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Developable varieties in positive characteristic

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Abstract. We find a characteristic-free algebraic condition for developability of uniruled varieties. As an application, we study developable varieties over positive characteristic fields. In particular, we generalize classification theorem of one parameter developable ruled varieties to arbitrary characteristic.

1. Introduction

In classical differential geometry, a ruled surface whose tangent planes are constant on each line is called developable ([7, Section 3.7]). We study developable varieties in the context of projective (algebraic) geometry.

There are at least two different definitions of developable varieties in algebraic geometry:

(A) The varieties with degenerate Gauss maps ([3, p. 142], [9]), and

(B) The ruled varieties whose tangent spaces are constant on each leaf ([2]).

These two definitions are equivalent in characteristic zero, at least for a suitable choice of the ruling in (B) (Remark 7.1). Recently, developable varieties of type (A) has been classified in characteristic zero ([1], [8], [9], [10]). When the characteristic is positive, (B) still implies (A), but (A) does not necessarily imply (B). In fact, we give an example of a non-ruled surface with the degenerate Gauss map (Example 7.2).

In this paper, we concentrate on the developable varieties of type (B). In the following, "developable" means developable of type (B). Our main results are a developability criterion in any characteristic, and a generalization of classification theorem of one parameter developable varieties of type (B).

If the ground field K is $\mathbf{C}$ then we have an analytic developability criterion ([2, p. 65]) and if char $K = 0$, an algebraic criterion ([8, Theorem 0.2]). (The paper [8] does not use the word 'developable'.) We establish the following algebraic developability criterion which is valid over any algebraically closed

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field. Below, a ruling is called uniruling if the projection from the incidence correspondence to the ruled variety is generically finite (Definition 2.1).

THEOREM 1.1. *Let* $B \subset \mathbf{G}(k, N)$ *be a closed variety of dimension* r, I_B *the incidence correspondence of* B *and let* $X = \bigcup_{E \in B} E \subset \mathbf{P}^N$ *be uniruled by* B.

(1) *If the uniruling is developable then for any local parameter system* $u_1, \dots, u_r$ *at* $x \in B_{\mathrm{sm}}$ *and any local basis* $v_0, \dots, v_k : V \to \mathbf{A}^{N+1}$, *there exists an open subset* $V' \subset V$ *(*x *may be outside* V'*) such that the rank of the* $(N+1) \times ((k+1) + r(k+1))$*-matrix satisfies*

$$\operatorname{rank}\left(v_i(s), \frac{\partial v_i}{\partial u_j}(s)\right)_{i,j} \leq k + r + 1 \quad (*)$$

for any point $s \in V'$.

(2) *Assume that the projection* $I_B \to X$ *is generically étale. Then, the uniruling is developable if and only if the equality holds in* $(*)$, *namely,*

$$\operatorname{rank}\left(v_i(s), \frac{\partial v_i}{\partial u_j}(s)\right)_{i,j} = k + r + 1 \quad (**).$$

REMARK 1.2. *Using the notation of Theorem 1.1,*

(1) *If* char $K = 0$ *or* dim $B = 1$ *then the projection* $I_B \to X$ *is always generically étale (Lemma 4.1). In particular, developability is equivalent to the condition* $(**)$ *for any local parameter system* $\{u_i\}$ *and any local basis* $\{v_i\}$.

(2) *If* dim $B = 2$, *the condition* $(**)$ *implies the generically étaleness, hence the developability follows (Lemma 5.1), but when* char $K > 0$, *we give examples of developable varieties without* $(**)$ *(Examples 5.2 and 6.1). If* dim $B \geq 3$, *we also construct non-developable unirulings with* $(**)$ *(Example 6.2).*

By Remark 1.2(1), we get the following classification theorem of one parameter developable ruled varieties by the analysis of focal locus.

THEOREM 1.3. *Let* $B \subset \mathbf{G}(k, N)$ *be a* 1*-dimensional closed subvariety giving developable uniruling,* I_B *the incidence correspondence of* B *with the projections* $f : I_B \to \mathbf{P}^N$ *and* $g : I_B \to B$, *and let* $X = f(I_B)$. *Define the focal subscheme* $\overline{F} \subset I_B$ *to be the locus where* rank $df <$ dim I_B. *Let* $F \subset I_B$ *be the closure of* $\overline{F} \backslash \{(x, E) \mid E$ *is a singular point of* B *or a focal leaf*$\}$, *and* $F' = f(F)$ *the focal variety. (See Definition 2.5). Then:*

(0) *For a general element* $E \in B$, $f(F \cap g^{-1}(E)) \subset F'$ *is a linear subspace of dimension* $k-1$, *and we have a rational map* $h : B \dashrightarrow \mathbf{G}(k-1, N)$, $E \mapsto f(F \cap g^{-1}(E))$.

(1) *Assume that* h *is a constant, then* F' *is a linear subspace of dimension*

$k-1$. In this case, there exists a curve $C \subset X$ which is contained in a linear subspace disjoint from F', such that X is the join variety of C and F'.

(2) *Assume that h is not a constant. Let B' be the closure of the image of h, then F' is uniruled by B'. Let $\gamma : F' \dashrightarrow \mathbf{G}(k,N)$ be the Gauss map of F', then the following conditions are equivalent:*
   (a) *$f|_F : F \to F'$ is generically étale.*
   (b) *$f|_F : F \to F'$ is birational.*
   (c) *$h : B \dashrightarrow B'$ is generically étale.*
   (d) *$h : B \dashrightarrow B'$ is birational.*
   (e) *$B = \gamma(F')$ and the diagram*

$$\begin{tikzcd} F \arrow[r, "g|_F"] \arrow[d, "f|_F"'] & B \\ F' \arrow[ur, dashed, "\gamma"'] & \end{tikzcd}$$

commutes.

It is known that if $K = \mathbf{C}$ then one parameter developable ruled varieties are osculating scrolls (use (2)) or join varieties of osculating scrolls and linear subspaces (use (1)) ([2, p. 77], Corollary 4.2). Our theorem is a generalization of this fact.

The plan of this paper is as follows. In Section 2 we introduce the notion of focal loci for families of linear subspaces and recall differential of rational functions. In Section 3 we prove our developability criterion Theorem 1.1. In Section 4 we prove the characteristic-free classification theorem of one parameter developable varieties. In Section 5 we study two parameter ruled varieties. In Section 6 we consider higher dimensional parameter. In Section 7 we study the fibers of the Gauss maps.

