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# On the steady Oseen problem in the whole space

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**ABSTRACT.** We deal with Oseen's equations in the whole space. A class of existence, uniqueness and regularity results for both the scalar and the vectorial equations are given. Isotropic weighted Sobolev spaces are used for describing the growth or the decay of functions at infinity.

## 1. Introduction

The Oseen equations are a linearized version of the Navier-Stokes equations describing a viscous and incompressible fluid in which a small body is moving. The purpose of this paper is to study the Oseen problem in the whole space $\mathbf{R}^n$, $n \geq 2$:

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + k \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f} \qquad &\text{in } \mathbf{R}^n, \\ \operatorname{div} \boldsymbol{u} = h \qquad &\text{in } \mathbf{R}^n. \end{aligned} \tag{1.1}$$

Here the unknowns are the velocity $\boldsymbol{u}$ of the fluid and the pressure $\pi$. The data are the viscosity $\nu$ of the fluid, the external force $\boldsymbol{f}$, the function $h$ and the positive real $k$. System (1.1) was proposed by Oseen (see [19]) in order to remove some physical paradoxes of the Stokes system, which corresponds to the case $k = 0$. One of the first works devoted to these equations is due to Finn [10]. Specifically, Finn treated Oseen's equations in three dimensional exterior domains when $(1 + |x|)\boldsymbol{f}$ is square integrable and $h = 0$. He proved the existence of a solution $\boldsymbol{u}$ such that $(1 + |x|)^{-1}\boldsymbol{u}$ is square integrable. Farwig [9] proved, among other results, the existence of a solution $(\boldsymbol{u}, \pi)$ of (1.1) when $\boldsymbol{f} \in L^p(\mathbf{R}^n)^n$ and $h \in W^{1,p}(\mathbf{R}^n)$. In that case the solution $(\boldsymbol{u}, \pi)$ satisfies $\boldsymbol{u} \in L^p_{loc}(\mathbf{R}^n)^n$, $\partial_i \partial_j u_k \in L^p(\mathbf{R}^n)$, $\partial_i \pi \in L^p(\mathbf{R}^n)$, $i = 1, 2, \ldots, n$. In [11]

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Galdi stated that if $\boldsymbol{f} \in W^{m,p}(\mathbf{R}^n)^n$, $h \in W^{m+1,p}(\mathbf{R}^n)$, $m \geq 0$, then the problem (1.1) has a solution in $W_{loc}^{m+2,p}(\mathbf{R}^n) \times W_{loc}^{m+1,p}(\mathbf{R}^n)$. In [7] Farwig investigates the system (1.1), set in a three dimensional exterior domain, using anisotropic weighted $L^2$ spaces. The use of anisotropic weights seems to be a natural approach because of the anisotropy introduced by the term $\partial_1 u$. However, such an approach contains some serious technical complications. The reader can refer to [16], [20], [8], [7], [3] for existence results in anisotropic weighted spaces. To our knowledge, most of the existing results in the literature concern the case $\boldsymbol{f} \in L^p(\mathbf{R}^n)^n$ or are around that case. Several questions concerning the existence, the uniqueness and regularity of the solution remain not treated, especially when the data $\boldsymbol{f}$ and $h$ are slowly decreasing or have a polynomial behavior at infinity. Among the results we present in this paper, we shall prove that if $\boldsymbol{f} = (f_i, \dots, f_n)$ and $h$ satisfies the conditions

$$(1 + |x|^2)^{(|\mu|-m)/2} D^\mu f_k \in L^p(\mathbf{R}^n) \qquad \text{for } |\mu| \leq m,$$
$$(1 + |x|^2)^{(|\mu|-m-1)/2} D^\mu h \in L^p(\mathbf{R}^n) \qquad \text{for } |\mu| \leq m + 1,$$

for some integer $m \geq 0$, then Problem (1.1) admits a solution $(\boldsymbol{u}, \pi)$, unique up to a class of polynomials, and satisfying

$$(1 + |x|^2)^{(|\mu|-m-2)/2} D^\mu u_k \in L^p(\mathbf{R}^n) \qquad \text{for } |\mu| \leq m + 2,$$
$$(1 + |x|^2)^{(|\mu|-m-1)/2} D^\mu \pi \in L^p(\mathbf{R}^n) \qquad \text{for } |\mu| \leq m + 1.$$

In all the paper, we deal with following problem obtained from (1.1) by means of a simple scaling argument

$$\begin{aligned} -\Delta \boldsymbol{u} + 2\frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f} \qquad \text{in } \mathbf{R}^n, \\ \operatorname{div} \boldsymbol{u} = h \qquad \text{in } \mathbf{R}^n. \end{aligned} \tag{1.2}$$

We are also interested in the scalar equation

$$-\Delta u + 2\frac{\partial u}{\partial x_1} = f \qquad \text{in } \mathbf{R}^n, \tag{1.3}$$

which is intimately linked to the system (1.2). The relation between this scalar equation and the general vectorial system (1.2) as well as the relation between their fundamental solutions are discussed in Section 3 hereafter. Observe for the moment that Oseen's system (1.2) can be formally decomposed into two problems: a Laplace equation for the pressure

$$\Delta \pi = \operatorname{div} \boldsymbol{f} + \Delta h - 2\frac{\partial h}{\partial x_1}, \tag{1.4}$$

and a scalar equation on each component of the velocity

$$-\Delta u_i + 2\frac{\partial u_i}{\partial x_1} = \tilde{f}_i, \tag{1.5}$$

where $\tilde{f}_i = f_i - \dfrac{\partial \pi}{\partial x_i}$. Consequently, one must choose a functional framework which allows to solve *both* the equations (1.4) and (1.5) for several behaviors at infinity. The use of weighted Sobolev spaces turned out to be convenient for treating problems in unbounded domains, and consequently seems to be the natural framework for treating Problem (1.4) (see for instance [2]). The main difficulty here lies in the choice of the *weights* since the convective term $\dfrac{\partial \boldsymbol{u}}{\partial x_1}$ in Equation (1.5) induces an anisotropic behavior of the velocity, while the pressure keeps an isotropic behavior as in the Stokes problem. Another difficulty is due to divergence condition div $\boldsymbol{u} = h$ which complicates seriously the problem. In Section 3 we expose how the system (1.2) can be treated by solving only the scalar equation (1.3) in such a way that the divergence condition is automatically fulfilled. For all these reasons, we shall treat in a first time and independently the scalar equation (1.3). We prove that there exists at least two kinds of solutions; tempered solutions, which are tempered distributions, and quasi-tempered solutions which are not necessarily tempered distributions. Only tempered solutions seem to be useful for solving the vectorial system (1.2).

