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Irrationality measure of sequences*

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ABSTRACT. The new concept of an irrationality measure of sequences is introduced in this paper by means of the related irrational sequences. The main results are two criteria characterising lower bounds for the irrationality measures of certain sequences. Applications and several examples are included.

1. Introduction

The concept of irrationality is very important in Diophantine approximations. There are several criteria for the irrationality of numbers, see for example, Erdös and Strauss [6], [7], Hančl and Rucki [14], Borwein [1], [2] or Borwein and Zhou [3]. Some interesting results concerning the Cantor series can be found in the paper of Tijdeman and Pingzhi Yuan [17]. Let us mention the book of Nishioka [16] which contains a nice survey of Mahler theory including many results on irrationality. If we want to approximate a real number by rationals then it is appropriate to introduce the so-called irrationality measure of numbers.

DEFINITION 1. Let ξ be an irrational number. Then the number

$$\limsup_{\substack{q\to\infty \\ q\in\mathbf{N}}} \log_q \left(\min_{p\in\mathbf{N}}\left|\xi - \frac{p}{q}\right|\right)^{-1}$$

is called the *irrationality measure* of the number ξ.

Let us note that for such a measure we have the following theorem.

THEOREM 1. *Any irrational number has an irrationality measure greater or equal to* 2.

The proof of Theorem 1 can be found in the book of Hardy and Wright in [15]. The result concerning the lower bound for the irrationality measure of

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the sum of infinite series which consist of terms of rational numbers is included in the paper of Duverney [4] for instance. In 1975 Erdös [5] defined irrational sequences in the following way.

Definition 2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$$

is an irrational number, then the sequence $\{a_n\}_{n=1}^{\infty}$ is called *irrational*. If $\{a_n\}_{n=1}^{\infty}$ is not an irrational sequence, then it is a *rational* sequence.

Erdös [5] also proved that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is irrational. Some generalizations and similar criteria can be found in [8], [9], [11] or [12]. To each irrational sequence $\{a_n\}_{n=1}^{\infty}$ we can associate the sums of infinite series $\left\{\sum_{n=1}^{\infty} \frac{1}{c_n a_n}, c_n \in \mathbf{N}\right\}$ which are all irrational numbers. If we want to approximate such a set by rationals then it is suitable to introduce the so-called irrationality measure of sequences in the following way.

Definition 3. Let $\{a_n\}_{n=1}^{\infty}$ be an irrational sequence. Let $\mathfrak{C}$ be the set of all sequences of positive integers, $\mathfrak{C} = \{\{c_n\}_{n=1}^{\infty}, c_n \in \mathbf{N}\}$. Then the number

$$\inf_{\{c_n\}_{n=1}^{\infty} \in \mathfrak{C}} \limsup_{\substack{q \to \infty \\ q \in \mathbf{N}}} \log_q \left(\min_{p \in \mathbf{N}} \left| \sum_{n=1}^{\infty} \frac{1}{a_n c_n} - \frac{p}{q} \right| \right)^{-1}$$

is called the *irrationality measure* of the sequence $\{a_n\}_{n=1}^{\infty}$.

Unfortunately it is impossible to find a version of Duverney's criterion (see [4]) for irrationality measure in the case of irrational sequences. We now introduce Theorem 2 and Theorem 3 which are new criteria.

2. Main result

Theorem 2. *Let ε, ε_1 and S be three positive real numbers such that*

$$\varepsilon_1 < \frac{\varepsilon}{1+\varepsilon} \tag{1}$$

and

$$S > \frac{1}{1-\varepsilon_1}. \tag{2}$$

Assume that $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are two sequences of positive integers such that* $\{a_n\}_{n=1}^{\infty}$ *is nondecreasing, and that*

$$\limsup_{n\to\infty} a_n^{1/(S+1)^n} > 1; \tag{3}$$

$$b_n = O(a_n^{\varepsilon_1}); \tag{4}$$

and for every sufficiently large positive integer n

$$a_n > n^{1+\varepsilon}. \tag{5}$$

Then the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ *is irrational and has the irrationality measure greater than or equal to* $\max(2, S(1-\varepsilon_1))$.