Notation

The base field K is an arbitrary characteristic algebraically closed field throughout this paper. Varieties are integral algebraic schemes over K. Points mean closed points. $\mathbf{G}(k,N)$ is the Grassmann manifold whose points are the k-dimensional linear subspaces in $\mathbf{P}^N$. When X is a variety, $K(X)$ is the function field of X. For a variety X and a point $x \in X$, T_xX is the Zariski tangent space $(m_x/m_x^2)^*$ at $x \in X$. If a local parameter system $u_1, \dots, u_r \in m_x$ is given then $\overline{u_1}^*, \dots, \overline{u_r}^* \in (m_x/m_x^2)^* = T_xX$ are the dual basis for $\overline{u_1}, \dots, \overline{u_r} \in (m_x/m_x^2)$. For a variety X we write the smooth locus of X by X_{sm}. For a projective variety $X \subset \mathbf{P}^N$ and a point $p \in X_{\mathrm{sm}}$, $\mathbf{T}_pX \subset \mathbf{P}^N$ is the projective embedded tangent space of X at $p \in X$. $[v] \in \mathbf{P}^N$ denotes the point of $\mathbf{P}^N$ corresponding to the equivalence class of $v \in \mathbf{A}^{N+1} \backslash 0$. Given a linear subspace $V \subset \mathbf{A}^{N+1}$, $\mathbf{P}(V) \subset \mathbf{P}^N$ means the linear subspace of $\mathbf{P}^N$ corresponding to V.

2. Preliminaries

Let $B \subset \mathbf{G}(k, N)$ be a closed subvariety of dimension r, and $I_B = \{(x, E) \in \mathbf{P}^N \times B \mid x \in E\}$ with the natural projections:

$$\begin{array}{ccccc} B & \xleftarrow{p_1} & B \times \mathbf{P}^N & \xrightarrow{p_2} & \mathbf{P}^N \\ & & \cup & & \\ B & \xleftarrow{g} & I_B & \xrightarrow{f} & \mathbf{P}^N. \end{array}$$

We call I_B the incidence correspondence of B. Let $X := f(I_B) = \bigcup_{E \in B} E \subset \mathbf{P}^N$. Then X is a closed variety of dimension $\leq k + r$.

DEFINITION 2.1. *We say that X is* uniruled *by B, or B determines the* uniruling *of X if* $\dim X = k + r$. *Then we call B the* base *and an element $E \in B$ a* leaf. *We say that the uniruling is* developable *if $\mathbf{T}_p X = \mathbf{T}_q X$ for any $E \in B$ and any points $p, q \in E \cap X_{\mathrm{sm}}$.*

We can identify this Grassmannian with the Grassmannian of linear subspaces of dimension $k + 1$ of $\mathbf{A}^{N+1}$. We construct another incidence correspondence $\widehat{I_B} \subset B \times \mathbf{A}^{N+1}$ and the projections as follows:

$$\begin{array}{ccccc} B & \xleftarrow{q_1} & B \times \mathbf{A}^{N+1} & \xrightarrow{q_2} & \mathbf{A}^{N+1} \\ & & \cup & & \\ B & \xleftarrow{g'} & \widehat{I_B} & \xrightarrow{f'} & \mathbf{A}^{N+1}. \end{array}$$

Let $\hat{X} \subset \mathbf{A}^{N+1}$ be the affine cone corresponding to X. Then $\hat{X} = f'(\widehat{I_B})$.

DEFINITION 2.2. *We say that a family $v_0, \dots, v_k : V \to \mathbf{A}^{N+1}$ of regular morphisms on some open set V of B is a* local basis for B *if the linear subspace of $\mathbf{A}^{N+1}$ generated by $v_0(s), \dots, v_k(s)$ is equal to the $(k+1)$-dimensional linear subspace given by s for any $s \in V$.*

REMARK 2.3. *We can always take a local basis for any base $B \subset \mathbf{G}(k, N)$. To explain this, we recall some basic properties of $\mathbf{G}(k, N)$ ([6]). Let $\Lambda := \{I \subset \{0, \dots, N\} \mid \#I = k + 1\}$ where $\#I$ is the cardinality of I. Let $\mathbf{G}(k, N)$ be embedded in $\mathbf{P}^M$ where $M = \binom{N+1}{k+1} - 1$ by Plücker embedding and $(\cdots : p_I : \cdots)$ be its coordinate. For each $I \in \Lambda$, $U_I = \mathbf{G}(k, N) \cap \{(\cdots : p_J : \cdots) \in \mathbf{P}^M \mid p_I \neq 0\}$ is isomorphic to the affine space $\mathbf{A}^{(N-k)(k+1)}$. Hence $\{U_I\}_{I \in \Lambda}$ is an affine cover of $\mathbf{G}(k, N)$. On each U_I, we can take a $(k+1)$-dimensional base $v_0, \dots, v_k$ in*

$\mathbf{A}^{N+1}$ *represented by coordinate functions on* U_I. *For example, if* $I = \{0, \ldots, k\}$ *then we can write*

$$v_0 = (1 \quad 0 \quad \cdots \quad 0 \quad p_{k+1,1,\ldots,k} \quad \cdots \quad p_{N,1,\ldots,k})$$
$$\vdots$$
$$v_k = (0 \quad 0 \quad \cdots \quad 1 \quad p_{0,\ldots,k-1,k+1} \quad \cdots \quad p_{0,\ldots,k-1,N}).$$

Hence, for any variety $B \subset \mathbf{G}(k, N)$, *by restricting these to* $B \cap U_I$, *we have a local basis* $v_0, \ldots, v_k : B \cap U_I \to \mathbf{A}^{N+1}$. *We call this the* standard local basis for B.

Remark 2.4. *We assume* $V \subset U_I$ *for some* $I \in \Lambda$. *If we have a local basis* $v_0, \ldots, v_k : V \to \mathbf{A}^{N+1}$, *then we have natural isomorphisms* $V \times \mathbf{A}^{k+1} \to (g')^{-1}(V) : (s, t_0, \ldots, t_k) \mapsto (s, \sum_{i=0}^{k} t_i v_i(s))$ *and* $V \times \mathbf{P}^k \to g^{-1}(V) : (s) \times (t_0 : \cdots : t_k) \mapsto (s, [\sum_{i=0}^{k} t_i v_i(s)])$. *We call these morphisms local trivializations.*

Let $U = g^{-1}(B_{\mathrm{sm}})$. This is the smooth locus of I_B. Let $W = (g')^{-1}(B_{\mathrm{sm}})$. This is the smooth locus of $\widehat{I_B}$.

