $\Re\mu > 0,\ \tau > 0$)

$$\begin{cases} \mathscr{W}_{\mu}^{\tau}f(t) = \frac{1}{\Gamma(\mu)}\int_t^{\infty} f(s)(\cosh(\tau s) - \cosh(\tau t)d\cosh(2s), \\ \mathscr{W}_{\mu}^{\tau} \circ \mathscr{W}_{\sigma}^{\tau} = \mathscr{W}_{\mu}^{\mu+\sigma}, \\ \mathscr{W}_{o}^{\tau} = Id, \end{cases}$$

Also $\mathscr{W}_{-n}^{\tau}f(t) = \left(\frac{-1}{d(\cosh\tau t)}\right)^n f(t)$.

Hence we obtain

$$f(t) = 2^{-l}(\cosh t{-}l)2^{-3(n-1)-1/2}\pi^{1/2}\frac{1}{\Gamma(n)}\mathscr{W}_{l-1/2}^{2} \circ \mathscr{W}_{-l-n+1}^{1}(Af).$$

By replacing the Weyl transformations by their integral expressions we get the desired result.

4.3 Computation of the distributions T_λ, T'_λ and $T_{2\lambda}$ and the trace formula.

In this subsection we will proceed to give the spherical Fourier transform for the distributions T_λ, T'_λ and $T_{2\lambda}$

1) $\boldsymbol{T_\lambda(\psi)}$

$$T_\lambda(\psi) = \frac{1}{d_l}\int_N \psi(n)dn = \frac{e^{-\rho t}}{2\pi d_l}\int_{\mathbf{R}} \hat{\psi}_l(v)dv$$

2) $\boldsymbol{T'_\lambda(\psi)}$.

As we consider the case $G = SU(n,1)$, we have

$$T'_\lambda(\psi) = \int_{N_\lambda}\left[\int_{N_{2\lambda}} \chi_l(\kappa_1(n_\lambda n_{2\lambda}))\psi(a_t(n_\lambda n_{2\lambda}))d_{N_{2\lambda}}\right] log(\|n_\lambda\|)d_{N_\lambda},$$

where for $g = (g_{ij})_{ij} \in SU(n,1)$, $g = \kappa_1(g)\kappa_2(g)a_t(g)\kappa'$, we have $\kappa_1(g) = \frac{(g)_{00}}{|(g)_{00}|}$ and $cosh(t) = |(g)_{00}|$.

Hence

$$\begin{cases} \kappa_1(n_{s,\xi}) = \frac{1+is+\frac{1}{2}|\xi|^2}{\left|1+is+\frac{1}{2}|\xi|^2\right|} \\ \cosh(t(n_{s,\xi})) = \left|1+is+\frac{1}{2}|\xi|^2\right| \end{cases}$$

If we write $\psi(t) := \psi(a_t) = \tilde{\psi}(\cosh t)$, we get

$$T'_\lambda(\psi) = \int_{\mathbf{C}^{n-1}}\left[\int_{\mathbf{R}} \chi_l\left(\frac{1+is+\frac{1}{2}|\xi|^2}{\left|1+is+\frac{1}{2}|\xi|^2\right|}\right)\tilde{\psi}\left(\left|1+is+\frac{1}{2}|\xi|^2\right|\right)ds\right] log|\xi|d\xi.$$

Next, by using successively polar coordinates in $\mathbf{C}^{n-1}$ i.e., $\xi = rw$, $w \in S^{2n-3}$, $d\xi = r^{2n-3}\, drdw$ and afterwards by setting $x = r^2$, we obtain

$$T'_\lambda(\psi) = \frac{vol(S^{2n-3})}{4}\int_0^\infty \int_{-\infty}^{+\infty}\left(\frac{1+is+\frac{1}{2}|x|^2}{\left|1+is+\frac{1}{2}|x|^2\right|}\right)^l$$

$$\times\, \tilde{\psi}\left(\left|1+is+\frac{1}{2}|\xi|^2\right|\right) log|x|x^{n-1}\, dsdx,$$

and by using the Inverse of the τ_l-Abel transform applied to the function ψ we have

$$\psi(t) = const\int_t^\infty d(\cosh 2s)(\cosh 2s - \cosh 2t)^{l-3/2}\left(\frac{-d}{d(\cosh s)}\right)^{l+n-1}(A_l\psi)(s)ds.$$

As $\psi(t) = \tilde{\psi}(\cosh t)$ where $\tilde{\psi} : [1,+\infty[\mapsto \mathbf{R}$ and by setting $\tau = \cosh s$, $d\tau = \sinh s\, ds$, we obtain

$$\tilde{\psi}(\tau) = 2^{-2l-3n-5/2}\sqrt{\pi}\frac{1}{(n-1)!}(\cosh\, t)^{-l}\frac{1}{\Gamma\left(l-\frac{1}{2}\right)}\left(l-\frac{1}{2}\right)^{-1/2}$$

$$\times\, \tau^{-l}\int_\tau^\infty (t^2-\tau^2)^{l-1/2}\left(\frac{d}{dt}\right)^{l+n}(A_l\tilde{\psi})(t)dt.$$

Hence, by substituting it into the integral formula giving $T_\lambda'(\psi)$ we get

$$T_\lambda'(\psi) = c\int_0^\infty \int_{-\infty}^{+\infty} \left(\frac{1 + is + \frac{1}{2}|x|^2}{\left|1 + is + \frac{1}{2}|x|^2\right|} \right)^l x^{n-1} \log(x) \left(s^2 + \left(1 + \frac{x}{2}\right)^2 \right)^{-l/2}$$

$$\times \left[\int_{(s^2+(1+x/2))^{1/2}} \left(\tau^2 - s^2 - \left(1 + \frac{x}{2}\right)^2 \right)^{l-1/2} \left(\frac{d}{d\tau}\right)^{l+n} (A_l\tilde{\psi})(\tau)d\tau \right] dsdx.$$

Now by using Fubini's Theorem, we get

$$T_\lambda'(\psi) = \int_0^\infty x^{n-2}\ log\ x \int_{1+x/2}^\infty \left(\frac{d}{d\tau}\right)^{l+n} (A_l\tilde{\psi})(\tau)$$

$$\times \int_{-[\tau^2-(1+x/2)^2]^{1/2}}^{[\tau^2-(1+x/2)^2]^{1/2}} \left(1 - is + \frac{x}{2}\right)^{-l} \left[\tau^2 - \left(1 + \frac{x}{2}\right)^2\right]^{l-1/2} dsd\tau dx.$$

If we denote by $A(\tau, x)$ the inner integral in the expression of $T_\lambda'(\psi)$ and we set $s = \left[\tau^2 - \left(1 + \frac{x}{2}\right)^2\right]^{1/2} t$, $ds = \left[\tau^2 - \left(1 + \frac{x}{2}\right)^2\right]^{1/2} dt$, i.e.,

$$A(\tau, x) = \int_{-[\tau^2-(1+x/2)^2]^{1/2}}^{[\tau^2-(1+x/2)^2]^{1/2}} \left(1 - is + \frac{x}{2}\right)^{-l} \left[\tau^2 - \left(1 + \frac{x}{2}\right)^2\right]^{l-1/2} ds$$

$$= \left[\tau^2 - \left(1 + \frac{x}{2}\right)^2\right]^l \int_{-1}^1 \left[1 + \frac{x}{2} - it\left(\tau^2 - \left(1 + \frac{x}{2}\right)^2\right)^{1/2}\right]^{-l} (1 - t^2)^{l-1/2} dt,$$

we put $y = \frac{1+t}{2}$ to obtain

$$A(\tau, x) = 2^l \left[\tau^2 - \left(1 + \frac{x}{2}\right)^2\right]^l \left[1 + \frac{x}{2} + i\left(\tau^2 - \left(1 + \frac{x}{2}\right)^2\right)^{1/2}\right]$$

$$\times \int_0^1 \left[1 - \frac{2i\left(\tau^2 - \left(1 + \frac{x}{2}\right)^2\right)^{1/2}}{1 + \frac{x}{2} + i\left(\tau^2 - \left(1 + \frac{x}{2}\right)^2\right)^{1/2}}\right]^{-l} y^{l-1/2}(1 - y)^{l-1/2} dy.$$

By using the integral representation for the hypergeometric function of Gauss ([5], page 59):

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} dt,$$

we get as expression for $A(\tau,x)$:

$$A(\tau,x)=2^l\left[\tau^2-\left(1+\frac{x}{2}\right)^2\right]^l\left[1+\frac{x}{2}+i\left(\tau^2-\left(1+\frac{x}{2}\right)^2\right)^{1/2}\right]\frac{\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(2l+1)}$$
$$\times\,{}_2F_1\left(l,l+\frac{1}{2};2l+1,\frac{2i\left(\tau^2-\left(1+\frac{x}{2}\right)^2\right)^{1/2}}{1+\frac{x}{2}+i\left(\tau^2-\left(1+\frac{x}{2}\right)^2\right)^{1/2}}\right).$$

Next, by applying successively the following transformation formulas

$$ {}_2F_1(a,b;2b;z)=\left(1-\frac{1}{2}z\right)^{-a}{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};b+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right) \tag{1}$$

$$ {}_2F_1\left(a-\frac{1}{2},a;2a;z\right)=2^{2a-1}(1+(1-z)^{1/2})^{1-2a} \tag{2}$$

we get the following simple expression for $A(\tau,x)$.

$$A(\tau,x)=\frac{\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(2l+1)}\left(\tau-\left(1+\frac{x}{2}\right)\right)^l.$$

Hence, by substituting it in the integral defining the distribution $T'_\lambda(\psi)$ we get

$$T'_\lambda(\psi)=c'(l,n)\int_0^\infty x^{n-2}\,log\,x\int_{1+x/2}^\infty\left(\frac{d}{d\tau}\right)^{l+n}(A_l\tilde{\psi})(\tau)\left(\tau-\left(1+\frac{x}{2}\right)\right)^l d\tau dx.$$