Theorem 3. *Let* ε *and* S *be two positive real numbers with* $S > 1$. *Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are two sequences of positive integers, such that* $\{a_n\}_{n=1}^{\infty}$ *is nondecreasing,* (3) *and* (5) *for every sufficiently large positive integer n hold, and that for every positive real number* β

$$b_n = o(a_n^{\beta}). \tag{6}$$

Then the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ *is irrational and has the irrationality measure greater than or equal to* $\max(2, S)$.

3. Proofs

Lemma 1. *Let* ε_1 *be a positive real number such that* $\varepsilon_1 < 1$. *Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are two sequences of positive integers with* $\{a_n\}_{n=1}^{\infty}$ *non-decreasing, such that*

$$b_n = O(a_n^{\varepsilon_1}) \tag{7}$$

and

$$a_n > 2^n \tag{8}$$

for every sufficiently large n. Then for every ε_2 *with* $\varepsilon_2 > \varepsilon_1$ *and sufficiently large n*

$$\sum_{j=0}^{\infty} \frac{b_{n+j}}{a_{n+j}} < \frac{1}{a_n^{1-\varepsilon_2}}. \tag{9}$$

Proof (of Lemma 1). Let n be a sufficiently large positive integer such that (8) holds. From equation (7) we obtain that there exists a positive real number K which does not depend on n and such that

$$b_n \le K a_n^{\varepsilon_1}. \tag{10}$$

Inequality (10) implies that

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{b_{n+j}}{a_{n+j}} &\le \sum_{j=0}^{\infty} \frac{K a_{n+j}^{\varepsilon_1}}{a_{n+j}} = \sum_{j=0}^{\infty} \frac{K}{a_{n+j}^{1-\varepsilon_1}} \\ &= \sum_{n \le n+j < \log_2 a_n} \frac{K}{a_{n+j}^{1-\varepsilon_1}} + \sum_{n+j \ge \log_2 a_n} \frac{K}{a_{n+j}^{1-\varepsilon_1}}. \end{aligned} \tag{11}$$

Now we will estimate the both sums on the right hand side of inequality (11). For the first sum of (11), we obtain that

$$\sum_{n \le n+j < \log_2 a_n} \frac{K}{a_{n+j}^{1-\varepsilon_1}} \le \frac{K \log_2 a_n}{a_n^{1-\varepsilon_1}}. \tag{12}$$

For the second sum of (11), inequality (8) yields

$$\begin{aligned}\sum_{n+j \ge \log_2 a_n} \frac{K}{a_{n+j}^{1-\varepsilon_1}} &\le \sum_{n+j \ge \log_2 a_n} \frac{K}{2^{(n+j)(1-\varepsilon_1)}} \\ &\le \frac{K}{a_n^{(1-\varepsilon_1)}} \sum_{j=0}^{\infty} \frac{1}{2^{j(1-\varepsilon_1)}} \le \frac{L}{a_n^{1-\varepsilon_1}}, \end{aligned} \tag{13}$$

where L is a suitable positive real constant which does not depend on n. From (11), (12) and (13) we obtain that for every ε_2 with $\varepsilon_2 > \varepsilon_1$ and for every sufficiently large positive integer n

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{b_{n+j}}{a_{n+j}} &\le \sum_{n \le n+j < \log_2 a_n} \frac{K}{a_{n+j}^{1-\varepsilon_1}} + \sum_{n+j \ge \log_2 a_n} \frac{K}{a_{n+j}^{1-\varepsilon_1}} \\ &\le \frac{K \log_2 a_n + L}{a_n^{1-\varepsilon_1}} \le \frac{1}{a_n^{1-\varepsilon_2}}. \end{aligned}$$

Thus (9) holds. The proof of Lemma 1 is complete. □

LEMMA 2. *Let S, ε_1, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy all conditions in Theorem 2. Then there exists a positive real number α such that for every sufficiently large n*

$$\sum_{j=0}^{\infty} \frac{b_{n+j}}{a_{n+j}} \le \frac{1}{a_n^{\alpha}}. \tag{14}$$

Proof (of Lemma 2). From (4) we obtain that there exists a positive real constant K, such that

$$b_n \leq K a_n^{\varepsilon_1}. \tag{15}$$

Inequality (15) implies

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{b_{n+j}}{a_{n+j}} &\leq \sum_{j=0}^{\infty} \frac{K a_{n+j}^{\varepsilon_1}}{a_{n+j}} = \sum_{j=0}^{\infty} \frac{K}{a_{n+j}^{1-\varepsilon_1}} \\ &= \sum_{n \leq n+j < a_n^{1/(1+\varepsilon)}} \frac{K}{a_{n+j}^{1-\varepsilon_1}} + \sum_{n+j \geq a_n^{1/(1+\varepsilon)}} \frac{K}{a_{n+j}^{1-\varepsilon_1}}. \end{aligned} \tag{16}$$