Definition 2.5. *Let* $\mathcal{T}_U$ *and* $\mathcal{T}_{\mathbf{P}^N}$ *be the tangent sheaves. The morphism* $df : \mathcal{T}_U \to f^*(\mathcal{T}_{\mathbf{P}^N})$ *of sheaves on* U *is induced from* $f|_U : U \to \mathbf{P}^N$. *The condition*

$$\operatorname{rank} df_{(E,x)} < k + r = \dim I_B$$

defines a closed subscheme of U. *This is called the* focal subscheme *of* B. *And the closure of the image of the focal subscheme under* f *in* X *is called the* focal locus *of* B.

We call an element $E \in B$ *a* focal leaf *if* $g^{-1}(E)$ *is contained in the focal subscheme in* I_B. *If* $f : I_B \to X$ *is generically étale then there exists a nonempty open set* $V \subset B_{\mathrm{sm}}$ *such that each element* $E \in V$ *is not a focal leaf. Then, the closure of* $f(F \cap g^{-1}(V)) \subset X$ *is called the* focal variety *of* B.

Let $\mathcal{T}_W$ *and* $\mathcal{T}_{\mathbf{A}^{N+1}}$ *be the tangent sheaves. The morphism of sheaves on* W $df' : \mathcal{T}_W \to (f')^*(\mathcal{T}_{\mathbf{A}^{N+1}})$ *is induced from* $f'|_W : W \to \mathbf{A}^{N+1}$. *The condition*

$$\operatorname{rank} df'_{(E,x)} < k + r + 1 = \dim \widehat{I_B}$$

defines a closed subscheme of W. *This is called the* affine focal subscheme *of* B.

Remark 2.6. *The focal subscheme* F *and the affine focal subscheme* $\hat{F}$ *is given by locally same equations, more precisely, for each* $s \in B_{\mathrm{sm}}$ *there exists an affine open neighbourhood* $V = \mathrm{Spec} R \subset B$ *at* s *such that the defining ideal of* $F \cap g^{-1}(V) \subset g^{-1}(V) \cong V \times \mathbf{P}^k \cong \mathrm{Proj} R[x_0, \ldots, x_k]$ *is equal to the defining ideal of* $\hat{F} \cap (g')^{-1}(V) \subset (g')^{-1}(V) \cong V \times \mathbf{A}^{k+1} \cong \mathrm{Spec} R[x_0, \ldots, x_k]$.

To describe the focal subscheme, we recall the notion of differential of rational functions.

LEMMA 2.7. *Let B be a variety of dimension r and $u_1, \dots, u_r \in \mathcal{O}_{B,x}$ a local parameter system at a point $x \in B_{\mathrm{sm}}$. Then, for each j there exists a unique K-derivation $\dfrac{\partial}{\partial u_j} : K(B) \to K(B)$ such that $\dfrac{\partial}{\partial u_j}(u_i) = \delta_{i,j}$ where $\delta_{i,j}$ is Kronecker's delta. Furthermore, they have the following usual properties:*

(a) $\dfrac{\partial}{\partial u_j}(\mathcal{O}_{B,x}) \subset \mathcal{O}_{B,x}$.

(b) *If $\{s_1, \dots, s_r\}$ is another local parameter system at x then*

$$\frac{\partial}{\partial u_j} = \sum_{i=1}^{r} \frac{\partial}{\partial s_i} \frac{\partial s_i}{\partial u_j}$$

as K-derivations from $K(B)$ to itself.

(c) *There exists a nonempty open set $V \subset B$ such that for any point $y \in V$, the set of functions $u_1 - u_1(y), \dots, u_r - u_r(y) \in m_y$ is also a local parameter system at $y \in V$. If y is a point in this open set then $\dfrac{\partial}{\partial u_j} = \dfrac{\partial}{\partial(u_j - u_j(y))}$.*

(d) *Let $f \in \mathcal{O}_{B,x}$ and $d_x f$ the homomorphism between tangent spaces $T_x B \to T_{f(x)}\mathbf{A}^1 = \mathbf{A}^1$ induced from the local function $f : U \to \mathbf{A}^1$. Then, $d_x f(\overline{u_j}^*) = \dfrac{\partial f}{\partial u_j}(x) \in K$.*

PROOF. Existence and uniqueness of $\dfrac{\partial}{\partial u_j}$ are proven by separability of the field extension $K(B)/K(u_1, \dots, u_r)$. If $\hat{\mathcal{O}}$ is the completion of $\mathcal{O}_{B,x}$ by m_x then we have $K(B) \cap \hat{\mathcal{O}} = \mathcal{O}_{B,x}$ in the quotient field of $\hat{\mathcal{O}}$. This implies (a). (b), (c), (d) are easily proven. □

Let $B \subset \mathbf{G}(k, N)$ be a closed variety of dimension r, $v_0, \dots, v_k : V \to \mathbf{A}^{N+1}$ a local basis for B, and let $\Phi : V \times \mathbf{A}^{k+1} \to \mathbf{A}^{N+1}$ be the composition of the local trivialization map in the sense of Remark 2.4 and the natural projection $f' : \widehat{I_B} \to \mathbf{A}^{N+1}$.

DEFINITION 2.8. *Let $\hat{X} = f'(\widehat{I_B})$. We call Φ an* adapted parameterization *of $\hat{X}$.*

LEMMA 2.9. *Let $T_0, \dots, T_k$ be coordinates on $\mathbf{A}^{k+1}$. For any local parameter system $u_1, \dots, u_r$ at any point $s \in V_{\mathrm{sm}}$ and any $(t_0, \dots, t_k) \in \mathbf{A}^{k+1}$, the functions $u_1, \dots, u_r$, $T_0 - t_0, \dots, T_k - t_k$ form a system of local parameter at $(s, t_0, \dots, t_k)$. Let $d_{(s,t)}\Phi : T_{(s,t)}(B \times \mathbf{A}^{k+1}) \to T_{\Phi(s,t)}\mathbf{A}^{N+1} = \mathbf{A}^{N+1}$ be the homo-*

morphism between tangent spaces induced from $\Phi : V \times \mathbf{A}^{k+1} \to \mathbf{A}^{N+1}$. *Then,*

$$d_{(s,t)}\Phi(\overline{u_i}^*) = \sum_{l=0}^{k} t_l \frac{\partial v_l}{\partial u_i}(s), \qquad i = 1, \dots, r$$

$$d_{(s,t)}\Phi(\overline{T_j - t_j}^*) = v_j(s), \qquad j = 0, \dots, k.$$

Proof. Let $v_l = (v_l^0, \dots, v_l^N)$ for $l = 0, \dots, k$ and $Y_0, \dots, Y_N$ be coordinates of $\mathbf{A}^{N+1}$. We have

$$d_{(s,t)}\Phi(\overline{u_1}^*)(Y_0 - (t_0 v_0^0(s) + \cdots + t_k v_k^0(s))) = \sum_{l=0}^{k} t_l d_s v_l(\overline{u_1}^*)(Y_0 - v_l^0(s)),$$

$$d_{(s,t)}\Phi(\overline{T_j - t_j}^*)(Y_0 - (t_0 v_0^0(s) + \cdots + t_k v_k^0(s))) = v_j^0(s).$$

By Lemma 2.7 (d), we have the result. □

3. Developability criterion

In this section we prove Theorem 1.1. First we note the rank conditions in Theorem 1.1.