Also, as $\lim_{\tau\mapsto\infty}\left(\frac{d}{d\tau}\right)^n(A_l\tilde{\psi})(\tau)\mapsto 0$, we have:

$$\int_{1+x/2}^\infty\left(\frac{d}{d\tau}\right)^{l+n}(A_l\tilde{\psi})(\tau)\left(\tau-\left(1+\frac{x}{2}\right)\right)^l d\tau=(-1)^{-1}l!\int_{1+x/2}^\infty\left(\frac{d}{d\tau}\right)^n(A_l\tilde{\psi})(\tau)d\tau.$$

Therefore

$$\begin{aligned}T'_\lambda(\psi)&=c'(l,n)(-1)^{l+1}l!\int_0^\infty x^{n-2}\log x\left(\frac{d}{d\tau}\right)^{n-1}(A_l\tilde{\psi})\left(1+\frac{x}{2}\right)\\&=c'(l,n)(-1)^l l!2^{n-2}\int_0^\infty(A_l\tilde{\psi})'\left(1+\frac{x}{2}\right)\left(\frac{d}{d\tau}\right)^{n-2}[x^{n-2}\log x].\end{aligned}$$

As

$$\begin{aligned}\left(\frac{d}{d\tau}\right)^{n-2}[x^{n-2}\log x]&=(n-2)!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-2}\right)\\&=(n-2)!\,\log x+\frac{\Gamma'(n-2)}{\Gamma(n-2)}+\gamma;\end{aligned}$$

(γ is the Euler constant)

we get

$$T'_{\lambda}(\psi) = 2^{n-1}(n-2)!c'(l,n) \times \left(\frac{\Gamma'(n-2)}{\Gamma(n-2)} + \gamma\right) A_l \psi(0)$$
$$- 2^{n-2}(n-2)!c'(l,n) \int_0^{\infty} (A_l \tilde{\psi})' \left(1 + \frac{x}{2}\right) \log x \, dx.$$

Now, by using the sublemma in ([15], page 121) we get the following final expression for $T'_{\lambda}(\psi)$:

$$T'_{\lambda}(\psi) = const\left[\frac{1}{\pi}\left(\frac{\Gamma'(n-2)}{\Gamma(n-2)} + \gamma\right) \int_{-\infty}^{+\infty} \hat{\psi}_l(v) dv + \hat{\psi}_l(0)\right.$$
$$\left. - \frac{2}{\pi} \int_{-\infty}^{+\infty} \hat{\psi}_l(v) \frac{\Gamma'(1+iv)}{\Gamma(1+iv)} \, dv\right].$$

3) $\boldsymbol{T_{2\lambda}(\psi)}$

We have $c_{\Gamma_{2\lambda}} T_{2\lambda}(\psi) = \lim_{s \mapsto 0}\left(\frac{d}{ds} s L^*_{\psi}(s)\right)$, where the function $L^*_{\psi}(s)$ is defined as follows.

$$L^*_{\psi}(s) = \int_{-\infty}^{+\infty} \left(\sum_{\eta \in \Gamma_{2\lambda};\, \eta \neq 1} \chi_l(\kappa_1(a_t \eta a_{-t}))\psi(a(a_t \eta a_{-t}))e^{2nt(1+s)} \, dt.\right.$$

We have for $\eta = n(v, 0) \in \Gamma_{2\lambda} \subset N_{2\lambda}$:

$$\kappa_1(a_t \eta a_{-t}) = \frac{1 + e^{2t}\left(iv + \frac{1}{2}|\xi|^2\right)}{\left|1 + e^{2t}\left(iv + \frac{1}{2}|\xi|^2\right)\right|} \qquad \text{and} \qquad a(a_t \eta a_{-t}) = \left|1 + e^{2t}\left(iv + \frac{1}{2}|\xi|^2\right)\right|.$$

Hence

$$L^*_{\psi}(s) = \sum_{\eta = n(v,0) \in \Gamma_{2\lambda};\, v \neq 0} \int_{-\infty}^{+\infty} \left(\frac{1 + e^{2t} iv}{|1 + e^{2t} iv|}\right)^l \tilde{\psi}((1 + v^2 e^{4t})^{1/2}) e^{2nt(1+s)} \, dt$$
$$= \sum_{\eta = n(v,0) \in \Gamma_{2\lambda};\, v > 0} \int_{-\infty}^{+\infty} \left(\frac{1 + e^{2t} i|v|}{|1 + e^{2t} iv|}\right)^l \tilde{\psi}((1 + v^2 e^{4t})^{1/2}) e^{2nt(1+s)} \, dt$$
$$+ \sum_{\eta = n(v,0) \in \Gamma_{2\lambda};\, v < 0} \int_{-\infty}^{+\infty} \left(\frac{1 - e^{2t} i|v|}{|1 + e^{2t} iv|}\right)^l \tilde{\psi}((1 + v^2 e^{4t})^{1/2}) e^{2nt(1+s)} \, dt.$$

Next by setting $x = e^{2t}|v|$, $dt = \frac{dx}{2x}$, we get

$$L_\psi^*(s) = \frac{1}{2}\left(\sum_{\eta=n(v,0)\in\Gamma_{2\lambda};\,v>0} \frac{1}{|v|^{n(1+s)}}\right)\int_0^{+\infty}\left(\frac{1+ix}{|1+ix|}\right)^l \psi((1+x^2)^{1/2})x^{n(1+s)-1}\,dx$$

$$+\frac{1}{2}\left(\sum_{\eta=n(v,0)\in\Gamma_{2\lambda};\,v<0} \frac{1}{|v|^{n(1+s)}}\right)\int_0^{+\infty}\left(\frac{1-ix}{|1+ix|}\right)^l \tilde{\psi}((1+x^2)^{1/2})x^{n(1+s)-1}\,dx$$

$$=\frac{1}{4}\left(\sum_{\eta=n(v,0)\in\Gamma_{2\lambda};\,v\neq 0} \frac{1}{|v|^{n(1+s)}}\right)\int_0^{+\infty}\left\{\left(\frac{1+ix}{|1+ix|}\right)^l + \left(\frac{1-ix}{|1+ix|}\right)^l\right\}$$

$$\times\, \tilde{\psi}((1+x^2)^{1/2})x^{n(1+s)-1}\,dx.$$

Since L_ψ^* is regular at $s=0$, we simply have

$$c_{\Gamma_{2\lambda}}T_{2\lambda}(\psi) = L_\psi^*(0) = c_{\Gamma_{2\lambda}}\int_0^{+\infty}\left\{\left(\frac{1+ix}{|1+ix|}\right)^l + \left(\frac{1-ix}{|1+ix|}\right)^l\right\}\tilde{\psi}((1+x^2)^{1/2})x^{n-1}\,dx,$$

where $c_{\Gamma_{2\lambda}}$ is the constant term in the Epstein zeta function associated to the Lattice $\Gamma_{2\lambda}$. Now as

$$\tilde{\psi}((1+x^2)^{1/2}) = c(n,l)(1+x^2)^{-l/2}\int_{(1+x^2)}^{\infty}\left(\frac{d}{d\tau}\right)^{l+n}\tilde{A}_l\psi(\tau)(\tau^2-1-x^2)^{l-1/2}d\tau,$$

we get

$$T_{2\lambda}(\psi) = c(n,l)\int_0^{\infty}(1+x^2)^{-l/2}\left(\int_{(1+x^2)}^{\infty}\left(\frac{d}{d\tau}\right)^{l+n}\tilde{A}_l\psi(\tau)(\tau^2-1-x^2)^{l-1/2}d\tau\right)$$

$$\times\left\{\left(\frac{1+ix}{|1+ix|}\right)^l + \left(\frac{1-ix}{|1+ix|}\right)^l\right\}x^{n-1}\,dx,$$

and using the Fubini and afterwards the change of variable $x=(\tau^2-1)^{1/2}y$ we get

$$T_{2\lambda}(\psi) = c(n,l)\int_1^{\infty}\left(\frac{d}{d\tau}\right)^{l+n}\tilde{A}_l\psi(\tau)$$

$$\times\int_0^{(\tau^2-1)^{1/2}}(\tau^2-1-x^2)^{l-1/2}\{(1-ix)^{-l}+(1+ix)^{-l}\}x^{n-1}\,dx$$

$$=\int_1^{\infty}\left(\frac{d}{d\tau}\right)^{l+n}\tilde{A}_l\psi(\tau)(\tau^2-1)^{l+(n-1)/2}$$

$$\times\left[\int_0^1(1-y^2)^{l-1/2}\{(1-i(\tau^2-1)^{1/2}y)^{-l}+(1+i(\tau^2-1)^{1/2}y)^{-l}\}y^{n-1}\,dy\right]d\tau.$$

Now, by using the Picard's integral representation for the first Appell's hypergeometric function $F_1(a, b, b', c, x, y)$ given by ([5], page 231)

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_1(a, b, b', c, x, y) = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}du,$$

we get

$$\begin{aligned} T_{2\lambda}(\psi) = c_{n,l} \frac{\Gamma(n)\Gamma\left(l+\frac{1}{2}\right)}{\Gamma\left(n+l+\frac{1}{2}\right)} \int_1^\infty \left(\frac{d}{d\tau}\right)^{l+n} \tilde{A}_l \psi(\tau)(\tau^2-1)^{l+(n-1)/2} \\ \times \left\{ F_1\left(n, l, -l+\frac{1}{2}, l+\frac{1}{2}+n, i(\tau^2-1)^{1/2}, -1\right) \right. \\ \left. + F_1\left(n, l, -l+\frac{1}{2}, l+\frac{1}{2}+n, -i(\tau^2-1)^{1/2}, -1\right) \right\} d\tau. \end{aligned}$$

For $l = 0$ the involved Appell's function reduces to the following simple expression

$$F_1\left(n, 0, \frac{1}{2}, \frac{1}{2}+n, i(\tau^2-1)^{1/2}, -1\right) = {}_2F_1\left(n, \frac{1}{2}, \frac{1}{2}+n, -1\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)},$$

and by replacing it we find the result for $T_{2\lambda}(\psi)$ given in [15] for $l = 0$.