Now we will estimate the both sums on the right hand side of inequality (16). For the first sum, we obtain that

$$\sum_{n \leq n+j < a_n^{1/(1+\varepsilon)}} \frac{K}{a_{n+j}^{1-\varepsilon_1}} \leq \frac{K a_n^{1/(1+\varepsilon)}}{a_n^{1-\varepsilon_1}} = \frac{K}{a_n^{1-1/(1+\varepsilon)-\varepsilon_1}} = \frac{K}{a_n^{\varepsilon/(1+\varepsilon)-\varepsilon_1}}. \tag{17}$$

For the second sum, inequality (5) implies that there exist positive real constants V and R not depending on n, such that

$$\begin{aligned}\sum_{n+j \geq a_n^{1/(1+\varepsilon)}} \frac{K}{a_{n+j}^{1-\varepsilon_1}} &\leq \sum_{n+j \geq a_n^{1/(1+\varepsilon)}} \frac{K}{(n+j)^{(1+\varepsilon)(1-\varepsilon_1)}} \\ &\leq \int_{a_n^{1/(1+\varepsilon)}}^{\infty} \frac{V\,dx}{x^{(1+\varepsilon)(1-\varepsilon_1)}} \leq \frac{R}{a_n^{\varepsilon/(1+\varepsilon)-\varepsilon_1}}. \end{aligned} \tag{18}$$

From (16), (17) and (18) we obtain that

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{b_{n+j}}{a_{n+j}} &\leq \sum_{n \leq n+j < a_n^{1/(1+\varepsilon)}} \frac{K}{a_{n+j}^{1-\varepsilon_1}} + \sum_{n+j \geq a_n^{1/(1+\varepsilon)}} \frac{K}{a_{n+j}^{1-\varepsilon_1}} \\ &\leq \frac{K}{a_n^{\varepsilon/(1+\varepsilon)-\varepsilon_1}} + \frac{R}{a_n^{\varepsilon/(1+\varepsilon)-\varepsilon_1}} = \frac{K+R}{a_n^{\varepsilon/(1+\varepsilon)-\varepsilon_1}}. \end{aligned} \tag{19}$$

Let $\alpha = \frac{1}{2}\left(\frac{\varepsilon}{1+\varepsilon} - \varepsilon_1\right)$. Then inequality (1) implies that $\alpha > 0$. This and (19) yield that for every sufficiently large n

$$\sum_{j=0}^{\infty} \frac{b_{n+j}}{a_{n+j}} \leq \frac{1}{a_n^{\alpha}}.$$

Thus (14) holds. The proof of Lemma 2 is complete. □

Proof (of Theorem 2). Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then there exists a bijection $\phi : \mathbf{N} \to \mathbf{N}$ such that for every $n \in \mathbf{N}$, $A_n = a_{\phi(n)} c_{\phi(n)}$ and the sequence $\{A_n\}_{n=1}^{\infty}$ is nondrecreasing. From this description of a bijection ϕ and from the fact that $\{a_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive integers we obtain that for every $n \in \mathbf{N}$

$$A_n = a_{\phi(n)} c_{\phi(n)} \geq a_n. \tag{20}$$

Let B_n denote $b_{\phi(n)}$ for all $n \in \mathbf{N}$. This and (20) imply that the sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ satisfy all asumptions of Theorem 2 too. From this fact, (2) and Theorem 1 we obtain that to prove Theorem 2 it is sufficient to prove that for every real number S_1 with

$$S_1 > \varepsilon_1 S, \tag{21}$$

and with $S_1 - \varepsilon_1 S$ sufficiently small we have

$$\liminf_{N \to \infty} \left(\prod_{n=1}^{N-1} a_n \right)^{S - S_1} \sum_{n=N}^{\infty} \frac{b_n}{a_n} = 0. \tag{22}$$