Remark 3.1. *The rank conditions in Theorem 1.1 do not depend on choices of local parameter systems nor local bases. The reasons are as follows:*

(1) *If we have two local parameter system* $u_1, \dots, u_r$ *and* $s_1, \dots, s_r$ *at some point* P *and a local basis* $v_0, \dots, v_k$ *on some open set, then we have*

$$\langle \{v_i, \partial v_i/\partial u_j \mid i, j\} \rangle = \langle \{v_i, \partial v_i/\partial s_j \mid i, j\} \rangle$$

near P.

(2) *If we have a local parameter system* $u_1, \dots, u_r$ *and two local bases* $v_0, \dots, v_k$ *and* $w_0, \dots, w_k$ *on some open sets, then we have*

$$\langle \{v_i, \partial v_i/\partial u_j \mid i, j\} \rangle = \langle \{w_i, \partial w_i/\partial u_j \mid i, j\} \rangle$$

on some nonempty open subset of V.

Now, we prove the developability criterion.

Proof of Theorem 1.1. (1). Let $\hat{X}$ be the affine cone of X, $v_0, \dots, v_k : V \to \mathbf{A}^{N+1}$ a local basis for B and let $\Phi : V \times \mathbf{A}^{k+1} \to \hat{X}$ be the adapted parameterization of $\hat{X}$. There exists an open subset $V' \subset V$ such that for any $s \in V'$ there exists a point $t \in \mathbf{A}^{k+1}$ such that $\Phi(s,t) \in \hat{X}_{\mathrm{sm}}$. Let $u_1, \dots, u_r$ be a local parameter system at $s_0 \in V_{\mathrm{sm}}$. We take an open subset $V'' \subset V'$ on which $\partial v_i/\partial u_j$ can be defined for any i, j. Let $s \in V''$. Let $q_i(s) = (s, 0, \dots, 0, 1, 0, \dots, 0) \in V \times \mathbf{A}^{k+1}$ whose 1 is at the $(i+2)$-nd place for

$i = 0, \dots, k$. By developability, we have a unique tangent space on a linear subspace corresponding to s. We write it $T_s \subset \mathbf{A}^{N+1}$. Let $T_{s,i} = \left\langle v_0(s), \dots, v_k(s), \frac{\partial v_i}{\partial u_1}(s), \dots, \frac{\partial v_i}{\partial u_r}(s) \right\rangle$ for $i = 0, \dots, k$. We have only to prove $T_{s,0} + \cdots + T_{s,k} \subset T_s$.

First, we consider the case that $\Phi(q_i(s)) \in \hat{X}_{\mathrm{sm}}$ for any i. Then $T_{s,i} = \mathrm{Im}(d_{q_i(s)}\Phi) \subset T_s$. Hence $T_{s,0} + \cdots + T_{s,k} \subset T_s$.

Next, we consider the general case. We can take another local basis $w_0, \dots, w_k : V'' \to \mathbf{A}^{N+1}$ with the adapted parameterization $\Psi : V'' \times \mathbf{A}^{k+1} \to \mathbf{A}^{N+1}$ satisfying that $\Psi(q_i(s)) \in \hat{X}_{\mathrm{sm}}$, as follows. Let $t = (t_0, \dots, t_k) \in \mathbf{A}^{k+1}$ such that $\Phi(s, t) \in \hat{X}_{\mathrm{sm}}$. Then there exist $\varepsilon_0, \dots, \varepsilon_k \in K \backslash 0$ such that $\Phi(p_i(s)) \in \hat{X}_{\mathrm{sm}}$, where $p_i(s) = (s, t_0, \dots, \overbrace{t_i + \varepsilon_i}^{(i+2)\text{-nd}}, \dots, t_k)$, $i = 0, \dots, k$. Let A be the matrix

$$A := \begin{pmatrix} t_0 + \varepsilon_0 & t_1 & \cdots & t_k \\ t_0 & t_1 + \varepsilon_1 & \cdots & t_k \\ \vdots & \vdots & \ddots & \vdots \\ t_0 & t_1 & \cdots & t_k + \varepsilon_k \end{pmatrix}.$$

Then the matrix A maps $(0, \dots, 1, \dots, 0)$ to $(t_0, \dots, t_i + \varepsilon_i, \dots, t_k)$ for any i. Let a_{ij} be the (i, j) element of A and $w_0 = \sum_{j=0}^{k} a_{0j} v_j, \dots, w_k = \sum_{j=0}^{k} a_{kj} v_j$. Then $w_0, \dots, w_k$ is a local basis on V'' which we required. Now $\Psi(q_i(s)) = \Phi(p_i(s))$ for any i. Then we have

$$\sum_{i=0}^{k} T_{s,i} = \left\langle w_0(s), \dots, w_k(s), \frac{\partial w_0}{\partial u_1}(s), \dots, \frac{\partial w_0}{\partial u_r}(s), \dots, \frac{\partial w_k}{\partial u_1}(s), \dots, \frac{\partial w_k}{\partial u_r}(s) \right\rangle.$$

We apply the fact in the particular case to $w_0, \dots, w_k$, so that we have $\sum_{i=0}^{k} T_{s,i} \subset T_s$. We complete the proof of (1).