Afterwards by using successively the formula ([2] page 15)

$$F_1(a, b, b', \gamma; x, y) = \sum_{m=0}^\infty \frac{(a, m)(b, m)}{(\gamma, m)(1, m)} {}_2F_1(a+m, b', \gamma+m, y)x^m,$$

and the formula ([5] page 104)

$${}_2F_1(a, b, 1+a-b, -1) = 2^{-a} \frac{\Gamma(1+a-b)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1-b+\frac{a}{2}\right)\Gamma\left(\frac{1+a}{2}\right)} \qquad 1+a-b \notin \mathbf{Z}^-,$$

we obtain

$$\begin{aligned} F_1\left(n, l, -l+\frac{1}{2}, l+\frac{1}{2}+n, i(\tau^2-1)^{1/2}, -1\right) \\ + F_1\left(n, l, -l+\frac{1}{2}, l+\frac{1}{2}+n, -i(\tau^2-1)^{1/2}, -1\right) \\ = \frac{\Gamma\left(l+\frac{1}{2}+n\right)}{\Gamma(n)\Gamma(l)} \sum_{m=2j}^\infty 2^{-n-2j+1} \frac{\Gamma(n+2j)\Gamma(l+2j)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2j+1)\Gamma\left(l+\frac{1}{2}+\frac{n}{2}+j\right)\Gamma\left(\frac{1}{2}\,\frac{n}{2}+j\right)}(1-\tau^2)^j \\ = \frac{\Gamma\left(l+\frac{1}{2}+n\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)\Gamma\left(\frac{n}{2}+l+\frac{1}{2}\right)} \sum_{j=0}^\infty \frac{(n, j)\left(\frac{l}{2}, j\right)\Gamma\left(\frac{l}{2}+\frac{1}{2}, j\right)}{\left(\frac{1}{2}, j\right)\left(\frac{n}{2}+\frac{l}{2}+l, j\right) j!}(1-\tau^2)^j \end{aligned}$$

$$= \frac{\Gamma\left(l+\frac{1}{2}+n\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)\Gamma\left(\frac{n}{2}+l+\frac{1}{2}\right)} {}_3F_2\left(\frac{n}{2},\frac{l}{2},\frac{l}{2}+\frac{1}{2};\frac{l}{2},\frac{n}{2}+\frac{l}{2}+l;1-\tau^2\right)$$

$$= \frac{\Gamma\left(l+\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n)\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)} G^{1,3}_{3,3}\left((1-\tau^2)\middle| \begin{matrix} 1-\frac{n}{2} & 1-\frac{l}{2} & \frac{1}{2}-\frac{l}{2} \\ 0 & \frac{1}{2} & -\frac{n}{2}-l+\frac{1}{2} \end{matrix}\right).$$

where $G^{m,n}_{p,q}(\ )$ is the Meijer's G-function (Erdely [5] page 207).

Now we set $\tau = \cosh t,\ D = \frac{d}{d\tau} = \frac{1}{\sinh t}\frac{d}{dt}$ so that

$$\left(\frac{d}{d\tau}\right)^{l+n} \tilde{A}_l\psi(\tau) = D^{n+l}A_l\psi(t),$$

and as ψ is an even function and $A_l\psi(t) = \int_0^\infty \hat{\psi}_l(v)\cos(vt)dv$ we have

$$D^{n+l}A_l\psi(t) = \frac{1}{\pi}\int_0^\infty \hat{\psi}_l(v)D^{n+l}\cos(vt)dv.$$

By replacing all the terms by their expression we get the following form for $T_{2\lambda}(\psi)$:

$$T_{2\lambda}(\psi) = c_{n,l}\int_0^\infty\int_0^\infty \hat{\psi}_l(v)D^{n+l}\cos(vt)dv\sinh t^{2l+n}$$
$$\times G^{1,3}_{3,3}\left(-\sinh^2 t\middle| \begin{matrix} 1-\frac{n}{2} & 1-\frac{l}{2} & \frac{1}{2}-\frac{l}{2} \\ 0 & \frac{1}{2} & -\frac{n}{2}-l+\frac{1}{2} \end{matrix}\right)dt.$$

Next, if we apply the formula ([5] page 209)

$$x^\sigma G^{m,n}_{p,q}\left(x\middle| \begin{matrix} a_r \\ b_r \end{matrix}\right) = G^{m,n}_{p,q}\left(x\middle| \begin{matrix} a_r+\sigma \\ b_r+\sigma \end{matrix}\right),$$

we get

$$T_{2\lambda}(\psi) = c_{n,l}(-1)^{l+n/2}\int_0 o^\infty \int_0^\infty \hat{\psi}_l(v)D^{n+l}\cos(vt)dvG^{1,3}_{3,3}$$
$$\times\left(-\sinh^2 t\middle| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix}\right)dt.$$

where

$$D^N\cos(vt) = (-1)^N\frac{v^2(v^2+1)\dots(v^2+(N-1)^2)}{1.3\dots(2N-1)}\times\mathscr{Y}^{2N+1}_v(t) \qquad (t\in\mathbf{R}),$$

and

$$\mathscr{Y}^N_v(t) = \frac{2^{(N-2)/2}\Gamma\left(\frac{N}{2}\right)}{(\sinh t)^{(N-2)/2}}.\mathscr{B}^{1-N/2}_{-1/2+iv}(\cosh t),$$

where $\mathscr{B}_\alpha^\beta$ is the Legendre function of the first kind and $\mathscr{Y}_\nu^N$ is the zonal spherical function on $SO(n,1)$ attached to the class one principal series representation associated with ν.

Hence, for any $m \geq 0$ there exists $c_M > 0$ and an integer $M > 0$ such that

$$\left|\left(\frac{d}{dt}\right)^m \mathscr{Y}_\nu^N(t)\right| \leq c_M(1+\nu^2)^M \frac{(1+t^2)^M}{e^{(N-1)(t/2)}} \qquad \forall t \geq 0 \text{ and } \nu \in \mathbf{R}.$$

By using the above growth estimate as well as the following asymptotic behavior ([5], page 212) when $t \to \infty$ for

$$\begin{aligned} &G_{3,3}^{1,3}\left(-\sinh^2 t \left| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix} \right.\right) \\ &\quad = G_{3,3}^{3,1}\left(\frac{-1}{\sinh^2 t} \left| \begin{matrix} 1-l-\frac{n}{2} & \frac{1}{2}-l-\frac{n}{2} & \frac{1}{2} \\ -l & -\frac{l+n}{2} & \frac{1}{2}-\frac{n+l}{2} \end{matrix} \right.\right) \\ &\quad \sim O(|\sinh t|^{-1+n+l}) \qquad \text{when } t \to \infty, \end{aligned}$$

we see that

$$\int_0^\infty \int_0^\infty |\hat{\psi}_l(\nu)| \left| G_{3,3}^{1,3}\left(-\sinh^2 t \left| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix} \right.\right)\right| d\nu dt < \infty.$$

Hence, we can use Fubini's theorem to obtain

$$T_{2\lambda}(\psi) = c_{n,l} \int_0^\infty \hat{\psi}_l(\nu) \times J_l(\nu) d\nu,$$

where $J_l(\nu)$ is an entire function of polynomial growth given by

$$J_l(\nu) = \int_0^\infty G_{3,3}^{1,3}\left(-\sinh^2 t \left| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix} \right.\right) D^{n+l} \cos(\nu t) dt.$$

By collecting all the terms involved in the trace formula for the operator $\pi^\Gamma(\varphi)$, we obtain the following theorem giving an explicit trace formula for the convolution operator associated to a function ψ in the algebra $I_{p,\tau_l}(G)$.

Theorem 4.2. *For* $\psi \in I_{p,\tau_l}(G)$, $0 < p < 1$ *the operator* $Tr\,\pi^\Gamma(\psi)$ *on* $L^2_{dis}(\Gamma\backslash G, \tau_l)$ *is given by the formula*

$$\sum_{j\geq 0} n_j^l \hat{\psi}(U_{\lambda_j}^l)$$

$$= vol(\Gamma\backslash G)[Z_\Gamma]\psi(e) + \sum_{\{\gamma\}\in C\Gamma_s\backslash[Z_\Gamma]} vol(\Gamma_\gamma\backslash G_\gamma)\int_{G\backslash G_\gamma}\psi(x\gamma x^{-1})d(G\backslash G_\gamma)$$

$$+ k_1\int_{-\infty}^{+\infty}\hat{\psi}_l(v)dv + \frac{1}{4}(r - tr\ M_l(0))\hat{\psi}_l(0) - \frac{r}{2\pi}\int_{-\infty}^{+\infty}\hat{\psi}_l(v)\frac{\Gamma'(1+iv)}{\Gamma(1+iv)}\,dv$$

$$+ k_2\int_{-\infty}^{+\infty}\hat{\psi}_l(v)J_l(v)dv + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\hat{\psi}_l(v)\ tr\left(M_l(-iv)\times\frac{d}{ds}M_l(iv)\right)dv,$$

where $M_l(s)\in End(L^2(\Gamma_M\backslash M, \tau_l|_M))$ *is the intertwining operator of* Γ *and* $\{U_{\lambda_j}^l\}$ *the set of* τ_l*-spherical representations occurring discretely in* $L^2(\Gamma\backslash G, \tau_l)$.