In a forthcoming paper, we will use our present results in order to solve Oseen's equations in an exterior domain.

In the sequel, we set

$$T = -\Delta + 2\frac{\partial}{\partial x_1}$$

and

$$T^* = -\Delta - 2\frac{\partial}{\partial x_1}$$

The remaining of this paper is organized as follows

- Section 2 is devoted to a brief presentation of some basic definitions and properties of weighted Sobolev spaces, used as a functional framework for solving both the scalar and the vectorial Oseen equations.
- In Section 3, the relation between the scalar equation (1.5) and the vectorial system (1.2) is discussed. Some properties of their fundamental solutions are shown.

- Section 4 deals with the scalar equation (1.5). Existence of solutions and the well posedness of the problem are treated in several functional frameworks.
- Section 5 is devoted to the study of the vectorial system (1.2). After giving a characterization to the solutions of homogeneous probelem, i.e. with $\boldsymbol{f} = \mathbf{0}$ and $h = 0$, we prove a complete class of existence, uniqueness and regularity results for the nonhomogeneous problem.

## 2. Notation and functional framework

### 2.1 Notation

In the sequel, $n \geq 2$ is an integer and $p$ is a real in the interval $]1, +\infty[$. The dual number of $p$ denoted $p'$ is defined by the relation $1/p + 1/p' = 1$. We use bold characters for vector functions or distributions. For $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$ we write

$$|\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Given a real $\alpha$, we denote by $[\alpha]$ its integer part. For any $k \in \mathbf{Z}$, $\mathbf{P}_k$ stands for the space of polynomials of degree lower than or equal to $k$ and $\mathbf{P}_k^{\Delta}$ the subspace of harmonic polynomials of $\mathbf{P}_k$. If $k$ is a negative integer, we set by convention $\mathbf{P}_k = \{0\}$. We recall that $\mathscr{D}(\mathbf{R}^n)$ is the well-known space of $\mathscr{C}^\infty(\mathbf{R}^n)$ functions with a compact support and $\mathscr{D}'(\mathbf{R}^n)$ its dual space, namely the space of distributions. We denote by $\mathscr{S}(\mathbf{R}^n)$ the Schwartz space of functions $\phi \in \mathscr{C}^\infty(\mathbf{R}^n)$ with rapid decrease at infinity, by $\mathscr{S}'(\mathbf{R}^n)$ its dual, i.e. the space of tempered distributions, and by $\mathscr{S}_1'(\mathbf{R}^n)$ the space of all the distributions $u \in \mathscr{D}'(\mathbf{R}^n)$ such that $e^{-x_1} u \in \mathscr{S}'(\mathbf{R}^n)$.

The Fourier transform of any complex valued Lebesgue integrable function $u : \mathbf{R} \to \mathbf{C}$ is defined by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-i\langle \boldsymbol{x}, \xi \rangle} u(\boldsymbol{x}) dx,$$

where $\xi \in \mathbf{R}^n$. If $u \in \mathscr{S}'(\mathbf{R}^n)$ then its Fourier transform $\hat{u} \in \mathscr{S}'(\mathbf{R}^n)$ is defined by $\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle$ for any function $\phi \in \mathscr{S}(\mathbf{R}^n)$. The Fourier transform is an invertible mapping from $\mathscr{S}(\mathbf{R}^n)$ into $\mathscr{S}(\mathbf{R}^n)$ and from $\mathscr{S}'(\mathbf{R}^n)$ into $\mathscr{S}'(\mathbf{R}^n)$.

Given a Banach space $B$ with its dual space $B'$ and a closed subspace $X$ of $B$, we denote by $B' \perp X$ the subspace of $B'$ orthogonal to $X$, namely

$$B' \perp X = \{f \in B', \forall v \in X, \langle f, v \rangle = 0\} = (B/X)'.$$

For any real $\alpha > 0$, the Bessel kernel $g_\alpha$ is defined as the function whose Fourier transform is

$$\hat{g}_\alpha(\boldsymbol{\xi}) = (2\pi)^{-n/2}(1+|\xi|^2)^{-\alpha/2}.$$

The kernel $g_\alpha$ can be expressed in terms of Bessel functions by the formula

$$g_\alpha(\boldsymbol{x}) = \gamma_\alpha |x|^{(\alpha-n)/2} K_{(n-\alpha)/2}(|x|).$$

with $\gamma_\alpha = (2\pi)^{-n/2} 2^{-\alpha/2+1} \Gamma(\alpha/2)$. Here $\Gamma$ denotes the classical Gamma function and $K_\lambda$ denotes the modified Bessel function of third kind. Since for any integer $m \geq 0$, we know that

$$K_{m+1/2}(t) = \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{k=0}^{m} \frac{(m+k)!}{k!(m-k)!} \frac{1}{(2t)^k},$$

we deduce an explicit expression of $g_\alpha$ when $\alpha < n$ and $n - \alpha$ is odd. In particular if $\alpha = 2$ and $n = 3$ one has

$$g_2(\boldsymbol{x}) = \frac{1}{4\pi|x|} e^{-|x|}.$$

More generally, the following estimate holds (see [24] or [23])

$$\forall \boldsymbol{x} \in \mathbf{R}^n, \qquad |g_\alpha(\boldsymbol{x})| \leq \frac{c_\alpha}{|x|^{n-\alpha}(1+|x|)^{(\alpha+1-n)/2}} e^{-|x|}. \tag{2.1}$$

Hence,

$$g_\alpha \in W_s^{0,p}(\mathbf{R}^n) \quad \text{for } s \in \mathbf{R},\ 1 \leq p \leq +\infty \text{ and } (n-\alpha)p < n. \tag{2.2}$$

In the sequel, the expression $a \lesssim b$ (resp. $a \simeq b$) means that there exists a constant $c$ not depending on the functions $a$ and $b$ such that $a \leq cb$.