(2) ($\Rightarrow$) Follows from (1) and the assumption. ($\Leftarrow$) Let $u_1, \dots, u_r$ be a local parameter system and $v_0, \dots, v_k : V \to \mathbf{A}^{N+1}$ a local basis. For any $s \in V$, let

$$T_s = \left\langle v_0(s), \dots, v_k(s), \frac{\partial v_0}{\partial u_1}(s), \dots, \frac{\partial v_0}{\partial u_r}(s), \dots, \frac{\partial v_k}{\partial u_1}(s), \dots, \frac{\partial v_k}{\partial u_r}(s) \right\rangle.$$

For the adapted parameterization $\Phi : V \times \mathbf{A}^{k+1} \to \hat{X}$ induced by the local basis, by Lemma 2.9, $\mathrm{Im}(d_{(s,t)}\Phi) \subset T_s$ for any $(s, t) \in V_{\mathrm{sm}} \times \mathbf{A}^{k+1}$. By generically étaleness of $I_B \to X$, we have an open subset $V' \subset V$ such that for any $s \in V'$ there exists $t \in \mathbf{A}^{k+1}$ such that $\dim \mathrm{Im}(d_{(s,t)}\Phi) = k + r + 1$. Fix $s \in V'$. The set of points $(t_0, \dots, t_k)$ such that $\dim \mathrm{Im}(d_{(s, t_0, \dots, t_k)}\Phi) = k + r + 1$ is open in $s \times \mathbf{A}^{k+1} \simeq \mathbf{A}^{k+1}$. If $\dim \mathrm{Im}(d_{(s,t)}\Phi) = k + r + 1$ then $\mathrm{Im}(d_{(s,t)}\Phi) = T_s$. We see projective tangent spaces are constant on each element of $V' \subset B$, hence the uniruling is developable. $\square$

4. One parameter developable ruled varieties

In this section, we describe the focal subscheme by using the standard local basis when the dimension of a base $B \subset \mathbf{G}(k, N)$ is one, and classify one parameter developable uniruled varieties.

Let $B \subset \mathbf{G}(k, N)$ be a closed subvariety of dimension 1, I_B the incidence correspondence of B with the projections $f : I_B \to \mathbf{P}^N$ and $g : I_B \to B$, and let $X = f(I_B)$ be uniruled by B. Let $I = \{0, \dots, k\}$. We can assume $B \cap U_I \neq \emptyset$. We take the standard local basis $v_0, \dots, v_k : B \cap U_I \to \mathbf{A}^{N+1}$. Let u be a local parameter at some smooth point of B. Let $V \subset B$ be an open set that dv_i/du can be defined for all $i = 0, \dots, k$. Then l-th coordinate of dv_i/du is 0 if $0 \leq l \leq k$. Hence, the focal subscheme of B is locally isomorphic to the closed subscheme of $V \times \mathbf{P}^k$ given by

$$\sum_{i=0}^{k} t_i \frac{dv_i}{du}(s) = 0.$$

(The isomorphism is given by the local trivialization map in the sense of Remark 2.4.)

Lemma 4.1. *Let $B \subset \mathbf{G}(k, N)$ be a closed subvariety of dimension* 1 *and I_B the incidence correspondence of B with the projections. Then the focal subscheme of B is a proper closed subscheme of the smooth locus of I_B.*

Proof. We have only to prove that $dv_i/du \neq 0$ for some local parameter u and some i. But at a smooth point of B, one can choose one of the coordinate function minus constant as a local parameter, then we have $v_i = (0, \dots, 1, \dots, 0, \dots, u + c, \dots)$ for some i with $c \in K$. This implies $dv_i/du \neq 0$. $\square$

Proof of Theorem 1.3. (0). Now we describe $h : B \dashrightarrow \mathbf{G}(k-1, N)$. We can assume that $\dfrac{dv_k}{du} \neq 0$. When B gives developable uniruling, by Theorem 1.1, there are functions $\lambda_0, \dots, \lambda_{k-1}$ in the function field $K(B)$ such that $\dfrac{dv_i}{du} = \lambda_i \dfrac{dv_k}{du}$ for $i = 0, \dots, k-1$. Hence the focal subscheme is locally given by $t_k + \sum_{i=0}^{k-1} t_i \lambda_i = 0$. Moreover,

$$f(F \cap g^{-1}(E)) = \mathbf{P}(\langle \{ v_i(E) - \lambda_i(E) v_k(E) \mid i = 0, \dots, k-1 \} \rangle).$$

(1) We can take C as the image of the morphism $V \to \mathbf{P}^N$, $s \mapsto [v_k(s)]$.

(2) First, we will prove that $(a) \Rightarrow (e) \Rightarrow (c) \Rightarrow (a)$. We have the natural birational morphism $\phi : V \times \mathbf{P}^{k-1} \to F$, $(s) \times (t_0 : \cdots : t_{k-1}) \mapsto (s) \times [\sum_{i=0}^{k-1} t_i (v_i(s) - \lambda_i(s) v_k(s))]$. The composition of $\phi : V \times \mathbf{P}^{k-1} \to F$ and $f : F \to F'$ is given by $(s) \times (t_0 : \cdots : t_{k-1}) \mapsto [\sum_{i=0}^{k-1} t_i (v_i(s) - \lambda_i(s) v_k(s))]$. We consider the affine lifting of this, $\widehat{f \circ \phi} : V \times \mathbf{A}^k \to \mathbf{A}^{N+1}$, $(s, t_0, \dots, t_{k-1}) \mapsto$

$\sum_{i=0}^{k-1} t_i(v_i(s) - \lambda_i(s)v_k(s))$. The image of the differential of this morphism at $(s, t_0, \dots, t_{k-1})$ is $\left\langle v_0(s) - \lambda_0(s)v_k(s), \dots, v_{k-1}(s) - \lambda_{k-1}(s)v_k(s), \frac{d}{du}(\sum_{i=0}^{k-1} t_i(v_i(s) - \lambda_i(s)v_k(s)))\right\rangle$. Here,

$$(\textstyle\sum_{i=0}^{k-1} t_i(v_i - \lambda_i v_k))' = \sum_{i=0}^{k-1} t_i(v_i' - \lambda_i' v_k - \lambda_i v_k') = -(\sum_{i=0}^{k-1} t_i\lambda_i')v_k$$

($'$ means differential by u).

When we assume (a), $\widehat{f \circ \phi}$ is generically étale, hence, $\sum_{i=0}^{k-1} t_i\lambda_i'$ is not 0 for general $(s, t_0, \dots, t_{k-1})$. We have (e).

Now, we assume (e). Let $I_{B'}$ be the incidence correspondence of B' with projections $\hat{f} : I_{B'} \to \mathbf{P}^N$ and $\hat{g} : I_{B'} \to B'$. We take suitable open sets $V \subset B$ and $V' \subset B'$ which make the following morphisms well defined:

$\phi : V \times \mathbf{P}^{k-1} \to F$, $(s, t_0, \dots, t_{k-1}) \mapsto (s, [\sum_{i=0}^{k-1} t_i(v_i(s) - \lambda_i(s)v_k(s))])$,
$H : V \times \mathbf{P}^{k-1} \to V' \times \mathbf{P}^{k-1}$, $(s, t) \mapsto (h(s), t)$,
local trivialization $\psi : V' \times \mathbf{P}^{k-1} \to I_{B'}$ by the standard local basis for B'.