Inequality (21) implies that there is a positive real number ε_2 with $\varepsilon_2 > \varepsilon_1$ and such that $S_1 > \varepsilon_2 S$. Let us put

$$\delta = \frac{S_1 - \varepsilon_2 S}{2}.$$

Then $\delta > 0$ and we have

$$\begin{aligned} \varepsilon_2 + \frac{S - S_1}{S - \delta} &= 1 - \frac{S_1 - \varepsilon_2 S - \delta + \varepsilon_2 \delta}{S - \delta} \\ &= 1 - \frac{\frac{S_1 - \varepsilon_2 S}{2} + \varepsilon_2 \delta}{S - \delta} < 1. \end{aligned} \tag{23}$$

Inequality (3) implies that

$$\limsup_{n \to \infty} a_n^{1/(S+1-\delta)^n} = \infty. \tag{24}$$

Let A be a sufficiently large positive real number. From (24) we obtain that there exists a positive integer n such that

$$a_n^{1/(S+1-\delta)^n} > A.$$

Assume that α is a positive real number which satisfies condition (14) in Lemma 2. Let for every positive integer $k \geq \max(a_1, 3)$, w_k denote the least positive integer such that

$$a_{w_k}^{1/(S+1-\delta)^{w_k}} > k^{2/\alpha}. \tag{25}$$

Suppose that t_k is the greatest positive integer less than w_k such that

$$a_{t_k} \le k^{t_k}. \tag{26}$$

Let v_k be the least positive integer greater than t_k such that

$$a_{v_k}^{1/(S+1-\delta)^{v_k}} > k. \tag{27}$$

From the description of sequences $\{t_k\}_{k=a_1}^{\infty}$, $\{v_k\}_{k=a_1}^{\infty}$, $\{w_k\}_{k=a_1}^{\infty}$ and from (25), (26) and (27) we obtain that

$$t_k < v_k \le w_k, \tag{28}$$

$$\lim_{k\to\infty} v_k = \infty$$

and for every positive integers r and s with $v_k \le r \le w_k$ and $t_k < s < v_k$

$$a_r > k^r > 2^r \tag{29}$$

and

$$a_s < k^{(S+1-\delta)^s}. \tag{30}$$

The fact that the sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers is nondecreasing and inequality (26) imply that

$$\prod_{n=1}^{t_k} a_n \le a_{t_k}^{t_k} \le k^{t_k^2}. \tag{31}$$

From (28) and (30) we obtain

$$\begin{aligned} k^{(S+1-\delta)^{v_k}} &\ge k^{(S-\delta)((S+1-\delta)^{v_k-1}+(S+1-\delta)^{v_k-2}+\cdots+1)} \\ &= \left(\prod_{n=1}^{v_k-1} k^{(S+1-\delta)^n}\right)^{(S-\delta)} \ge \left(\prod_{n=1}^{v_k-1} a_n\right)^{(S-\delta)} \left(\prod_{n=1}^{t_k} a_n\right)^{-(S-\delta)}. \end{aligned}$$

This fact, (28) and (31) yield

$$\begin{aligned} k^{(S+1-\delta)^{v_k}} &\ge \left(\prod_{n=1}^{v_k-1} a_n\right)^{(S-\delta)} \left(\prod_{n=1}^{t_k} a_n\right)^{-(S-\delta)} \\ &\ge \left(\prod_{n=1}^{v_k-1} a_n\right)^{(S-\delta)} k^{-t_k^2(S-\delta)} \ge \left(\prod_{n=1}^{v_k-1} a_n\right)^{(S-\delta)} k^{-v_k^2(S-\delta)}. \end{aligned} \tag{32}$$

Inequality (32) implies that

$$\left(\prod_{n=1}^{v_k-1} a_n\right)^{(S-S_1)} \le k^{((S-S_1)/(S-\delta))(S+1-\delta)^{v_k}+(S-S_1)v_k^2}. \tag{33}$$

From (27), (29) and Lemma 1 we obtain the fact that

$$\sum_{n=v_k}^{w_k} \frac{b_n}{a_n} \le \frac{1}{a_{v_k}^{1-\varepsilon_2}} \le \frac{1}{k^{(1-\varepsilon_2)(S+1-\delta)^{v_k}}}. \tag{34}$$