## 2.2 Weighted Sobolev spaces. Some basic results

In the sequel, $\langle x \rangle$ denotes the basic weight defined by

$$\langle x \rangle = (1+|x|^2)^{1/2}. \tag{2.3}$$

For $1 \leq p \leq +\infty$, $L^p(\mathbf{R}^n)$ will refer to the space of (equivalence classes of) all measurable functions that are $p^{\text{th}}$ power integrable on $\mathbf{R}^n$. This space is equipped with the norm

$$\|u\|_{L^p(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n} |u|^p dx\right)^{1/p}.$$

Given two integers $m \geq 0$ and $k \in \mathbf{Z}$, we consider the weighted spaces

$$W^{m,p}(\mathbf{R}^n) = \{u \in \mathscr{D}'(\mathbf{R}^n); \forall \mu \in \mathbf{N}^n, |\mu| \le m, D^\mu u \in L^p(\mathbf{R}^n)\}$$

$$V_k^{m,p}(\mathbf{R}^n) = \{u \in \mathscr{D}'(\mathbf{R}^n); \forall \mu \in \mathbf{N}^n, |\mu| \le m, \ \langle x \rangle^k D^\mu u \in L^p(\mathbf{R}^n)\}$$

$$W_k^{m,p}(\mathbf{R}^n) = \{u \in \mathscr{D}'(\mathbf{R}^n); \forall \mu \in \mathbf{N}^n, |\mu| \le m, \langle x \rangle^{k-m+|\mu|} D^\mu u \in L^p(\mathbf{R}^n)\}$$

$$\mathscr{H}_k^{m,p}(\mathbf{R}^n) = \{u \in \mathscr{D}'(\mathbf{R}^n); \forall \mu \in \mathbf{N}^n, |\mu| \le m, e^{-x_1} \langle x \rangle^k D^\mu u \in L^p(\mathbf{R}^n)\}.$$

These spaces are equipped with the norms

$$\|u\|_{W^{m,p}(\mathbf{R}^n)} = \left( \sum_{|\mu| \le m} \|D^\mu u\|_{L^p(\mathbf{R}^n)}^p \right)^{1/p}.$$

$$\|u\|_{V_k^{m,p}(\mathbf{R}^n)} = \left( \sum_{|\mu| \le m} \|\langle x \rangle^k D^\mu u\|_{L^p(\mathbf{R}^n)}^p \right)^{1/p}.$$

$$\|u\|_{W_k^{m,p}(\mathbf{R}^n)} = \left( \sum_{|\mu| \le m} \|\langle x \rangle^{k-m+|\mu|} D^\mu u\|_{L^p(\mathbf{R}^n)}^p \right)^{1/p}.$$

$$\|u\|_{\mathscr{H}_k^{m,p}(\mathbf{R}^n)} = \left( \sum_{|\mu| \le m} \|\langle x \rangle^k e^{-x_1} D^\mu u\|_{L^p(\mathbf{R}^n)}^p \right)^{1/p}.$$

The spaces $W^{m,p}(\mathbf{R}^n)$, $W_k^{m,p}(\mathbf{R}^n)$, $V_k^{m,p}(\mathbf{R}^n)$ and $\mathscr{H}_k^{m,p}(\mathbf{R}^n)$ are Banach spaces. The space $\mathscr{D}(\mathbf{R}^n)$ is dense in $W^{m,p}(\mathbf{R}^n)$, in $V_k^{m,p}(\mathbf{R}^n)$ and in $W_k^{m,p}(\mathbf{R}^n)$ (see, e.g., Hanouzet [15]). We denote by $W^{-m,p'}(\mathbf{R}^n)$, $V_{-k}^{-m,p'}(\mathbf{R}^n)$ and $W_{-k}^{-m,p'}(\mathbf{R}^n)$ the dual spaces of $W^{m,p}(\mathbf{R}^n)$, $W_k^{m,p}(\mathbf{R}^n)$ and $V_k^{m,p}(\mathbf{R}^n)$ respectively. They are spaces of tempered distributions. It is quite clear that the local properties of the spaces $W_k^{m,p}(\mathbf{R}^n)$ and $V_k^{m,p}(\mathbf{R}^n)$ coincide with those of the Sobolev space $W_k^{m,p}(\mathbf{R}^n)$. We also have the obvious identities

$$V_0^{m,p}(\mathbf{R}^n) = W^{m,p}(\mathbf{R}^n), \qquad V_k^{0,p}(\mathbf{R}^n) = W_k^{0,p}(\mathbf{R}^n).$$

The spaces $W_k^{m,p}(\mathbf{R}^n)$ will play a particular role here. For a detailed study of these spaces, one can refer to [2], [15] and [17]. In this paper we need the semi-norm

$$|u|_{W_k^{m,p}(\mathbf{R}^n)} = \left( \sum_{|\mu| = m} \|\langle x \rangle^k D^\mu u\|_{L^p(\mathbf{R}^n)} \right)^{1/p}.$$

and the Green's formula

$$\forall u \in W_k^{1,p}(\mathbf{R}^n), \ \forall v \in W_{-k+1}^{1,p'}(\mathbf{R}^n), \qquad \int_{\mathbf{R}^n} \frac{\partial u}{\partial x_i} v \, dx = -\int_{\mathbf{R}^n} u \frac{\partial v}{\partial x_i} \, dx, \quad (2.4)$$

where $1 \le i \le n$, $1 < p < +\infty$ and $k \in \mathbf{Z}$. We also have the inclusion

$$\mathbf{P}_\ell \subset W_k^{m,p}(\mathbf{R}^n), \qquad \text{if } \ell < m - n/p - k. \tag{2.5}$$

In what follows, the space $W_k^{m,p}(\mathbf{R}^n)$ will be considered often in the case

$$\frac{n}{p} + k \notin \{1, \dots, m\}. \tag{2.6}$$

Indeed, this condition is sufficient to get some Hardy's type inequalities. Namely, if (2.6) is fulfilled, then (see [2])

$$\forall u \in W_k^{m,p}(\mathbf{R}^n), \qquad \inf_{\lambda \in \mathbf{P}_{j'}} \|u + \lambda\|_{W_k^{m,p}(\mathbf{R}^n)} \lesssim |u|_{W_k^{m,p}(\mathbf{R}^n)},$$

where $j' = \min(m-1, j)$ and $j = -[k + n/p - m] - 1$ is the highest degree of polynomials contained in $W_k^{m,p}(\mathbf{R}^n)$. If (2.6) does not hold, namely if $\dfrac{n}{p} + k \in \{1, \dots, m\}$, then similar inequalities can be obtained by adding logarithmic factor to the weights in the definition of $W_k^{m,p}(\mathbf{R}^n)$ (see [2]). In that case, all the forthcoming results remains valid provided some minor corrections are given.