REMARK 4.2. *In* [1], *for* Γ *cocompact, we have established an explicit trace formula that we have used to give the small eigenvalue for the associated Laplace operator.*

5. Selberg zeta function associated to $E_{\tau_l}^{\Gamma}$

In order to apply the trace formula of Theorem 3.2 to the study of the associated zeta function we will need the characterization of the space $\mathscr{C}^p(G,\tau_l)$, $0<p<1$ under the τ_l-spherical Fourier transform. For this we describe the result of Trombi concerning this characterization (for more details, see Trombi [12]).

5.1 Characterization of the space $\mathscr{C}^p(G,\tau_l)$, $0<p<1$

For $0<p<1$, let us consider the strip $\mathscr{F}(p,n) = \left\{v\in\mathbf{C}/|\Im v|\leq\left(\frac{2}{p}-1\right)n\right\}$ and $U(\tau_l) = \{v\in\mathbf{C}; v = ir, r\leq 0 \text{ and } c_l(v) = 0\}$. We put $U_p(\tau_l) = \mathscr{F}(p,n)\cap U(\tau_l)$. Then, for $|l|>n$ we have $U_p(\tau_l) = \{-ik/0\leq k\leq m\}$ where $m = min\left(|l|-n, \left(\frac{2}{p}-1\right)n\right)$. Also, for $w\in\hat{G}^2(\tau_l)\backslash\hat{G}^p(\tau_l)$ where $\hat{G}^p(\tau_l)$ is the set of elements w in $\hat{G}(\tau_l)$ whose matrix coefficient are L^p summable over G, let us denote by $I_n(w)$ the following set (it describes the numbers $v\in\mathbf{C}$ for which w is embedded in $U^{l,v}$)

$$I_n(w) = \{v\in\mathbf{C}/Hom_{(g,K)}(w, U^{l,v})\neq\{0\}\}.$$

REMARK 5.1 (cf [12]). *Let* $w\in\hat{G}^2$, $0<p<2$. *Then* $I_n(w)\cap\mathscr{F}(p,n)\neq 0$ *if and only if* $w\in\hat{G}^2\backslash\hat{G}^p$. *Hence, we have*

$$U_p(\tau_l) = \bigcup_{w\in\hat{G}^2(\tau_l)\backslash\hat{G}^p(\tau_l)} I_n(w).$$

Let $\mathscr{E}_p$ the linear space spanned by τ_l-Fourier coefficient of the irreducible characters involved in the decomposition of $\theta^{l,t\xi}$, $t \in W(A)$, $\xi \in U_p(\tau_l)$ where $\theta^{l,t\xi}$ is the distribution defined by the character $U^{l,t\xi}$ (note that for $w \in \hat{G}^2(\tau_l) \backslash \hat{G}^p(\tau_l)$, $w \in \mathscr{E}_p$).

We choose a basis $\mathscr{B}_p$ for the space $\mathscr{E}_p$ as follows:

$$\mathscr{B}_p = \{\theta^w; w \in \hat{G}^2(\tau_l) \backslash \hat{G}^p(\tau_l)\} \cup \{\text{linearly independent elements } \theta^{l,t\xi};$$
$$t \in W(A); \xi \in U_p(\tau_l)\}.$$

We put $\hat{C}_l^p(G) = \{\theta^{l,v}; v \in Int\, \mathscr{F}(p,n)\} \cup \{\theta^w, w \in \hat{G}^2(\tau_l) \cap \mathscr{F}(p,n)\} \cup \mathscr{B}_p$. For a function $L : \hat{C}_l^p(G) \to \mathbf{C}$ we put $L(\theta^{l,v}) = L(l,v)$. Then we define the functional space:

$$\mathscr{C}^p(C(G),\tau_l)_0$$
$$= \left\{ L : \hat{C}_l^p(G) \to \mathbf{C} \,\middle|\, \begin{array}{l} \text{i) } v \to L(l,v) \text{ is holomorphic on } Int\, \mathscr{F}(p,n) \\ \text{ii) } L(l,sv) = L(l,v) \ \forall s \in W(A) \\ \text{iii) } v_{u,\alpha}^p(L) < \infty \ \forall \alpha \in \mathbf{R} \text{ and } u \in \mathscr{S}(\mathbf{C}) \end{array} \right\},$$

where the semi-norms $v_{u,\alpha}^p$ are defined as follows. $v_{u,\alpha}^p(L) = \sup_{v \in Int\, \mathscr{F}(p,n)} |L(v:u)|(1+|v|^2)^\alpha$ for $\alpha \in \mathbf{R}$ and u in the symmetric algebra $S(\mathbf{C})$ of differential operators on $\mathbf{C}$.

Now, let $\mathscr{C}^p(C(G),\tau_l)$ be the subspace of $\mathscr{C}^p(C(G),\tau_l)_0$ consists of functions that satisfy in addition the following linear relation

$$L(l,t\xi) = \sum_{\theta \in \mathscr{B}_p} \hat{\alpha}_\theta(\theta^{l,t\xi}) L(\theta), \qquad \forall t \in W(A) \text{ and } \xi \in U_p(\tau_l) \tag{$*$}$$

where, as the elements of $\mathscr{B}_p$ are linearly independents, for each $\theta \in \mathscr{B}_p$ there exists $\alpha_\theta \in \mathscr{C}_c^\infty(G,\tau_l)$ such that $\int_G \alpha_\theta(g^{-1})\theta'(g)dg = 0$ if $\theta' \neq \theta$, $\theta' \in \mathscr{B}_p$; $\int_G \alpha_\theta(g^{-1})\theta(g)dg = 1$ and further $\int_G \alpha_\theta(g^{-1})(g)\theta_w\, dg = 0$ for $w \in \hat{G}^p(\tau_l)$.

We endow $\mathscr{C}^p(C(G),\tau_l)$ with the topology generated by the semi-norms

$$\mu_{\alpha,u}^p(L) = v_{\alpha,u}^p(L) + \left(\sum_{w \in \hat{G}^2(\tau_l)} |L(w)|^2 \right)^{1/2}.$$

Within these notations, we have the following theorem.

Theorem 5.1 (See [12]). *The map $\mathscr{F}_l : \mathscr{C}^p(G,\tau_l) \to \mathscr{C}^p(C(G),\tau_l)$ is surjective.*

5.2 Zeta function

In this subsection we will define the logarithmic derivative of the zeta function and study its analytic continuation.

Let ε_0 be a fixed real number. Let $g \in \mathscr{C}^\infty(\mathbf{R})$ defined as follows (for $a < \varepsilon_0$).

$$g(t) = g(|t|) = \begin{cases} 0 & \text{if } |t| \le a, \\ c & \text{if } |t| \ge \varepsilon_0. \end{cases}$$

We put $\tilde{\varepsilon}_l(j) = \frac{1}{2}((-1)^{n+l+j} + 1)$ and we define the polynomial P_l as follows

$$P_l(v) = \begin{cases} 1 \quad \text{if } |l| \le n, \\ \prod_{j=1}^m (v^2 + j^2)^{\tilde{\varepsilon}_l(j)} \quad \text{if } |l| > n. \end{cases}$$

Let D_l be the differential operator on $\mathbf{R}$ whose Fourier transform is P_l.

Then, for $s \in \mathbf{C}$ we define the function ${}_l h_s$ on A by setting ${}_l h_s(a_t) = D_l(g(|t|)\ exp(n-s)|t|)$ it is clear that ${}_l h_s$ is a smooth function on A.

Let $H(r) = \int_0^\infty g'(x)\ exp(irx)dx$. From the definition of $g, g' \in \mathscr{C}_c^\infty(\mathbf{R})$ and $g'(x) = 0$ for $|x| \ge \varepsilon_0$.

Hence, by applying the classical Paley Wiener theorem we have the following lemma.

Lemma 5.1. *H is entire, moreover $\forall n \ge 1$, $m \ge 0$, there exists a constant $c_{m,n}$ such that we have the following estimates.*

$$\left|\frac{d^m H(r)}{dr^m}\right| \le \begin{bmatrix} c_{m,n}(|r|+1)^{-n} \quad \mathit{if} \;\; \Im r \ge 0, \\ c_{m,n}(|r|+1)^{-n}\ exp(\varepsilon_0|\Im r|) \quad \mathit{if} \;\; \Im r < 0. \end{bmatrix}$$

Also, direct computations gives

Lemma 5.2. *For $\Re s > 2n$, we have*

$${}_l\hat{h}_s(v) = P_l(v)\left\{\frac{H(i(s-n)-v)}{s-n+iv} + \frac{H(i(s-n)+v)}{s-n-iv}\right\}.$$

As a consequence of the above two lemmas we have the following proposition.

Proposition 5.1. *Suppose that $\Re s > 2n$, then there exist a number p, $0 < p < 1$ and a function ${}_l g_s \in \mathscr{C}^p(G, \tau_l)$ such that $\mathscr{F}_l({}_l g_s)(v) = {}_l\hat{h}_s(v)$. Therefore, ${}_l g_s$ is admissible for the trace formula and $A_l({}_l g_s) = {}_l h_s$.*

Proof. Let us consider the function ${}_l\hat{h}_s$ defined on the set $\mathscr{C}(G)$ as follows.

$${}_l\hat{h}_s = \begin{cases} P_l(v)\left\{\frac{H(i(s-n)-v)}{s-n+iv} + \frac{H(i(s-n)+v)}{s-n-iv}\right\} \quad \text{for } \theta^{l,v};\ v \in \mathscr{F}(p,n), \\ 0 \quad \text{for } \theta^{w(v)} \in G^2(\tau_l) \backslash G^p(\tau_l). \end{cases}$$

It is sufficient to show that there exist $0 < p < 1$ such that ${}_l\hat{h}_s(v) \in \mathscr{C}^p(C(G), \tau_l)$ and the surjectivity in Theorem 5 of the τ_l-spherical Fourier transform will ensure the existence of the function ${}_l g_s \in \mathscr{C}^p(G, \tau_l)$ such that $\mathscr{F}_l({}_l g_s)(v) = {}_l\hat{h}_s(v)$.