Inequality (25), the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is nondecreasing and Lemma 2 yield

$$\sum_{n=w_k+1}^{\infty} \frac{b_n}{a_n} \le \frac{1}{a_{w_k+1}^{\alpha}} \le \frac{1}{a_{w_k}^{\alpha}} \le \frac{1}{k^{2(S+1-\delta)^{w_k}}}. \tag{35}$$

Since $2 > 1-\varepsilon_2$ then, (28), (34) and (35) imply that for every sufficiently large k

$$\begin{aligned}\sum_{n=v_k}^{\infty} \frac{b_n}{a_n} &= \sum_{n=v_k}^{w_k} \frac{b_n}{a_n} + \sum_{n=w_k+1}^{\infty} \frac{b_n}{a_n} \\ &\le \frac{1}{k^{(1-\varepsilon_2)(S+1-\delta)^{v_k}}} + \frac{1}{k^{2(S+1-\delta)^{w_k}}} \le \frac{2}{k^{(1-\varepsilon_2)(S+1-\delta)^{v_k}}}.\end{aligned} \tag{36}$$

From (33) and (36) we obtain that for every sufficiently large v_k

$$\left(\prod_{n=1}^{v_k-1} a_n\right)^{(S-S_1)} \sum_{n=v_k}^{\infty} \frac{b_n}{a_n} \le k^{(S-S_1)v_k^2-(1-(\varepsilon_2+(S-S_1)/(S-\delta)))(S+1-\delta)^{v_k}}. \tag{37}$$

Inequality (23) yields that $1-\left(\varepsilon_2+\frac{S-S_1}{S-\delta}\right) > 0$. From this fact and (37) we obtain the fact that

$$\lim_{k\to\infty}\left(\prod_{n=1}^{v_k-1} a_n\right)^{S-S_1} \sum_{n=v_k}^{\infty} \frac{b_n}{a_n} = 0.$$

This implies (22). The proof of Theorem 2 is now complete. □

Proof (of Theorem 3). Suppose that the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ has an irrationality measure less than S. Then there exists a positive real number $S_1 < \min\left(S-1, \frac{\varepsilon}{1+\varepsilon}\right)$ such that the irrationality measure of the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ is less than $S-S_1$. Let $\varepsilon_1 = \frac{S_1}{S}$. Then (6) implies that $b_n = o(a_n^{\varepsilon_1})$. Hence $b_n = O(a_n^{\varepsilon_1})$. From this and Theorem 2 we obtain that the sequence

$\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ is irrational and has irrationality measure greater than or equal to $\max(2; S(1-\varepsilon_1))$. This is a contradiction since $S(1-\varepsilon_1) = S - S\varepsilon_1 = S - S_1$. □

4. Examples and comments

COROLLARY 1. *Let ε_1 and S be positive real numbers such that $S(1-\varepsilon_1) > 2$. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, with*

$$\limsup_{n\to\infty} a_n^{1/(S+1)^n} > 1$$

and

$$b_n = O(a_n^{\varepsilon_1}).$$

Then the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ has irrationality measure greater than or equal to $S(1-\varepsilon_1)$.

Corollary 1 is an immediate consequence of Theorem 2.

EXAMPLE 1. Corollary 1 implies that the sequence

$$\left\{\frac{3^{5^n} + n^5}{2^{5^n} + n^5}\right\}_{n=1}^{\infty}$$

has irrationality measure greater than or equal to $\frac{4(\log_2 3 - 1)}{\log_2 3}$.

EXAMPLE 2. Let K be a positive integer with $K\left(1 - \frac{1}{\log_2 e}\right) > 2$. Denote that $\mathrm{lcm}(x_1, x_2, \ldots, x_n)$ is the least common multiple of the numbers $x_1, x_2, \ldots, x_n$. Then Corollary 1 yields that irrationality measure of the sequence

$$\left\{\frac{\mathrm{lcm}(1, 2, \ldots, (K+1)^n) + n}{2^{(K+1)^n} + n^2}\right\}_{n=1}^{\infty}$$

is greater than or equal to $K\left(1 - \frac{1}{\log_2 e}\right)$.

COROLLARY 2. *Let S be a positive real number with $S > 2$. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers, that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, that*

$$\limsup_{n\to\infty} a_n^{1/(S+1)^n} > 1$$

with

$$b_n = o(a_n^{\beta})$$

for every positive real number β. *Then the sequence* $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ *has irrationality measure greater than or equal to* S.