We have the following algebraic and topological inclusions ($m \ge 0$)

$$V_k^{m,p}(\mathbf{R}^n) \subset W_k^{m,p}(\mathbf{R}^n) \subset W_{k-1}^{m-1,p}(\mathbf{R}^n) \subset \cdots \subset W_{k-m}^{0,p}(\mathbf{R}^n). \tag{2.7}$$

For any $\mu \in \mathbf{N}^n$, the mapping

$$u \in W_k^{m,p}(\mathbf{R}^n) \to D^\mu u \in W_k^{m-|\mu|,p}(\mathbf{R}^n) \tag{2.8}$$

is continuous (see [15]).

The spaces $W_k^{m,p}(\mathbf{R}^n)$ have proved to be adequate for treating several elliptic problems in unbounded regions of spaces and for several kinds of behavior at infinity (see [2], [21], [4, 5], [13, 14]). Let us recall some basic but fundamental results concerning the Laplace operator in the whole space (see [2]):

Theorem 2.1. *Let $m \in \mathbf{Z}$ and $\ell \in \mathbf{N}^* = \mathbf{N} \backslash \{0\}$.*

1. *If $n/p \notin \{1, \dots, \ell + 1\}$, then the operator*

$$\Delta : W_{m-\ell}^{m+1,p}(\mathbf{R}^n)/\mathbf{P}_{[\ell+1-n/p]}^{\Delta} \mapsto W_{m-\ell}^{m-1,p}(\mathbf{R}^n),$$

*is an isomorphism.*

2. *If $n/p' \notin \{1, \dots, \ell + 1\}$, then the operator*

$$\Delta : W_{m+\ell}^{m+1,p}(\mathbf{R}^n) \mapsto W_{m+\ell}^{m-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[\ell+1-n/p']}^{\Delta}$$

*is an isomorphism.*

3. *If* $n/p' \neq 1$ *and* $n/p \neq 1$, *then the operator*

$$\Delta : W_m^{m+1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p]} \mapsto W_m^{m-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$$

*is an isomorphism.*

## 3. Relation between scalar and vectorial Oseen's equations. Properties of the fundamental solutions

In this section, our purpose is to discuss briefly the relation between the scalar equation (1.3) and the vectorial Oseen's system (1.2). Indeed, the difference between equation (1.3) and system (1.2) seems to be reinforced by the presence of the pressure $\pi$ and the divergence equation $\operatorname{div} \boldsymbol{u} = h$. This difference also appears in terms of the fundamental solutions. However, there is a simple method for solving system (1.2) by solving equation (1.3) and the Laplace equation, in such a way that the divergence condition is automatically fulfilled. This method reveals that the fundamental solutions are intimately linked by a simple relation.

Formally, let $\theta$ and $\boldsymbol{s} = (s_1, \dots, s_n)$ be solution of the Laplace equations

$$\Delta\theta = h \quad \text{in } \mathbf{R}^n, \qquad \Delta \boldsymbol{s} = \boldsymbol{f} + \nabla\left(h - 2\frac{\partial\theta}{\partial x_1}\right) \quad \text{in } \mathbf{R}^n.$$

Consider in addition a vector function $\boldsymbol{\Phi} = (\Phi_1, \dots, \Phi_n)$ whose components $\Phi_i$, $1 \leq i \leq n$, are solution of the scalar equations $T\Phi_i = s_i$. Then the pair $(\boldsymbol{u}, \pi)$ defined by

$$\boldsymbol{u} = \nabla\theta + \Delta\boldsymbol{\Phi} - \nabla(\operatorname{div} \boldsymbol{\Phi}),$$

$$\pi = \operatorname{div} \boldsymbol{s},$$

is solution of the system (1.2). In other words, the existence of solutions of (1.2) can be obtained by treating the Laplace equation and the scalar equation (1.3). On the other hand, concerning the fundamental solution $(\mathcal{O}_{i,j}, e_j)$ of (1.2) we know that

$$\mathcal{O}_{ij} = \left(\delta_{ij}\Delta - \frac{\partial^2}{\partial x_i \partial x_j}\right)\varphi$$

$$e_j = \frac{\partial}{\partial x_j}\left(-\Delta + 2\frac{\partial}{\partial x_1}\right)\varphi.$$

Here, $i, j = 1, \dots, n$ and $\varphi$ satisfies $T\varphi = \mathscr{E}$ where $\mathscr{E}$ is the fundamental solution of the Laplace equation. It is well known (see for instance [11]) that the fundamental solution $\mathcal{O}$ of the scalar equation (1.3) in $\mathbf{R}^n$ is given by

$$\mathcal{O}(\boldsymbol{x}) = (2\pi)^{-n/2}|x|^{1-n/2}e^{x_1}K_{n/2-1}(|x|).$$

In particular, if the dimension is odd, namely if $n = 2m + 1$ for some integer $m \geq 1$, then

$$\mathcal{O}(\boldsymbol{x}) = \frac{1}{2}\frac{1}{(2\pi)^m}\frac{1}{|x|^m}e^{x_1-|x|}\sum_{k=0}^{m-1}\frac{(m+k-1)!}{k!(m-k-1)!}\frac{1}{(2|x|)^k},$$

Thus

$$\mathcal{O}(x) = \frac{1}{2}e^{x-|x|} \quad \text{if } n = 1, \qquad \mathcal{O}(\boldsymbol{x}) = \frac{1}{4\pi}\frac{e^{x_1-|x|}}{|x|} \quad \text{if } n = 3.$$

In order to clarify the relation between $(\mathcal{O}_{ij}, e_j)$ and $\mathcal{O}$ it is convenient to use the notion of *Riesz transform* (see for instance [22]). Recall that the Riesz transforms of a function $u \in L^p(\mathbf{R}^n)$, $p > 1$, are defined by

$$\widehat{R_j u}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{u}, \qquad j = 1, \dots, n,$$

where $\widehat{R_j u}$ and $\hat{u}$ denote the Fourier transform of $R_j u$ and $u$ respectively. Among the properties of the operators $R_i$, $i = 1, \dots, n$, let us recall that they preserve the class $L^p(\mathbf{R}^n)$ and satisfy

$$R_i \circ R_j(\Delta u) = -\frac{\partial^2 u}{\partial x_i \partial x_j}$$

Consequently the relation between $(\mathcal{O}_{ij}, e_j)$ and $\mathcal{O}$ is summarized in terms of the Riesz transforms as follows

Lemma 3.1. *For each $i, j \leq n$,*

$$\mathcal{O}_{ij} = (\delta_{ij}I + R_i \circ R_j)\mathcal{O}. \tag{3.9}$$

Since $R_i$ maps $L^p(\mathbf{R}^3)$ into itself, one can easily get some properties of $\mathcal{O}_{ij}$ from those of $\mathcal{O}$. Let us enumerate some of them. We state the following proposition whose proof is given in appendix A.