We have the following commutative diagram:

$$\begin{tikzcd}
V \times \mathbf{P}^{k-1} \arrow[r, "\phi"] \arrow[dd, "H"'] \arrow[dr] & F \arrow[r, "g|_F"] \arrow[d, "f|_F"'] & B \arrow[dd, dashed, "h"] \\
& F' \arrow[ur, dashed, "\gamma"'] & \\
V' \times \mathbf{P}^{k-1} \arrow[r, "\psi"'] \arrow[ur] & I_{B'} \arrow[r, "\hat{g}"'] \arrow[u, "\hat{f}"'] & B'
\end{tikzcd}$$

This implies (c) because $\hat{g} : I_{B'} \to B'$ is generically smooth.

If (c) is true, then $H : V \times \mathbf{P}^{k-1} \to V' \times \mathbf{P}^{k-1}$; $(s, t) \mapsto (h(s), t)$ is generically étale. Therefore the composition $\hat{f} \circ \psi \circ H = f|_F \circ \phi$ is generically étale, this implies (a).

Next we prove that (e) $\Rightarrow$ (d) and (d), (e) $\Rightarrow$ (b). Now assume (e). We take a section $B' \to I_{B'}$ of $\hat{g}$ so that $w : B' \to F'$, the composition with $\hat{f}$, has its image not contained in the singular locus. We prove that $\gamma \circ w$ gives the converse of h. Let $s \in B$ be a general point. We have $(s, w \circ h(s)) \in F$, hence $s = \gamma(f(s, w \circ h(s))) = \gamma \circ w \circ h(s)$ by (e). Let $s' \in B'$ be a general point. There is $s \in B$ such that $h(s) = s'$. $(s, w(s')) \in F$, hence $s = \gamma(f(s, w(s'))) = \gamma \circ w(s')$ by (e). We have $s' = h(s) = h \circ \gamma \circ w(s')$. If we assume (d), then H is birational. On the other hand, $\hat{f} \circ \psi : V \times \mathbf{P}^{k-1} \to F'$ is birational because B' gives developable uniruling by (e). Therefore, the composition $\hat{f} \circ \psi \circ H = f|_F \circ \phi$ is birational. $\square$

We denote the l-th osculating scroll of a curve C by $Tan^{(l)}C$. We can recover the following well-known classification result in characteristic 0.

Corollary 4.2. *Let $B \subset \mathbf{G}(k, N)$ be a closed variety of dimension* 1 *giving developable uniruling, and $X = \bigcup_{E \in B} E$. If* char $K = 0$ *then there is a curve $C \subset \mathbf{P}^N$, an integer l, and a linear subspace $L \subset \mathbf{P}^N$ such that*

$$X = Tan^{(l)} C \# L$$

(where # means the join of the varieties).

Proof. We use Theorem 1.3 inductively. If we have the case (1) in Theorem 1.3 then we have nothing to prove. If we have the case (2), then all conditions (a)–(e) are true by the assumption about characteristic, mainly we use (e). Then we have $X = TanF'$ and F' is one parameter developable uniruled. We apply Theorem 1.3 to F'. Inductively, we have

$$X = Tan(Tan(\cdots Tan(C\#L)\cdots)) = Tan(Tan(\cdots(TanC)\cdots))\#L.$$

We know that $Tan(Tan(\cdots(TanC)\cdots)) = Tan^{(l)}C$ ([4, Corollary 2.3]). $\square$

Remark 4.3. *To prove our Corollary, we use mainly* (a) $\Rightarrow$ (e) *in* (2) *of Theorem 1.3, but the converse is also important. If* (a) *is false, then our theorem asserts that the focal variety is not tangent to general leaves at focal points.*

Remark 4.4. *We consider the case $k = 1$. Then, $g|_F : F \to B$ is always birational. Hence,* (e) *implies that the Gauss map on the focal variety $\gamma : F' \dashrightarrow B$ is also birational. Then, it is well known that F' is "reflexive". (See* [5] *for definition.)*

Remark 4.5. *Let the notation be as in Theorem 1.3. Then $f : I_B \to X$ is birational because this morphism is generically finite and a general fiber is one point. If X is not a linear subspace, we can prove that the focal locus is contained in the singular locus of X in the same way of the proof in* [8, Theorem 4.1].

We give a strange example that the focal variety is a linear subspace of codimension one in positive characteristic.

Example 4.6. *We set* char $K = p > 0$*, $N = 3$ and $k = 1$. Let $B \subset \mathbf{G}(1,3)$ be the closure of the image of the regular map $\mathbf{A}^1 \to \mathbf{G}(1,3) : u \mapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) = (1 : u : 0 : 0 : -u^p : -u^{p+1})$. Then the standard local basis is represented by*

$$v_0 = (1 \quad 0 \quad 0 \quad u^p)$$

$$v_1 = (0 \quad 1 \quad u \quad 0).$$

These differentials are

$$(0 \quad 0 \quad 0 \quad 0)$$

$$(0 \quad 0 \quad 1 \quad 0).$$

Therefore B gives a developable uniruling by Theorem 1.1 *and the focal variety is the closure of the image of $[v_0]$. Hence the focal variety is a line.*

5. Two parameter ruled varieties

In this section we prove that the rank condition $(**)$ in Theorem 1.1 implies generically étaleness of the projection $I_B \to X$ if the dimension of B is 2, hence $(**)$ is a sufficient condition for developability in this case. We give an example of a 2-dimensional B which gives developable uniruling but $(**)$ fails, hence $(**)$ is not a necessary condition.

LEMMA 5.1. *Let $B \subset \mathbf{G}(k, N)$ be a closed variety of dimension* 2, *I_B the incidence correspondence of B with the projections $f : I_B \to \mathbf{P}^N$ and $g : I_B \to B$, and let $X = f(I_B) \subset \mathbf{P}^N$ be uniruled by B. Assume $B \cap U_I \neq \varnothing$ where $I = \{0, \dots, k\}$. Let $v_0, \dots, v_k : B \cap U_I \to \mathbf{A}^{N+1}$ be the standard local basis for B. If there exists an open subset $V \subset B \cap U_I$ such that*

$$\operatorname{rank}\left(\frac{\partial v_0}{\partial u_1}(s) \cdots \frac{\partial v_k}{\partial u_1}(s) \frac{\partial v_0}{\partial u_2}(s) \cdots \frac{\partial v_k}{\partial u_2}(s)\right) = 2$$

for any $s \in V$ then the projection $f : I_B \to X$ is generically étale.