The functions $(s-n+-iv)^{-1}$ have all their derivatives bounded in a strip $|\Im v| \leq n+\varepsilon$ where $0<\varepsilon<\Re(s-2n)$ so if we take $0<p_0<1$ such that $\left(\frac{2}{p_0}-1\right)n = n+\varepsilon$ and we choose a number p such that $0<p<p_0<1$ we see that $(s-n\pm iv)^{-1}$ have all their derivatives bounded in a strip $|\Im v| \ll \left(\frac{2}{p}+1\right)n$ and also from Lemma 1 and lemma 2, the functions $P_l(v)H(i(s-n)\pm v)$ are holomorphic and rapidly decreasing functions of v, hence $v^p_{u,\alpha}({}_l\hat{h}_s) < \infty$ $\forall\alpha \in \mathbf{R}$ and $u \in \mathscr{S}(\mathbf{C})$. Also it is clear that ${}_l\hat{h}_s$ is an even function. Therefore ${}_l\hat{h}_s \in \mathscr{C}^p(C(G),\tau_l)_0$. Furthermore as $P_l(v)=0$ for all $v \in U_p(\tau_l)$, i.e. ${}_l\hat{h}_s|_{U_p(\tau_l)} = 0$. Then for $\xi \in U_p(\tau_l)$ we have

$$\sum_{\theta \in \mathscr{B}_p} \alpha_\theta(\theta^{l,\iota\xi})_l\hat{h}_s(\theta) = \sum_{w \in \hat{G}^2(\tau_l)\backslash\hat{G}^p(\tau_l)} \alpha_{\theta^w}(\theta^{l,\iota\xi})_l\hat{h}_s(\theta^w) = 0.$$

Hence, also the linear relation (*) holds for ${}_l\hat{h}_s$ and then the claim of the Proposition follows.

Hence the function ${}_l\hat{h}_s$ is admissible for the trace formula.

Applying the trace formula to ${}_lg_s$ we get

$$\sum_j n^l_j\,{}_l\hat{h}_s(\lambda_j) = vol(\Gamma\backslash G)[Z_\Gamma]_lg_s(e) + \sum_{[\gamma]\in C\Gamma_s} vol(\Gamma_\gamma\backslash G_\gamma)\int_{G_\Gamma\backslash G} {}_lg_s(x\gamma x^{-1})dx$$
$$+ k_1\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)dv + \frac{1}{4}(r - tr\ M_l(0))_l\hat{h}_s(0) - \frac{r}{2\pi}\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)\frac{\Gamma'(1+iv)}{\Gamma(1+iv)}\,dv$$
$$+ k_2\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)J_l(v)dv + \frac{1}{4\pi}\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)\ tr\left(M_l(-iv)\times\frac{d}{dv}M_l(iv)\right)dv.$$

It is known that

$$vol(\Gamma_\gamma\backslash G_\gamma)\int_{G_\Gamma\backslash G} {}_lg_s(x\gamma x^{-1})dx = \tau_l(\gamma)^{-1}l(\gamma)j(\gamma)^{-1}c(a(\gamma))_lh_s(l(\gamma)),$$

where $j(\gamma)$ is the positive integer such that $\gamma = \delta^{j(\gamma)}$ with δ primitive in Γ. $c(a(\gamma)) = \varepsilon^A(a)\xi(a(\gamma))^{-1}\prod_{\alpha\in P^+}(1-\xi_\alpha(a(\gamma))^{-1})^{-1}$, here for any $\mu \in a^*_{\mathbf{C}}$, ξ_μ stands for the character of A defined by $\xi_\mu(a) = \exp(\mu(log(a)))$, $\varepsilon^A(a)$ is the sign of $1-\xi_{\alpha_1,n+1}(a)^{-1}$.

Also, for hyperbolic elements the set $\{l(\gamma), \gamma \in C\Gamma_s\backslash\{e\}\}$ is bounded away from zero. Then when defining $g(t)$ if choose ε_0 smaller enough than all $l(\gamma)$, we have $g(l(\gamma)) = c$ for every $\gamma \in C\Gamma_s\backslash\{e\}$ and ${}_lh_s(a_t) = cP_l(i(n-s))\ exp(n-s)t$.

Definition 5.1. *We put*

$$\tilde{z}^\Gamma_l(s,g) = g(\varepsilon_0)P_l(i(n-s))\sum_{\gamma\in C\Gamma_s\backslash\{e\}} \tau_l(\gamma)^{-1}l(\gamma)j(\gamma)^{-1}c(a(\gamma))\ exp(n-s)l(\gamma).$$

The sum defining $\tilde{z}_l^{\Gamma}(s,g)$ is absolutely and uniformly convergent in any half plane $\Re s > 2n + \varepsilon$, hence it is holomorphic for $\Re s > 2n$. By replacing it in the trace formula we get

$$\begin{aligned}\tilde{z}_l^{\Gamma}(s,g) = {} & \sum_j n_j^l {}_l\hat{h}_s(\lambda_j) - vol(\Gamma\backslash G)[Z_\Gamma]_l g_s(e) \\ & - k_1 \int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)dv - \frac{1}{4}(r - tr\ M_l(0))_l\hat{h}_s(0) \\ & + \frac{r}{2\pi}\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)\frac{\Gamma'(1+iv)}{\Gamma(1+iv)}\,dv - k_2\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)J_l(v)dv \\ & - \frac{1}{4\pi}\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)\ tr\left(M_l(-iv)\times\frac{d}{dv}M_l(v)\right)dv.\end{aligned}$$

There are seven terms on the right side of the above formula. We shall call them respectively $A_1(s),\dots,A_7(s)$ and we shall study the analytic continuation of each of them separately.

For the terms $A_1(s)$ and $A_2(s)$, the proof is the same as in Wakayama [13].

For the terms $A_3(s), A_4(s), A_5(s)$ and $A_7(s)$ as their expression is almost the same as the ones for $l = 0$ (the scalar case) the proof of their analytic continuation does not differ in an essential way from the one in [6].

So except for $A_6(s)$, we only report the results of their analytic continuation with respect to s, i.e. their poles and the residues at these poles.

$$.A_1(s) = \sum_j n_j^l {}_l\hat{h}_s(\lambda_j)$$

The following lemma is needed to prove the analytic continuation for $A_1(s)$ to the whole complex line $\mathbf{C}$.

Lemma 5.3 (Wallach, [14]). *There exist $\alpha_0 > 0$ such that for every $\alpha > \alpha_0$ we have*

$$\sum_{\pi\in\hat{G}(\tau_l)} m_\Gamma(\pi)(1+|\pi(\Omega)|)^{-\alpha} < \infty.$$

Ω is the Casimir operator of G.

Thanks to the lemma above and Proposition 2, for $\Re s > 2n$ the series

$$A_1(s) = \sum_{j\ge 1} n_j^l P_l(\lambda_j)\left\{\frac{H(i(s-n)-\lambda_j)}{s-n+i\lambda_j} + \frac{H(i(s-n)+\lambda_j)}{s-n-i\lambda_j}\right\} \qquad \text{for } \Re s > 2n,$$

converge absolutely and uniformly in compact sets disjoint from $s_j^{\pm} = n \pm i\lambda_j$.

Hence $A_1(s)$ has a meromorphic continuation to the whole complex plane $\mathbf{C}$ with simple poles in $s_j^{\pm}$.

If $s_j^+ \neq s_j^-$, the residues of $A_1(s)$ at $s_j^{\pm}$ are $n_j^l P_l(\lambda_j)H(0)$.

If $s_j^+ = s_j^-$, the residues of $A_1(s)$ at $s_j^{\pm}$ are $2n_j^l P_l(\lambda_j)H(0)$.

When $P_l(\lambda_j) = 0$, we interpret that there is no pole at $s = n + i\lambda_j$.

$$.A_2(s) = -vol(\Gamma\backslash G)[Z_\Gamma]_l g_s(e),$$

where ${}_l g_s(e) = \frac{1}{2\pi}\int_{\mathbf{R}} P_l(v)\frac{H(i(s-n)+v)}{s-n-iv}\mu_l(v)dv$, by shifting to the complex plane and using rectangular contour, we apply residue theorem to get

$$ {}_l g_s(e) = i\sum_{k\geq 0}\frac{H(i(s-n)+r_k)}{s-n-ir_k}P_l(r_k)d_k \qquad \text{for } \Re s > 2n.$$

Hence $A_2(s)$ can be continued meromorphically to $\mathbf{C}$ with simple poles at $s_k = n + ir_k$ $(k \geq 0, k \in \mathbf{Z})$ and has the residues $-iH(0)P_l(r_k)d_k$ at s_k.

Where the numbers r_k $(r_k = ia_k, a_k \geq 1)$ are the poles in the upper half plane of the Plancherel measure $\mu_l(v) = [c_l(v) \times c_l(-v)]^{-1}$ and d_k the residues of μ_l at r_k (for a detailed expression for r_k and d_k see Wakayama [13]).

$$.A_3(s) = -\frac{1}{2}(r - tr\ M_l(0))P_l(0)\frac{H(i(s-n))}{s-n} \qquad \text{for } \Re s > 2n.$$

The right side defines a meromorphic continuation of $A_3(s)$ with simple poles at $s = n$ with residue $-\frac{1}{2}((r - tr\ M_l(0))P_l(0)H(0)$.

$$.A_4(s) = 0.$$

$$A_4(s) = k_1\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)dv = {}_l h_s(e) = 0.$$

$$.A_5(s) = \frac{r}{2\pi}\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)\frac{\Gamma'(1+iv)}{\Gamma(1+iv)}\,dv,$$

shifting again to the complex plane and applying the residue theorem we pick residues of the function $\frac{\Gamma'(1+iv)}{\Gamma(1+iv)}$ in the upper half plane to get

$$A_5(s) = -r\sum_{k\geq 1}P_l(ik)\frac{H(i(s-n+k))}{s-n+k} \qquad \text{for } \Re s > 2n.$$

The series in the right hand side defines a meromorphic continuation for $A_5(s)$ to $\mathbf{C}$ with simple poles at the points $s = n-k$, $k \geq 1$ with residues $-rP_l(ik)H(0)$.

$$.A_7(s) = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)\frac{\psi_l'(iv)}{\psi_l(iv)}\,dv$$

where $\psi_l(v) = \det M_l(v)$.