Proposition 3.2. *We have*

(a) *If $n = 3$, then $\mathcal{O} \in L^p(\mathbf{R}^3)$, $2 < p < 3$.*
(b) *If $n = 3$, then $\mathcal{O} - g_2 \in L^p(\mathbf{R}^3)$, $2 < p \leq +\infty$.*
(c) *If $n = 4$, $\mathcal{O} - g_2 \in L^p(\mathbf{R}^4)$, $2 < p < 4$.*
(d) *If $n = 5$, $\mathcal{O} - g_2 \in L^p(\mathbf{R}^5)$, $2 < p < 5/2$.*

Assertions (b)–(d) allow us to decompose the convolution $\mathcal{O} * f$ of the fundamental solution $\mathcal{O}$ (or $\mathcal{O}_{ij}$) with any function $f$ into the sum $g_2 * f + (\mathcal{O} - g_2) * f$. The advantage of this decomposition is twofold: firstly, as stated in Proposition 3.2, the function $\mathcal{O} - g_2$ has a better behavior than $\mathcal{O}$. The second advantage lies in the fact that $g_2 * u$ belongs to $W^{2,p}(\mathbf{R}^n)$ if $u$ belongs to $L^p$ (see Lizorkin spaces in Section 4.1 hereafter or in [6]).

The proof of the following corollary stems from assertion (b) of Proposition 3.2.

COROLLARY 3.3. *If* $n = 3$, $1 \le p < 3$ *and* $2 < q \le +\infty$, *then* $\mathcal{O}, \mathcal{O}_{ij} \in L^p(\mathbf{R}^3) + L^q(\mathbf{R}^3)$, $i, j = 1, 2, 3$.

In appendix B, a proof of the following corollary is given

COROLLARY 3.4. *Let* $f \in L^p(\mathbf{R}^3)$, $1 \le p < 2$ *and* $n = 3$. *We set*

$$p_1^* = \frac{2p}{2-p}, \quad p_2^* = \begin{cases} \dfrac{3p}{3-2p} & \text{if } p < 3/2 \\ +\infty & \text{if } 3/2 \le p < 2 \end{cases}, \quad p_3^* = \begin{cases} \dfrac{p}{p-1} & \text{if } 1 < p < 2 \\ +\infty & \text{if } p = 1 \end{cases}.$$

*Then, the following assertions are true*

(a) $\mathcal{O} * f \in L^r(\mathbf{R}^3)$ *for each* $r$, $p_1^* < r < p_2^*$. *Moreover, if* $p \ne 3/2$ *and* $p \ne 1$, *then* $\mathcal{O} * f \in L^r(\mathbf{R}^3)$ *with* $r = p_2^*$. *In all the cases,*

$$\|\mathcal{O} * f\|_{L^r(\mathbf{R}^3)} \lesssim \|f\|_{L^p(\mathbf{R}^3)}.$$

(b) $\mathcal{O} * f = h_1 + h_2$ *with* $h_1 \in L^r(\mathbf{R}^3)$, $p_1^* < r \le p_3^*$, $h_2 \in W^{2,p}(\mathbf{R}^3)$ *and*

$$\|h_1\|_{L^r(\mathbf{R}^3)} + \|h_2\|_{W^{2,p}(\mathbf{R}^3)} \lesssim \|f\|_{L^p(\mathbf{R}^3)}.$$

(c) *If* $1 < p < 3/2$, *then* $\mathcal{O} * f = h_1 + h_2$ *with* $h_1 \in L^q(\mathbf{R}^3)$ *and* $h_2 \in W^{2,p}(\mathbf{R}^3)$ *with*

$$q = \frac{(p+3)p}{3-2p}.$$

*The same assertions remain true if* $\mathcal{O}$ *is replaced by* $\mathcal{O}_{ij}$, $1 \le i, j \le 3$.

## 4. The scalar equation

Here we deal with the scalar equation

$$-\Delta u + 2\frac{\partial u}{\partial x_1} = f \qquad \text{in } \mathbf{R}^n. \tag{4.1}$$

A first approach for treating this equation is based on a simple but efficient idea. It consists of rewriting the equation in term of the new unknown $w(\boldsymbol{x}) = e^{-x_1} u(\boldsymbol{x})$. More precisely, let us consider the more general equation

$$-\Delta u + \nabla\phi.\nabla u = f \qquad \text{in } \mathbf{R}^n,$$

with $\Delta\phi = 0$. Setting $w(\boldsymbol{x}) = e^{-\phi(\boldsymbol{x})/2}u(\boldsymbol{x})$, $w$ satisfies the usual elliptic equation:

$$-\Delta w + a(\boldsymbol{x})w = F_1 \qquad \text{in } \mathbf{R}^n, \tag{4.2}$$

with $a(\boldsymbol{x}) = \frac{1}{4}|\nabla\phi|^2(\boldsymbol{x})$ and $F_1(\boldsymbol{x}) = e^{-\phi(\boldsymbol{x})/2}f(\boldsymbol{x})$. Notice that equation (4.1) corresponds to the particular case $\phi = 2x_1$. In this case, equation (4.2) writes

$$-\Delta w + w = F_1 \qquad \text{in } \mathbf{R}^n. \tag{4.3}$$

The main advantage of this new formulation is that the anisotropic character of equation (4.1) has disappeared. However, as we shall see, solutions obtained by this method could be different from those obtained by dealing directly with equation (4.1). This difference is mainly due to the fact that the space of tempered distribution $\mathscr{S}'(\mathbf{R}^n)$ is not preserved under multiplication by $e^{-x_1}$.