PROOF. We can choose a local parameter system u_1, u_2 such that some two coordinate functions are $u_1 + d_1$ and $u_2 + d_1$ $(d_1, d_2 \in K)$. Therefore, essentially, the forms of the standard local basis $v_0, \dots, v_k$ are 3 types:

$$\begin{aligned}
v_0 &= (1 \quad 0 \quad \cdots \quad 0 \quad u_1 + d_1 \quad u_2 + d_2 \quad * \quad \cdots \quad *)\\
v_1 &= (0 \quad 1 \quad \cdots \quad 0 \quad * \quad * \quad * \quad \cdots \quad *)\\
&\vdots\\
v_k &= (0 \quad 0 \quad \cdots \quad 1 \quad * \quad * \quad * \quad \cdots \quad *)\\
v_0 &= (1 \quad 0 \quad \cdots \quad 0 \quad u_1 + d_1 \quad * \quad * \quad \cdots \quad *)\\
v_1 &= (0 \quad 1 \quad \cdots \quad 0 \quad * \quad u_2 + d_2 \quad * \quad \cdots \quad *)\\
&\vdots\\
v_k &= (0 \quad 0 \quad \cdots \quad 1 \quad * \quad * \quad * \quad \cdots \quad *)
\end{aligned}$$

$$
\begin{array}{l}
v_0 = (1 \quad 0 \quad \cdots \quad 0 \quad u_1 + d_1 \quad * \quad * \quad \cdots \quad *) \\
v_1 = (0 \quad 1 \quad \cdots \quad 0 \quad u_2 + d_2 \quad * \quad * \quad \cdots \quad *) \\
\vdots \\
v_k = (0 \quad 0 \quad \cdots \quad 1 \qquad * \qquad * \quad * \quad \cdots \quad *)
\end{array}
$$

The first type satisfies $\operatorname{rank}\left(\frac{\partial v_0}{\partial u_1}\,\frac{\partial v_0}{\partial u_2}\right) = 2$. This means the differential of the adapted parameterization $\Phi : V \times \mathbf{A}^{k+1} \to \hat{X}$ is surjective at a point $(s, 1, 0, \ldots, 0)$. Next we consider the second type. If $\operatorname{rank}\left(\frac{\partial v_0}{\partial u_1}\,\frac{\partial v_0}{\partial u_2}\right) = 2$ or $\operatorname{rank}\left(\frac{\partial v_1}{\partial u_1}\,\frac{\partial v_1}{\partial u_2}\right) = 2$ then we have nothing to prove. Hence we may assume $\frac{\partial v_0}{\partial u_2} = \frac{\partial v_1}{\partial u_1} = 0$. Then $\operatorname{rank}\left(\frac{\partial v_0}{\partial u_1}\,\frac{\partial v_1}{\partial u_2}\right) = 2$. This means the differential of $\Phi : V \times \mathbf{A}^{k+1} \to \hat{X}$ is surjective at a point $(s, 1, 1, 0, \ldots, 0)$. Finally, we consider the third type. We may assume $\frac{\partial v_0}{\partial u_2} = \frac{\partial v_1}{\partial u_1} = 0$ and $\frac{\partial v_0}{\partial u_1} = \frac{\partial v_1}{\partial u_2}$. If $\operatorname{rank}\left(\frac{\partial v_1}{\partial u_2}\,\frac{\partial v_i}{\partial u_1}\right) = 2$ or $\operatorname{rank}\left(\frac{\partial v_0}{\partial u_1}\,\frac{\partial v_i}{\partial u_2}\right) = 2$ for some $i \geq 2$ then we have the result. If not, then the rank condition is not satisfied. We complete the proof. □

Example 5.2. *We set* $\operatorname{char} K = 2$ *and* u, v *coordinates on* $\mathbf{A}^2$. *Let* $B \subset \mathbf{G}(1, 4)$ *be the closure of the image of the regular morphism* $\mathbf{A}^2 \to U_I$ *such that the standard local basis is written by*

$$
\begin{array}{l}
v_0 = (1 \quad 0 \quad u \quad u^3v^2 + u^4 \quad 0) \\
v_1 = (0 \quad 1 \quad v \qquad u^2v^3 \qquad v^4).
\end{array}
$$

We have the variety $X = \bigcup_{E \in B} E \subset \mathbf{P}^4$. *We can check that* B *determines the unique uniruling, the rank condition* $(**)$ *in Theorem 1.1 is not satisfied and the projection from the incidence correspondence of* B *to* X *is not generically étale. The defining polynomial* F *of* X *is* $X_1X_2^8 + X_0^6X_1X_3^2 + X_0^2X_2^6X_4 + X_1^7X_4^2 + X_0^2X_1^3X_2^2X_4^2$. *Therefore, the Gauss map on* X *is given by*

$$
\begin{aligned}
&(\partial F/\partial X_0 : \partial F/\partial X_1 : \partial F/\partial X_2 : \partial F/\partial X_3 : \partial F/\partial X_4) \\
&\quad = (0 : X_2^8 + X_0^6X_3^2 + X_1^6X_4^2 + X_0^2X_1^2X_2^2X_4^2 : 0 : 0 : X_0^2X_2^6) \\
&\quad = (0 : s^2(su + tv)^6v^4 : 0 : 0 : s^2(su + tv)^6) \\
&\quad = (0 : v^4 : 0 : 0 : 1)
\end{aligned}
$$

at $[sv_0 + tv_1] \in X$. *Therefore the Gauss map does not depend on s, t. The uniruling is developable.*

6. Cases of higher dimensional bases

In this section we treat higher dimensional bases. We give two examples: bases giving developable uniruling without $(**)$, and non-developable uniruling with $(**)$.