The function ψ_l can be written as a ratio of two entire functions P, Q both of finite order and have no zeros in common. Let $\{q_k\}_{k\geq 1}$ be the zeroes of order b_k of Q where only a finite number $\{q_1, \dots q_j\}$ lies in $\Re(s) > 0$. Then we have

$$A_7(s) = \sum_{k\geq j+1} b_k P_l(iq_k) \frac{H(i(s-n-q_k))}{s-n-q_k} \qquad \text{for } \Re s > 2n.$$

The series on the right converges absolutely, uniformly in compact sets disjoint from $\{n+q_k; k > j+1\}$ and defines a meromorphic continuation of the left side to all of $\mathbf{C}$.

The poles of $A_7(s)$ thus continued are simple and are at the points $\{n+q_k/k \geq l+1\}$. The residue at the pole $n+q_k$ is $b_k P_l(iq_k)H(0)$.

5.3 Discussion of the term $A_6(s) = \int_{-\infty}^{+\infty} {}_l\hat{h}_s(v)J_l(v)dv$

Our aim in this subsection is to show that $A_6(s) = 0$ for $\Re s > 2n$ (and then we extend the particular case when $l = 0$ and n odd considered by R. Gangolli and G. Warner in [Ga]).

$$\begin{aligned} A_6(s) &= \int_{-\infty}^{+\infty} P_l(v)\left\{\frac{H(i(s-n)-v)}{s-n+iv} + \frac{H(i(s-n)+v)}{s-n-iv}\right\} J_l(v)dv \\ &= 2\int_{-\infty}^{+\infty} P_l(v)\frac{H(i(s-n)+v)}{s-n-iv} J_l(v)dv, \end{aligned}$$

where

$$J_l(v) = \int_0^\infty G_{3,3}^{1,3}\left(-\sinh^2 t \left| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix}\right.\right) D^{n+l}\cos(vt)dt.$$

Hence, by using the expression for $D^N \cos(vt)$ we get

$$\begin{aligned} J_l(v) &= (-1)^{n+l}\frac{v^2(v^2+1)\dots(v^2+(l+n-1)^2)}{1\times 3\times\cdots\times(2l+2n-1)} \\ &\quad \times \int_0^\infty G_{3,3}^{1,3}\left(-\sinh^2 t \left| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix}\right.\right) \mathscr{Y}_v^{2l+2n+1}(t)dt, \end{aligned}$$

also with the help of the growth estimate for the spherical function $\mathscr{Y}_v^{\cdot}$, there exists $M > 0$ integer and a constant $c_M > 0$ such that

$$\begin{aligned} |J_l(v)| &\leq c_M \frac{v^2(v^2+1)\dots(v^2+(l+n-1)^2)}{1\times 3\times\cdots\times(2l+2n-1)} \times (1+v^2)^M \\ &\quad \times \int_0^\infty G_{3,3}^{1,3}\left(-\sinh^2 t \left| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix}\right.\right) e^{-(l+n)t}(1+t^2)^M dt, \end{aligned}$$

where the integral $\int_0^\infty G_{3,3}^{1,3}\left(-\sinh^2 t \middle| \begin{matrix} 1+l & 1+\frac{l+n}{2} & \frac{1+l+n}{2} \\ l+\frac{n}{2} & l+\frac{n}{2}+\frac{1}{2} & \frac{1}{2} \end{matrix}\right) e^{-(l+n)t}(1+t^2)^M dt$ is convergent.

Hence $|J_l(v)| \le c'_M \frac{v^2(v^2+1)\dots(v^2+(l+n-1)^2)}{1\times 3\times\dots\times(2l+2n-1)} \times (1+v^2)^M$.

Now let $T>0$, by using Cauchy's theorem on the contour consisting of the semi circle of radius T centered at 0 we have

$$\int_{-T}^{+T} P_l(v)\frac{H(i(s-n)+v)}{s-n-iv}J_l(v)dv = \int_{\Omega_T} P_l(v)\frac{H(i(s-n)+v)}{s-n-iv}J_l(v)dv,$$

Ω_T is the semi-circle traversed positively of length T.

Using Lemma 1 as well as the growth estimate for $J_l(v)$; we have $\left|\int_{\Omega(\Gamma)} P_l(v)\frac{H(i(s-n)+v)}{s-n-iv}J_l(v)dv\right| = O(|T|^{-N})$, where N is an integer that we choose at will. Letting $T\to\infty$, we see that $A_6(s)=0$ for $\Re s > 2n$.

As seen in [6], the manifold $\Gamma\backslash G/K$ has the homotopy type of a compact manifold M and $vol(\Gamma\backslash G)$ is a rational multiple of the Euler characteristic E of M. Also looking at the expression of id_k (cf [13]), we see that these numbers are rational with dominator depending only on (G,K) not on k and l. Hence there exist a Positive integer κ such that $i\, vol(\Gamma\backslash G)d_k = e_k\frac{E}{\kappa}$, e_k integers. We put $H(0)=\kappa$.

PROPOSITION 5.2. *We summarize the informations about the poles of $\tilde{z}_l^\Gamma$ in the following table:*

poles	*residues*	
$s=n\pm i\lambda_j$	$\kappa n_j^l P_l(\lambda_j)$	$j\ge 1;\ \lambda_j\ne 0$
$s=n$	$2\kappa n_j^l P_l(0) - \frac{1}{2}\kappa(r - tr\ M_l(0))P_l(0)$	$j\ge 1;\ \lambda_j = 0$
	$-\frac{1}{2}\kappa(r - tr\ M_l(0))P_l(0)$	$\lambda_j\ne 0;\ j\ge 1$
$s=n-k$	$-r\kappa P_l(ik)$	$k\ge 1;\ k\ne n$
$s=0$	$\kappa P_l(in)(1-r)$	
$s=2n$	$\kappa P_l(in)$	
$s=n+ir_k$	$-e_k[Z_\Gamma]EP_l(r_k)$	$k\ge 1$
$s=n+q_k$	$\kappa b_k P_l(iq_k)$	$k\ge j+1$

5.4 Functional equation

We set

$$\phi_l(t) = \kappa\, vol(\Gamma\backslash G)P_l(it)\mu_l(it),$$

$$\tilde{\theta}_l(s) = \tilde{z}_l(s) - r\kappa\frac{\Gamma'(1+s-n)}{\Gamma(1+s-n)}P_l(i(n-s)) + \sum_{k=1}^{j}\kappa\frac{b_k}{(s-n-q_k)}P_l(i(s-n)),$$

$$g_l(s) = \tilde{\theta}_l(s) + \tilde{\theta}_l(2n - s) + \phi_l(s - n) - \kappa \frac{\psi_l'(n - s)}{\psi_l(n - s)} P_l(i(n - s)).$$

Then we have:

$\tilde{\theta}_l(s)$		$\tilde{\theta}_l(2n - s)$	
poles	residues	poles	residues
$s = n \pm i\lambda_j$	$\kappa n_j^l P_l(\lambda_j)$	$n \pm i\lambda_j$	$-\kappa n_j^l P_l(\lambda_j)$
$s = n$	$\frac{-1}{2}\kappa(r - tr\ M_l(0))P_l(0)$	n	$\frac{1}{2}\kappa(r - tr\ M_l(0))P_l(0)$
$s = n;\ \lambda_j = 0$	$2\kappa n_j^l P_l(0)$ $-\frac{1}{2}\kappa(r - tr\ M_l(0))P_l(0)$	$n;\ \lambda_j = 0$	$-2\kappa n_j^l P_l(0)$ $+\frac{1}{2}\kappa(r - tr\ M_l(0))P_l(0)$
$s = 0;\ 2n$	$\kappa P_l(in)$	$0;\ 2n$	$-\kappa P_l(in)$
$s = n + ir_k$	$-e_k[Z_\Gamma]EP_l(r_k)$	$n - ir_k$	$e_k[Z_\Gamma]EP_l(r_k)$
$s = n + q_k$; $k \geq 1$	$\kappa b_k P_l(iq_k)$	$n - q_k$	$-\kappa b_k P_l(iq_k)$

The function $\tilde{\theta}_l(s) + \tilde{\theta}_l(2n - s)$ has simple poles in $\{n \pm ir_k; n \pm q_k\}$ with residues respectively $\mp e_k EP_l(r_k)$ and $\pm \kappa b_k P_l(iq_k)$.

The poles of $\phi_l(t)$ are in $s = n \pm ir_k$ with residues $\pm e_k EP_l(r_k)$ and the poles of $\frac{\psi_l'(n-s)}{\psi_l(n-s)}$ are in $s = n \pm q_k$ with residues $\pm b_k$.

Hence the function $g_l(s) = \tilde{\theta}_l(s) + \tilde{\theta}_l(2n - s) + \phi_l(s - n) - \kappa \frac{\psi_l'(n-s)}{\psi_l(n-s)} \cdot P_l(i(n - s))$ is an entire function.

Proposition 5.3.

$$\tilde{\theta}_l(s) + \tilde{\theta}_l(2n - s) + \phi_l(s - n) - \kappa \frac{\psi_l'(n - s)}{\psi_l(n - s)} P_l(i(n - s))$$

$$= -4\pi\kappa k_1 P_l(i(n - s)) - 4\pi\kappa k_2 P_l(i(n - s))J_l(s).$$

where k_1, k_2 and J_l are as defined in the former section.