These considerations lead one to distinguish two kinds of solutions of (4.1); *tempered solutions*, which are tempered distributions, and *quasi-tempered* solutions. A solution $u$ of (4.1) will be called quasi-tempered if $e^{-x_1}u \in \mathscr{S}'(\mathbf{R}^n)$. The former solutions are obtained by solving (4.1) directly and may not be unique. On the contrary, the latter solutions are unique in general. It is worth nothing that only tempered solutions of the equation (4.1) turn out to be useful in the treatment of the vectorial Oseen's system (1.2).

*Remark (Relation with the Laplace equation).* Let $u \in \mathscr{S}_1'(\mathbf{R}^n)$ be a solution of (4.1) and set $w = e^{-x_1}u$. Consider the finite measure $\mu_2$ defined by $\mu_2 = (2\pi)^{n/2}\delta_0 - (2\pi)^{n/2}g_2(\boldsymbol{x})d\boldsymbol{x}$, with $\delta_0$ the Dirac measure at the origin. Let $\hat{\mu}_2$ be the Fourier transform of $\mu_2$. We have $\hat{\mu}_2(\boldsymbol{\xi}) = |\xi|^2(1+|\xi|^2)^{-1} = 1-(1+|\xi|^2)^{-1}$. Next, let $v \in \mathscr{S}'(\mathbf{R}^n)$ be a solution of the Laplace equation $-\Delta v = e^{-x_1}f = F$ in $\mathbf{R}^n$. Then, on the one hand, we have $|\xi|^2\hat{v} = \hat{F}$. On the other hand, since $-\Delta w + w = F$, we get $\hat{w}(\boldsymbol{\xi}) = (1+|\xi|^2)^{-1}\hat{F}$. Thus, $\hat{w}(\boldsymbol{\xi}) = \hat{\mu}_2(\boldsymbol{\xi})\hat{v}$, and $w(\boldsymbol{x}) = (2\pi)^{-n/2}\mu_2 * v(\boldsymbol{x}) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} v(\boldsymbol{x}-\boldsymbol{y})d\mu_2(\boldsymbol{y})$. Namely, $w = v - g_2 * v$ and $u = e^{x_1}(v - g_2 * v) = (2\pi)^{-n/2}e^{x_1}\mu_2 * v$.

## 4.1 Quasi-tempered solutions of scalar Oseen equation

Our aim here is to show existence of quasi-tempered solutions of the scalar Oseen equation (4.1). The main result of this paragraph is the following

THEOREM 4.1. *Let $m, k \in \mathbf{Z}$ be two integers and $p > 1$ a real. Then, the operator*

$$T : \mathscr{H}_k^{m+2,p}(\mathbf{R}^n) \to \mathscr{H}_k^{m,p}(\mathbf{R}^n)$$

*is an isomorphism.*

**Proof of Theorem 4.1**

Firstly, observe that the mapping

$$v \mapsto w = e^{-x_1} v$$

is an isomorphism between $\mathscr{H}_k^{m,p}(\mathbf{R}^n)$ and $V_k^{m,p}(\mathbf{R}^n)$. Moreover, in the sense of distributions one can easily prove that

$$Tv = e^{x_1}(I - \Delta)e^{-x_1}v.$$

This remark allows one to deal only with the operator $I - \Delta$. We start with the following lemma.

Lemma 4.2. *Let $k \geq 0$ be an integer and $p > 1$ be a real. Then, the operator*

$$I - \Delta : W^{k+2,p}(\mathbf{R}^n) \to W^{k,p}(\mathbf{R}^n)$$

*is an isomorphism.*

Proof. We need the following identity (see [6])

$$\mathscr{L}^{m,p}(\mathbf{R}^n) = W^{m,p}(\mathbf{R}^n), \qquad \forall m \in \mathbf{N},$$

where $\mathscr{L}^{m,p}(\mathbf{R}^n)$ is the Lizorkin space defined by

$$\mathscr{L}^{m,p}(\mathbf{R}^n) = \{u \in \mathscr{S}'(\mathbf{R}^n); u = g_m * v, v \in L^p(\mathbf{R}^n)\}.$$

In terms of Fourier transform, the unique solution of the equation

$$(I - \Delta)w = h,$$

is given by $w = \mathscr{F}^{-1}((1 + |\xi|^2)^{-1}\mathscr{F}(h)) = \mathscr{F}^{-1}(\hat{g}_2\mathscr{F}(h))$. Hence,

$$\begin{aligned}
h \in W^{m,p}(\mathbf{R}^n) &\Leftrightarrow h \in \mathscr{L}^{m,p}(\mathbf{R}^n) \\
&\Leftrightarrow (\hat{g}_m)^{-1}\hat{h} \in \mathscr{F}(L^p(\mathbf{R}^n)) \\
&\Leftrightarrow (\hat{g}_m)^{-1}(\hat{g}_2)^{-1}\hat{w} \in \mathscr{F}(L^p(\mathbf{R}^n)) \\
&\Leftrightarrow (\hat{g}_{m+2})^{-1}\hat{w} \in \mathscr{F}(L^p(\mathbf{R}^n)) \\
&\Leftrightarrow w \in W^{m+2,p}(\mathbf{R}^n). \quad \blacksquare
\end{aligned}$$

Lemma 4.3. *Let $k \in \mathbf{Z}$ and $m \in \mathbf{Z}$ be two integers and $p > 1$ a real. Then, the operator*

$$I - \Delta : V_k^{m+2,p}(\mathbf{R}^n) \to V_k^{m,p}(\mathbf{R}^n)$$

*is an isomorphism.*

Proof. We know that the operator $I - \Delta$ is one to one from $\mathscr{S}'(\mathbf{R}^n)$ into $\mathscr{S}'(\mathbf{R}^n)$. Hence, for $F \in V_k^{m,p}(\mathbf{R}^n)$, there exists a unique $w \in \mathscr{S}'(\mathbf{R}^n)$ such that

$$-\Delta w + w = F \qquad \text{in } \mathbf{R}^n.$$

Suppose first that $k \geq 0$. Let us prove by induction on $k$ the following

$$F \in V_k^{0,p}(\mathbf{R}^n) \Rightarrow (w \in V_k^{2,p}(\mathbf{R}^n) \text{ and } \|w\|_{V_k^{2,p}(\mathbf{R}^n)} \lesssim \|F\|_{V_k^{0,p}(\mathbf{R}^n)}). \tag{4.4}$$