EXAMPLE 6.1. *We set* $\operatorname{char} K = 2$ *and* u, v, w *coordinates on* $\mathbf{A}^3$. *Let* $B \subset \mathbf{G}(1,5)$ *be the closure of the image of the regular morphism* $\mathbf{A}^3 \to U_I$ *such that the standard local basis is written by*

$$v_0 = (1 \quad 0 \quad u \quad v \quad u^2 \quad v^2)$$

$$v_1 = (0 \quad 1 \quad w \quad w^2 \quad 0 \quad u^4).$$

We have $X = \bigcup_{E \in B} E \subset \mathbf{P}^5$. *We can check that* B *determines the unique developable uniruling, the rank condition* $(**)$ *in Theorem 1.1 is not satisfied and the projection from the incidence correspondence of* B *to* X *is not generically étale. The defining polynomial of* X *is* $F = X_0X_2^4 + X_0X_1^2X_3^2 + X_0^3X_4^2 + X_1^3X_4^2 + X_0^2X_1^2X_5$. *The Gauss map on* X *is given by*

$$\begin{aligned}
&(\partial F/\partial X_0 : \partial F/\partial X_1 : \partial F/\partial X_2 : \partial F/\partial X_3 : \partial F/\partial X_4 : \partial F/\partial X_5) \\
&\quad = (X_2^4 + X_1^2X_3^2 + X_0^2X_4^2 : X_1^2X_4^2 : 0 : 0 : 0 : X_0^2X_1^2) \\
&\quad = (v^2 : u^4 : 0 : 0 : 0 : 1)
\end{aligned}$$

at $[sv_0 + tv_1] \in X$. *The Gauss map on* X *does not depend on s, t. Hence, the uniruling is developable.*

EXAMPLE 6.2. *Let* $r \geq 3$, $\operatorname{char} K = p > 0$ *and* $u_1, \dots, u_r$ *be coordinates on* $\mathbf{A}^r$. *Let* $B \subset \mathbf{G}(k, k+r+1)$ *be the closure of the image of the regular morphism* $\mathbf{A}^r \to U_I$ *such that the standard local basis is written by*

$$\begin{aligned}
v_0 &= (1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad u_1 \quad u_2 \quad \cdots \quad u_{r-2} \quad u_{r-1} \quad 0 \quad u_r^p) \\
v_1 &= (0 \quad 1 \quad 0 \quad \cdots \quad 0 \quad u_r \quad 0 \quad \cdots \quad 0 \quad 0 \quad u_{r-1} \quad u_r^p) \\
v_2 &= (0 \quad 0 \quad 1 \quad \cdots \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \quad u_r^p) \\
&\vdots \\
v_k &= (0 \quad 0 \quad 0 \quad \cdots \quad 1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \quad u_r^p).
\end{aligned}$$

We have $X = \bigcup_{E \in B} E \subset \mathbf{P}^{k+r+1}$. *We can easily check that* B *determines a uniruling, the rank condition* $(**)$ *in Theorem 1.1 is true, and the projection from* *the incidence correspondence of B to X is not generically étale. The defining polynomial F of X is $X_0X_{k+r} - X_1X_{k+r-1}$. The Gauss map on X is given by*

$$\left(\frac{\partial F}{\partial X_0} : \cdots : \frac{\partial F}{\partial X_{k+r+1}}\right)$$
$$= (X_{k+r} : -X_{k+r-1} : \overbrace{0 : \cdots : 0}^{2\sim k+r-2} : -X_1 : X_0 : 0)$$
$$= (t_1u_r^p : -t_0u_{r-1} : 0 : \cdots : 0 : -t_1 : t_0 : 0)$$

at $(t_0 : t_1 : \cdots : t_k : t_0u_1 + t_1u_r : t_0u_2 : \cdots : t_0u_{r-1} : t_1u_{r-1} : t_0u_{r-1}^p + \cdots + t_ku_{r-1}^p) \in X$. *This map depends on t_0, t_1. Hence this uniruling is not developable. (But, X is a cone with the vertex $L : X_0 = X_1 = X_{k+r-1} = X_{k+r} = 0$, and with the induced $(k+r-2)$-dimensional space ruling, X is developable.)*

Problem 6.3. *Find a condition for developability when the projection $I_B \to X$ is not generically étale.*

7. The fibers of the Gauss maps

In characteristic 0, the generic fibers of the Gauss maps are linear subspaces ([3, Corollary 4.4.12]). This is based on the result that the contact locus is a linear subspace (see [2], [5] for definition). This implies that varieties with degenerate Gauss maps are ruled by Gauss fibers. We call this Gauss fiber ruling.

Remark 7.1. *Now we give the Gauss fiber ruling if* $\operatorname{char} K = 0$. *Let $X \subset \mathbf{P}^N$ be a closed subvariety of dimension n, $X^* \subset \mathbf{P}((\mathbf{A}^{N+1})^*)$ the dual variety and let δ be the dimension of X^*. We have the following rational maps:*

- *the Gauss map* $\gamma : X \dashrightarrow \mathbf{G}(n, N) \cong \mathbf{G}^*(N-n-1, N)$,
- $[v_0], \dots, [v_{N-n-1}] : \gamma(X) \dashrightarrow X^*$ *given by a local basis* $v_0, \dots, v_{N-n-1} : \mathbf{G}^*(N-n-1, N) \dashrightarrow (\mathbf{A}^{N+1})^*$,
- *the Gauss map* $\gamma^* : X^* \dashrightarrow \mathbf{G}^*(\delta, N)$,
- *a local basis* $w_0, \dots, w_\delta : \mathbf{G}^*(\delta, N) \dashrightarrow (\mathbf{A}^{N+1})^*$.

These give the rational maps $V_{i,j} = w_j \circ \gamma^* \circ [v_i] \circ \gamma : X \dashrightarrow (\mathbf{A}^{N+1})^*$ for $i = 0, \dots, N-n-1$, $j = 0, \dots, \delta$. The linear subspace $\langle \{V_{i,j} | j\} \rangle \subset (\mathbf{A}^{N+1})^*$ gives the equations of the contact locus of $[v_i] \in X^*$. The Gauss fiber of a general point $E \in \gamma(X)$ is the intersection of the contact loci of $[v_0(E)], \dots, [v_{N-n-1}(E)]$. Hence, let $r = \dim \gamma(X)$, *then the image of the rational map*

$$X \dashrightarrow \mathbf{G}^*(N-n+r-1, N) \cong \mathbf{G}(n-r, N)$$
$$x \mapsto \mathbf{P}(\langle \{V_{i,j}(x) \mid i, j\} \rangle)$$

gives the Gauss fiber ruling.

In positive characteristic, this is not always true. We give an example which does not admit Gauss fiber ruling.

Example 7.2. *Let* char $K = 3$. *We consider the hypersurface* X *in* $\mathbf{P}^3$ *given by* $F = XZ^6 - (Y^6 + Z^6 + W^6)W$. *For a general point* $(x : y : z : w) \in X$, *the tangent space given by* $z^6X - (y^6 + z^6 + w^6)W = 0$. *The intersection of* X *and this plane is the line* $X = W = 0$ *and the plane curve* $z^6X - (y^6 + z^6 + w^6)W = (y^2 + w^2)Z^2 - z^2(Y^2 + W^2) = 0$. *If* X *is uniruled, then there is a leaf which contains the point* $(x : y : z : w)$ *and is contained in this plane. Hence,* X *can not be uniruled.*

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