Corollary 2 is an immediate consequence of Theorem 3.

Example 3. Corollary 2 yields that the sequence

$$\left\{\frac{3^{4^n} + 2^n}{3^{3^n} + 5^n}\right\}_{n=1}^{\infty}$$

has irrationality measure greater than or equal to 3.

Example 4. Let S be a positive real number with $S \geq 2$. Assume that $\pi(x)$ is the number of primes less than or equal to x. As an immediate consequence of Corollary 2 we obtain that the sequence

$$\{\pi((S+1)^n)! + 1\}_{n=1}^{\infty}$$

has irrational measure greater than or equal to S.

Example 5. Let K be a positive integer such that $K > 3$. Also let $[x]$ be the greatest integer less than or equal to x. Then Theorem 2 implies that the sequence

$$\left\{\frac{2^{n+3(K+1)^{2^{2^{[\log_2 \log_2 n]}}}} + n^2}{2^{1+(K+1)^{2^{2^{[\log_2 \log_2 n]}}}} + n}\right\}_{n=1}^{\infty}$$

has irrationality measure greater than or equal to $\frac{2K}{3}$.

Example 6. Let K be a positive real number such that $K > 2$. Then Theorem 3 yields that the sequence

$$\left\{\frac{2^{n+(K+1)^{2^{2^{[\log_2 \log_2 n]}}}} + n!}{2^{\pi((K+1)^{2^{2^{[\log_2 \log_2 n]}}})} + n^n}\right\}_{n=1}^{\infty}$$

has irrationality measure greater than or equal to K.

Definition 4. Let x be an irrational number. If the irrationality measure of the number x is infinity then x is called *Liouville* number. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers, the sum of the series $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ is a Liouville number, then the sequence $\{a_n\}_{n=1}^{\infty}$ is called *Liouville*.

Corollary 3. *Let ε and ε_1 be two positive real numbers with $\varepsilon_1 < \frac{\varepsilon}{1+\varepsilon}$. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers, that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, and that*

$$b_n = O(a_n^{\varepsilon_1}),$$

with, for every positive real number S,

$$\limsup_{n\to\infty} a_n^{1/S^n} = \infty$$

and for every sufficiently large positive integer n, $a_n > n^{1+\varepsilon}$. Then the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ is Liouville.

Corollary 3 is an immediate consequence of Theorem 2 and Definition 4.

Example 7. As an immediate consequence of Corollary 3 we obtain that the sequences

$$\left\{\frac{2^{2n!}+n}{2^{n!}+n!}\right\}_{n=1}^{\infty} \qquad \text{and} \qquad \left\{\frac{2^{3n^n}+1}{2^{n^n}+1}\right\}_{n=1}^{\infty}$$

are Liouville.

Corollary 4. *Let ε be a positive real number. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers. Suppose that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, and that for every positive real number β*

$$b_n = o(a_n^{\beta}).$$

Finally assume that for every positive real number S,

$$\limsup_{n\to\infty} a_n^{1/S^n} = \infty$$

and for every sufficiently large positive integer n, $a_n > n^{1+\varepsilon}$. Then the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ is Liouville.

Corollary 4 is an immediate consequence of Corollary 3.

Example 8. As an immediate consequence of Corollary 4 we obtain that the sequences

$$\left\{\frac{n^{n!}+1}{2^{n!}+1}\right\}_{n=1}^{\infty} \qquad \text{and} \qquad \left\{\frac{2^{(n+1)!}+1}{2^{n!}+1}\right\}_{n=1}^{\infty}$$

are Liouville.

REMARK 1. *In either Corollary* 3 *or Corollary* 4 *choose* $b_n = 1$ *for every* $n \in \mathbf{N}$. *Then we obtain Erdös theorem which states the following. Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive integers such that for every positive integer* S,

$$\limsup_{n\to\infty} a_n^{1/S^n} = \infty.$$

Let also ε *be a positive real number such that for every sufficiently large positive integer* n, $a_n > n^{1+\varepsilon}$. *Then the sum of the series* $\sum_{n=1}^{\infty} \frac{1}{a_n}$ *is a Liouville number. For more details see* [5] *or* [10], *for instance.*

OPEN PROBLEM. We do not know if the sequence $\{4^{4^n}\}_{n=1}^{\infty}$ has the irrationality measure greater than 3.

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