Proof. First we perform the change of variable $s = n + iz$ $(z = i(n - s))$, the condition $\Re s > 2n$ is equivalent to $\Im z < -n$ and we keep the notations $\tilde{\theta}_l(z) = \tilde{\theta}_l(n + iz)$, $\phi_l(z) = \phi_l(n + iz)$ and $\frac{\psi_l'}{\psi_l}(z) = \frac{\psi_l'}{\psi_l}(n + iz)$. Then

$$\tilde{\theta}_l(z) = \kappa P_l(z) \sum_{\gamma \in C\Gamma_s} l(\gamma) j(\gamma)^{-1} c(a(\gamma)) \exp(-izl(\gamma))$$

$$- \kappa \frac{\Gamma'(1 + iz)}{\Gamma(1 + iz)} P_l(z) + \sum_{k=1}^{j} \kappa \frac{b_k}{(iz - q_k)} P_l(z).$$

The above sum is absolutely and uniformly convergent in $\Im z < -n - \delta$ $(\delta > 0)$. Also, we have

$$\tilde{\theta}_l(z) = -i \sum_{\lambda_j \in Q_l} n_j^l P_l(\lambda_j) \left\{ \frac{H(-z-\lambda_j)}{z+\lambda_j} + \frac{H(-z+\lambda_j)}{z-\lambda_j} \right\}$$
$$- vol(\Gamma \backslash G)[Z_\Gamma] \sum_{k \geq 0} \frac{H(-z+r_k)}{z-r_k} P_l(r_k) d_k$$
$$- \frac{1}{2} P_l(0)(r - tr\ M_l(0)) \frac{H(-z)}{iz} + \sum_{k \geq 0} b_k P_l(iq_k) \frac{H(-z-iq_k)}{iz-q_k}.$$

Let fix $\varepsilon > 0$ and let b be an even holomorphic function that is rapidly decreasing in the strip $\left\{ z / |\Im z| \leq \left(\frac{2}{p} - 1 \right) n + 2\varepsilon \right\}$. We consider the rectangular contour O_x in the complex z-plane with vertices $\pm x \pm iy$ and sides E_x^+, B_x^+, E_x^-; B_x^- $\left(y = \left(\frac{2}{p} - 1 \right) n + \varepsilon, x > 0, x \in Q_l \right)$. By applying the residue theorem to the function $\tilde{\theta}_l(z) b(z)$ ($b(z)$ is holomorphic) we get

$$\int_{O_x} b(z) \tilde{\theta}_l(z) dz = 2\pi i \left\{ -i\kappa \sum_{v_j \in Q_l \cap O_x} n_j^l P_l(\lambda_j)(b(\lambda_j) + b(-\lambda_j)) \right.$$
$$+ iE[Z_\Gamma] \sum_{r_k \leq (2/p-1)n} b(r_k) e_k + \frac{i}{2} \kappa (r - tr\ M_l(0)) P_l(0) b(0)$$
$$\left. - i\kappa \sum_{k \geq 1; |q_k| \leq (2/p-1)n} b_k P_l(iq_k) b(iq_k) \right\}.$$

Put $O_\infty = \lim_{x \to \infty} O_x$. Then

$$(1) \quad \int_{O_\infty} b(z) \tilde{\theta}_l(z) dz = 4\pi\kappa \sum_{\lambda_j \in Q_l} n_j^l P_l(\lambda_j) b(\lambda_j) + -2\pi E[Z_\Gamma] \sum_{r_k \leq (2/p-1)n} e_k P_l(r_k) b(r_k)$$
$$- \pi\kappa (r - tr\ M_l(0)) P_l(0) b(0) + 2\pi\kappa \sum_{k \geq 1; |q_k| \leq (2/p-1)n} b_k P_l(iq_k) b(iq_k).$$

On the other hand the evenness of b and the relation $\tilde{\theta}_l(z) - \tilde{\theta}_l(-z) = 2\tilde{\theta}_l(z) + \phi_l(iz) - \kappa \frac{\psi_l'(-iz)}{\psi_l(-iz)} P_l(z) - g_l(z)$ gives

$$\int_{O_x} b(z) \tilde{\theta}_l(z) dz = \int_{B_x^+} b(z) \tilde{\theta}_l(z) dz + \int_{B_x^-} b(z) \tilde{\theta}_l(z) dz$$
$$+ \int_{E_x^+} b(z) \tilde{\theta}_l(z) dz + \int_{E_x^-} b(z) \tilde{\theta}_l(z) dz$$
$$= 2 \int_{B_x^-} b(z) \tilde{\theta}_l(z) dz - \kappa \int_{B_x^-} b(z) \frac{\psi_l'(-iz)}{\psi_l(-iz)} P_l(z) dz + \int_{B_x^-} b(z) \phi_l(iz) dz$$
$$- \int_{B_x^-} b(z) g_l(z) dz + I_x^+ + I_x^-,$$

where $I_x^\pm = \int_{E_x^\pm} b(r) \tilde{\theta}_l(r) dr$.

We let $x \to \infty$, $\lim_{x\to\infty} I_{\pm}^x = 0$ and we put $L_y = \lim_{x\to\infty}\{-x - iy; -x + iy\}$ (the complex line $\{-x - iy; x - iy\}$ when $x \to \infty$) to obtain

$$(2) \qquad \int_{O_\infty} b(z)\tilde{\theta}_l(z)dz = 2\int_{L_{-y}} b(z)\tilde{\theta}_l(z)dz + \int_{L_{-y}} b(z)\phi_l(iz)dz - \kappa\int_{L_{-y}} b(z)\frac{\psi_l'(-iz)}{\psi_l(-iz)}P_l(z)dz - \int_{L_{-y}} b(z)g_l(z)dz.$$

Also

$$\begin{aligned}\frac{1}{2\pi}\int_{L_{-y}} b(z)\tilde{\theta}_l(z)dr &= \kappa\sum_{\gamma\in C\Gamma_s}\chi_l(\gamma)j(\gamma)^{-1}l(\gamma)c(a(\gamma))\frac{1}{2\pi}\int_{L_{-y}} b(z)P_l(z)\exp(-izl(\gamma))dz \\ &\quad - \kappa\frac{r}{2\pi}\int_{L_{-y}} b(z)\frac{\Gamma'(1+iz)}{\Gamma(1+iz)}P_l(z)dz + \kappa\sum_{k=0}^{j} b_k P_l(iq_k)b(iq_k).\end{aligned}$$

If we put $\beta_l(r) = P_l(r)b(r)$, we can check that $\beta_l \in \mathscr{C}^p(C(G),\tau_l)$ and it follows that there exists a function $f \in \mathscr{C}^p(G,\tau_l)$ such that

$$\mathscr{F}_l(f) = \beta_l \qquad \text{and} \qquad \frac{1}{2\pi}\int_{L_{-y}} b(z)P_l(z)\exp(-izl(\gamma))dr = A_l(f)(l(\gamma)).$$

Since $\phi_l(iz)$ is a tempered function and b is rapidly decreasing we use again the residue theorem to get

$$\begin{aligned}\frac{1}{2\pi}\int_{L_{-y}} b(z)\phi_l(iz)dz &= \frac{1}{2\pi}\int_{\mathbf{R}} b(z)\phi_l(iz)dz - E[Z_\Gamma]\sum_{|r_k|\le(2/p-1)n} b(r_k)e_kP_l(r_k) \\ &= 2\kappa\, vol(\Gamma\backslash G)[Z_\Gamma]f(e) - E[Z_\Gamma]\sum_{|r_k|\le(2/p-1)n} b(r_k)e_kP_l(r_k).\end{aligned}$$

Also, we have

$$\begin{aligned}-\kappa\int_{L_y}\beta_l(z)\frac{\psi_l'(-iz)}{\psi_l(-iz)}\,dz &= \kappa\int_{-\infty}^{+\infty}\beta_l(z)\frac{\psi_l'(iz)}{\psi_l(iz)}\,dz \\ &\quad + 2\pi\kappa\sum_{k\ge j+1;\,|q_k|\le(2/p-1)n} b_k\beta_l(iq_k) - 2\pi\kappa\sum_{j=1}^{j} b_k\beta_l(iq_k).\end{aligned}$$

After substituting, we compare (1) and (2) to obtain

$$\frac{1}{4\kappa\pi}\int_{-\infty}^{+\infty} b(z)g_l(z)dz = [Z_\Gamma]\ vol(\Gamma\backslash G)f(e) + \sum_{\gamma\in C\Gamma_s} l(\gamma)j(\gamma)^{-1}c(a(\gamma))A_l(f)(l(\gamma))$$

$$- \sum_{v_j\in Q_l} n_j^l\beta_l(\lambda_j)) + \frac{1}{4}(r - tr\ M_l(0))\beta_l(0)$$

$$+ \frac{1}{4\pi}\int_{-\infty}^{+\infty} \beta_l(z)\frac{\psi_l'(iz)}{\psi_l(iz)}\,dz - \frac{r}{2\pi}\int_{-\infty}^{+\infty}\frac{\Gamma'(1+iz)}{\Gamma(1+iz)}\beta_l(z)dz.$$

Now by applying the trace formula to the function f, we see that

$$\frac{1}{4\kappa\pi}\int_{-\infty}^{+\infty} b(z)g_l(z)dz = -k_1\int_{-\infty}^{+\infty}\beta_l(z)dz - k_2\int_{-\infty}^{+\infty}\beta_l(z)J_l(z)dz.$$

Since $\beta_l(z)$ can be varied over a wide class of functions, we see that $g_l(z) = -4\pi k_1\kappa P_l(z) - 4\pi k_2\kappa P_l(z)J_l(z)$ for z real and as the involved functions are entire the claim of Proposition 4 is proved.