This follows immediately from Lemma 4.2 when $k = 0$. Suppose that (4.4) holds for $0, \dots, k$ and suppose that $F \in V_{k+1}^{0,p}(\mathbf{R}^n)$. Necessarily $w \in V_k^{2,p}(\mathbf{R}^n)$. Setting $\tilde{w} = \langle x \rangle^{k+1} w$, one proves easily that

$$-\Delta \tilde{w} + \tilde{w} = \langle x \rangle^{k+1} F - 2\nabla \langle x \rangle^{k+1} . \nabla w - (\Delta \langle x \rangle^{k+1}) w.$$

The right hand side belongs to $L^p(\mathbf{R}^n)$ since $w \in V_k^{2,p}(\mathbf{R}^n)$ and $|D^\alpha \langle x \rangle^k| \lesssim \langle x \rangle^{k-|\alpha|}$ for any multi-index $\alpha$. It follows that $\tilde{w} \in V_0^{2,p}(\mathbf{R}^n)$, and, consequently, $w \in V_{k+1}^{2,p}(\mathbf{R}^n)$ since

$$\forall |\alpha| \leq 2; \quad |D^\alpha w| \lesssim \sum_{|v| \leq 2,\, v \leq \alpha} |D^v \langle x \rangle^{-k-1} D^{\alpha - v} \tilde{w}| \lesssim \sum_{|v| \leq 2,\, v \leq \alpha} |\langle x \rangle^{-k-1-|v|} D^{\alpha - v} \tilde{w}|.$$

This completes the proof of (4.4). Similarly, let us prove by induction on $k \geq 0$ the following

$$F \in V_{-k}^{0,p}(\mathbf{R}^n) \Rightarrow (w \in V_{-k}^{2,p}(\mathbf{R}^n) \text{ and } \|w\|_{V_{-k}^{2,p}(\mathbf{R}^n)} \lesssim \|f\|_{V_{-k}^{0,p}(\mathbf{R}^n)}). \tag{4.5}$$

This holds clearly for $k = 0$. Suppose that it holds for $0, \dots, k$ and let $F \in V_{-k-1}^{0,p}(\mathbf{R}^n)$. Setting $\tilde{F} = \langle x \rangle^{-k-1} F \in L^p(\mathbf{R}^n)$, $\tilde{w} = (I - \Delta)^{-1} \tilde{F} \in V_0^{2,p}(\mathbf{R}^n)$ and $h = w - \langle x \rangle^{-k-1} \tilde{w}$, one gets after a few calculation

$$-\Delta h + h = (\Delta \langle x \rangle^{k+1}) \tilde{w} + \nabla \langle x \rangle^{k+1} . \nabla \tilde{w}.$$

The right hand side of the last identity belongs to $V_{-k}^{0,p}(\mathbf{R}^n)$. From induction hypothesis we deduce that $h \in V_{-k}^{2,p}(\mathbf{R}^n) \hookrightarrow V_{-k-1}^{2,p}(\mathbf{R}^n)$. It follows that $w \in V_{-k-1}^{2,p}(\mathbf{R}^n)$ since $\langle x \rangle^{k+1} \tilde{w} \in V_{-k-1}^{2,p}(\mathbf{R}^n)$. This completes the proof of (4.5).

At this stage, the assertion of Lemma 4.3 is proved when $m = 0$ and $k \in \mathbf{Z}$. Now, suppose that $m \geq 1$, and let $w \in \mathscr{S}'(\mathbf{R}^n)$ be the unique solution of $-\Delta w + w = F$ with $F \in V_k^{m,p}(\mathbf{R}^n)$. Then, for each multi-index $\alpha$, $|\alpha| \leq m$, $D^\alpha w$ satisfies $-\Delta(D^\alpha w) + D^\alpha w = D^\alpha F \in V_k^{0,p}(\mathbf{R}^n)$. Hence, $D^\alpha w \in V_k^{2,p}(\mathbf{R}^n)$ for each $\alpha$, $|\alpha| \leq m$. We conclude that $w \in V_k^{m+2,p}(\mathbf{R}^n)$ which provides the proof of Lemma 4.3 for $m \geq 0$. The proof for $m \leq -2$ is based on a classical duality argument. It remains to treat the case $m = -1$. Let $F \in V_k^{-1,p}(\mathbf{R}^n)$ for some $k \in \mathbf{Z}$. Then, $F \in V_k^{-2,p}(\mathbf{R}^n)$. Hence, there exists $w \in V_k^{0,p}(\mathbf{R}^n)$ such that $-\Delta w + w = F$ (here we used the result of Lemma 4.3 for $m = -2$). Since in addition, $\partial_i F \in V_k^{-2,p}(\mathbf{R}^n)$, $i = 1, \dots, n$, we deduce that $\partial_i w \in V_k^{0,p}(\mathbf{R}^n)$ for $i = 1, \dots, n$ since $-\Delta(\partial_i w) + \partial_i w = \partial_i F$. Thus, $w \in V_k^{1,p}(\mathbf{R}^n)$. ■

## 4.2 Tempered solutions of the scalar Oseen equation

Our aim here is to look for solutions of the scalar equation (4.1) which are tempered distributions. This is a first step toward the resolution of the vectorial Oseen's equations (1.2). In the sequel, for each integer $m \in \mathbf{Z}$, we consider the space

$$\tilde{W}_k^{m,p}(\mathbf{R}^n) = \left\{ u \in W_k^{m,p}(\mathbf{R}^n), \frac{\partial u}{\partial x_1} \in W_k^{m-2,p}(\mathbf{R}^n) \right\},$$

equipped with the norm

$$\|u\|_{\tilde{W}_k^{m,p}(\mathbf{R}^n)} = \left\{ \|u\|^p_{W_k^{m,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|^p_{W_k^{m-2,p}(\mathbf{R}^n)} \right\}^{1/p}.$$

When $m \geq 0$ and $k \in \mathbf{Z}$, we set $\tilde{\mathscr{W}}_{-k}^{-m,p'}(\mathbf{R}^n)$ the dual space of $\tilde{W}_k^{m,p}(\mathbf{R}^n)$. Clearly, we have the embedding

$$\tilde{W}_{-k}^{-m,p'}(\mathbf{R}^n) \hookrightarrow W_{-k}^{-m,p'}(\mathbf{R}^n) \hookrightarrow \tilde{\mathscr{W}}_{-k}^{-m,p'}(\mathbf{R}^n).$$