Now we put $z_l^0(s) = \tilde{z}_l(s)(P_l(i(s-n)))^{-1}$, $\theta_l^0(s) = \tilde{\theta}_l(s)(P_l(i(s-n)))^{-1}$, and $\phi_l^0(s) = -\phi_l(s-n)(P_l(i(s-n)))^{-1} = -\kappa\ vol(\Gamma\backslash G)\mu_l(i(s-n))$. Suppose that $|l| > n$, then we have $P_l(i(s-n)) = \prod_{j=1}^m\{-(s-n)^2 + j^2\}^{\tilde{\varepsilon}_l(j)}$. Then $z_l^0, \theta_l^0(s)$ and $\phi_l^0(s)$ might have additional simple poles at $s = n\pm j$ $(j\in\{1,\dots,m\})$.

Let $r_l^\pm(j) = Res_{s=n\pm j}\ z_l^0(s)$. The functional equation

$$\theta_l^0(s) + \theta_l^0(2n-s) = \phi_l^0(s) + \kappa\frac{\psi_l'}{\psi_l}(n-s) - 4\pi\kappa k_1 - 4\pi\kappa k_2 J_l(s),$$

gives $r_l^+(j) - r_l^-(j) = d_l(j) + \kappa a_l(j)$ where $d_l(j) = Res_{s=n+j}\ \phi_l^0(s)$ and $a_l(j) = Res_{s=n+j}\ \frac{\psi_l'}{\psi_l}(n-s)$.

We further put

$$F_l(s) = \sum_{j=1}^m\left\{\frac{r_l^+(j)}{s-n-j} + \frac{r_l^-(j)}{s-n+j}\right\}^{\tilde{\varepsilon}_l(j)},$$

$$G_l(s) = \sum_{j=1}^m\left\{\frac{d_l(j)}{s-n-j} - \frac{d_l(j)}{s-n+j}\right\}^{\tilde{\varepsilon}_l(j)},$$

$$A_l(s) = \sum_{j=1}^m\left\{\frac{a_l(j)}{s-n-j} - \frac{a_l(j)}{s-n+j}\right\}^{\tilde{\varepsilon}_l(j)},$$

$z_l(s) = \tilde{z}_l^0(s) - F_l(s)$, $\theta_l(s) = \tilde{\theta}_l^0(s) - F_l(s)$, $\bar{\phi}_l(s) = \phi_l^o(s) - G_l(s)$ and $\frac{\bar{\psi}_l'}{\bar{\psi}_l}(s) = \frac{\psi_l'}{\psi_l}(s) - A_l(s)$.

If we replace κ by 4κ we have the following proposition.

PROPOSITION 5.4. *$z_l(s)$ is a meromorphic function with simple poles and integer residues. The spectral poles of $z_l(s)$ are located at $s = n \pm i\lambda_j$ with residues $4\kappa n_j^l$. Also there exists a series of other poles given in the following table:*

Poles	*residues*
$s = n$	$-2\kappa(r - tr\ M_l(0))$
$s = n + ir_k$	$-4E[Z_\Gamma]e_k = 4i\ vol(\Gamma\backslash G)[Z_\Gamma]d_k$
$s = n + q_k$	$4\kappa b_k;\ k \geq j+1$
$n - k$	$-4\kappa r;\ k \geq 1$

Also we have the following functional equation.

$$\theta_l(s) + \theta_l(2n - s) = \bar{\phi}_l(s) + 4\kappa\frac{\bar{\psi}_l'(n-s)}{\bar{\psi}_l(n-s)} + k_1' + k_2' J_l(s).$$

REMARK 5.2. *We see that the functions $z_l^\Gamma(s)$ $(\theta_l(s))$ have only simples poles with integer residues. Hence we can find meromorphic functions $Z_l^\Gamma(s)$ $(\Omega_l(s))$ defined up to a scalar such that*

$$\frac{d}{ds}(\log Z_l^\Gamma(s)) = z_l^\Gamma(s) \qquad \text{and} \qquad \frac{d}{ds}(\log \Omega_l(s)) = \theta_l(s),$$

where

$$\Omega_l(s) = Z_\Gamma(s) \times \Gamma(1 + s - n)^{-4r\kappa} \times \prod_{k=1}^{j}(s - n - q_k)^{4\kappa b_k}.$$

Now, let us denote by m_0 the even positive integer $-2\kappa(r - tr\ M_l(0))$. As Z_l^Γ is defined up to a multiplicative constant, we normalize it by requiring that $\lim_{s\to n}(s-n)^{-m_0} Z_l^\Gamma(s) = 1$, which fixes Z_l^Γ completely and we call it Selberg zeta function associated to (G, Γ, τ_l).

As an immediate consequence, we have

LEMMA 5.4. *The following functional equation holds for Ω_l:*

$$\Omega_l(2n - s) = \Omega_l(s)\bar{\psi}_l(n-s)^{4\kappa} \times \exp\left[\int_0^{s-n} \bar{\phi}_l(t)dt + k_1'\int_n^s J_l(t)dt + k_2'(s-n)\right].$$

In the following theorem we summarize all properties of the Selberg zeta function Z_l^Γ:

THEOREM 5.2. 1) *Z_l^Γ is a meromorphic function having no poles in $\Re s > 2n$ and it has the following functional equation*

$$Z_l^{\Gamma}(2n-s) = Z_l^{\Gamma}(s) \times \left(\frac{\Gamma(1-s+n)}{\Gamma(1+s-n)}\right)^{4\kappa} \times [\bar{\psi}_l(n-s)]^{4\kappa} \times \prod_{k=1}^{j}\left(\frac{s-n-q_k}{n-s-q_k}\right)^{4\kappa b_k}$$

$$\times \exp\left[\int_0^{s-n} \bar{\phi}_l(t)dt + k_1' \int_n^s J_l(t)dt + k_2'(s-n)\right].$$

2) *Let* $S_l = \{\lambda_j \mid$ *the* τ_l*-spherical representation* U^{l,λ_j} *occurs in* $L^2_{disc}(\Gamma\backslash G)\}$. Z_l^{Γ} *has spectral zeros at* $s = n \pm i\lambda_j$, $(\lambda_j \in S_l, \lambda_j \neq 0)$ *with order* $4\kappa n_j^l$. *These zeroes determine the location of* λ_j *corresponding to the* τ_l*-spherical representation* U^{l,λ_j} *occurring in* $L^2_{disc}(\Gamma\backslash G)$.

   There is also a spectral zero of order 4κ *at* $2n$.

3) *There are also a series of topological poles or zeroes (depending on the sign of* id_k*) of* Z_l^{Γ} *at* $\{n + ir_k/k \geq 1\}$ *with order* $|4e_kE|[Z_\Gamma]$, *where* $\{r_k, k \geq 1\}$ *are the poles of the Harish-Chandra measure* $c_l(r)^{-1}c_l(-r)^{-1}$ *in* $\Im r \geq 0$ *and* d_k *their order.*

4) n *may be a pole or a zero of* Z_l^{Γ} *depending on the sign of the related residue:*
   If $\lambda_j = 0 \in S_l$, *its order is* $|8\kappa n_j^l - 2\kappa(r - tr\ M_l(0))|$.
   If $\lambda_j = 0 \notin S_l$, *its order is* $|2\kappa(r - tr\ M_l(0))|$.

5) $s = 0$ *is a zero (resp. pole) of order* $|4(\kappa - e_0E)|$ *of* Z_l^{Γ} *when* $4(\kappa - e_0E)$ *is positive (resp. negative).*

6) Z_l^{Γ} *has poles at the points* $\{n - k; k \in \mathbf{Z}\}$ *of order* $4\kappa r$.

7) Z_l^{Γ} *has zeroes of order* $4\kappa b_k$ *at the points* $\{n + q_k/k \geq j+1\}$. *Where* q_k; $k \geq j+1$ *are the poles of the function* $\psi_l(s) = det\ M_l(s)$ *lying in* $\Re(s) < 0$ *and* b_k *their order.*

8) *If we put* $f_l(s) = \exp[\int_n^s(-F_l(z))dz]$ $(F_l(s) = \sum_{j=1}^m \left\{\frac{r_l^+(j)}{s-n-j} + \frac{r_l^-(j)}{s-n+j}\right\})$ *for* $\Re s > 2n$. *Since the residues* $r_l^{\pm}(j)$ *at the poles* $z = n \pm j$ $(1 \leq j \leq m)$ *of the meromorphic function* $F_l(z)$ *need not be integers,* $f_l(s)$ *is well defined only in:* $\mathbf{C}\backslash(-\infty, 0]$. *Therefore we take and fix a particular path in the half plane* $\Re s > 2n$, *when the above integral is interpreted as a contour integral. With these, the function* Z_l^{Γ} *has the following miscellaneous formula*

$$Z_l^{\Gamma}(s) = Cf_l(s) \prod_{\delta \in Prim_\Gamma} \prod_{\lambda \in L} [1 - \tau_l(a(\delta))^{-1}\xi_\lambda^{-1}(a(\delta))^{-1} \exp(-sl(\delta))]^{4\kappa m_\lambda}.$$

   Here, C *is some constant determined by our normalization of* Z_l^{Γ}, $Prim_\Gamma$ *is the set of primitive hyperbolic conjugacy classes,* L *is the semi-Lattice of linear forms in* $\mathfrak{a}_{\mathbf{C}}^*$ *of the form* $\sum_{i=1}^t m_i\alpha_i$, $\alpha_i \in P_+$ *and* m_i *being non-negative integers. For* $\lambda \in L$, m_λ *is the number of distinct t-tuples* $(m_1, \ldots, m_t)$ *of non-negative integers such that* $\lambda = \sum_{i=1}^t m_i\alpha_i$ *and* ξ_λ *is the character of* $\mathfrak{a}$ *corresponding to* λ.

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