In what follows, $\mathbf{Q}_k$ denotes the sum $\mathbf{H}'_k + \mathbf{P}_{k-1}$, where $\mathbf{H}'_k$ is the space of *homogeneous* polynomials of degree $k$ and depending only on $x_2, \dots, x_n$. We denote by $\mathbf{Q}_\ell^+$ (resp. $\mathbf{Q}_\ell^-$) the subspace of all the polynomials $p \in \mathbf{Q}_\ell$ satisfying $Tp = 0$ (resp. $T^*p = 0$). Notice that the mapping $p(x_1, x_2, \dots, x_n) \to p(-x_1, x_2, \dots, x_n)$ is one to one from $\mathbf{Q}_\ell^+$ into $\mathbf{Q}_\ell^-$. We have the lemma

Lemma 4.4. *Let $m \geq 0$, $1 < p < +\infty$ and $k \in \mathbf{Z}$ such that $k + n/p \notin \{0, \dots, m\}$. A function $u \in \tilde{W}_k^{m,p}(\mathbf{R}^n)$ satisfies $Tu = 0$ if and only if $u \in \mathbf{Q}_\ell^+$ with $\ell = -[k + n/p - m] - 1$.*

Proof. If $u$ is a tempered distribution such that $Tu = 0$, then $(|\xi|^2 - i\xi_1)\hat{u} = 0$. Since $|\xi|^2 - i\xi_1$ vanishes only at $\xi = 0$, we deduce that $u$ is polynomial. If in addition $u \in \tilde{W}_k^{m,p}(\mathbf{R}^n)$ then it belongs necessarily to $\mathbf{Q}_\ell$.

In the following theorems, we give some isomorphism results concerning the operator $T$, which in consequence yield existence results for the scalar Oseen equation. In the first theorem, we look for a solution that belongs to $\tilde{W}_0^{1,p}(\mathbf{R}^n)$. Theorems 4.6, 4.7 and 4.8 concern the regularity of solutions.

Theorem 4.5. *Assume that $n/p \neq 1$ and $n/p' \neq 1$. Then, the operator*

$$T : \tilde{W}_0^{1,p}(\mathbf{R}^n)/\mathbf{P}_{[1-n/p]} \to W_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$$

*is an isomorphism.*

Theorem 4.6. *Let $m \geq 2$ be an integer, and suppose that $n/p \neq \{1, \dots, m\}$, then the operator*

$$T : \tilde{W}_0^{m,p}(\mathbf{R}^n)/\mathbf{Q}^+_{[m-n/p]} \to W_0^{m-2,p}(\mathbf{R}^n)$$

*is an isomorphism.*

THEOREM 4.7. *Let $m \geq 2$ be an integer, and suppose that $n/p' \neq \{1, \dots, m\}$ and $n/p \neq \{1, 2\}$ if $m$ is even and $n/p \neq 1$. Let $\tilde{W}_0^{-m+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[m-n/p']}$ be the space of all the functions $u \in \tilde{W}_0^{-m+2,p}(\mathbf{R}^n)$ satisfying the conditions*

$$\forall p \in \mathbf{P}_{[m-2-n/p']}, \langle u, p \rangle = 0,$$

$$\forall p \in \mathbf{P}_{[m-n/p']}, \left\langle \frac{\partial u}{\partial x_1}, p \right\rangle = 0.$$

*Then, the operator*

$$T : \tilde{W}_0^{-m+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[m-n/p']} \to W_0^{-m,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m-n/p']}$$

*is an isomorphism.*

By duality and transposition, Theorem 4.6 yields

THEOREM 4.8. *Suppose that $m \geq 2$, $1 < p < +\infty$ and $n/p' \neq \{1, \dots, m\}$, then the operator*

$$T : W_0^{-m+2,p}(\mathbf{R}^n) \to \tilde{\mathscr{W}}_0^{-m,p}(\mathbf{R}^n) \perp \mathbf{Q}^-_{[m-n/p']}$$

*is an isomorphism.*

The following proposition plays a prominent role in the proof of Theorems 4.5–4.6,

PROPOSITION 4.9. *Let $1 < p < +\infty$, $m \geq 2$ such that $n/p \notin \{1, \dots, m\}$, and set $\ell = \min(m-1, [m-n/p])$. Then,*

$$\inf_{q \in \mathbf{Q}_\ell} \|u + q\|_{\tilde{W}_0^{m,p}(\mathbf{R}^n)} \lesssim |u|_{W_0^{m,p}(\mathbf{R}^n)} + \left|\frac{\partial u}{\partial x_1}\right|_{W_0^{m-2,p}(\mathbf{R}^n)}$$

*for each $u \in \tilde{W}_0^{m,p}(\mathbf{R}^n)$.*

**Proof of Proposition 4.9**

Observe first that the semi-norm $[u]_{m,p} = |u|_{W_0^{m,p}(\mathbf{R}^n)} + \left|\frac{\partial u}{\partial x_1}\right|_{W_0^{m-2,p}(\mathbf{R}^n)}$ is a norm on $\tilde{W}_0^{m,p}(\mathbf{R}^n)/\mathbf{Q}_\ell$. Indeed, if $[u]_{m,p} = 0$, then $|u|_{W_0^{m,p}(\mathbf{R}^n)} = 0$ and $u$ is polynomial. Since $u$ belongs to $W_0^{m,p}(\mathbf{R}^n)$, its degree is necessarily less or equal to $\ell$. Moreover, since $\frac{\partial u}{\partial x_1}$ belongs to $W_0^{m-2,p}(\mathbf{R}^n)$, the degree of $\frac{\partial u}{\partial x_1}$ is less or equal to $\ell - 2 = \min(m-3, [m-2-n/p])$. Hence, $u \in \mathbf{Q}_\ell$.

Now, let us prove that this semi-norm is equivalent to the norm of $\tilde{W}_0^{m,p}(\mathbf{R}^n)/\mathbf{Q}_\ell$. We need the following lemma (see [2]).

LEMMA 4.10. *Let $k \in \mathbf{Z}$ and $s \geq 1$ be two integers such that $n/p + k \neq \{1, \dots, s\}$. Let $q = \min(s-1, [s-k-n/p])$. Then, the semi-norm $|v|_{W_k^{s,p}(\mathbf{R}^n)}$ defines on $W_k^{s,p}(\mathbf{R}^n)/\mathbf{P}_q$ a norm which is equivalent to the quotient norm.*

Suppose first that $\ell \geq 0$, then from Lemma 4.10 it follows that

$$\forall v \in W_0^{m-\ell,p}(\mathbf{R}^n), \qquad \inf_{c \in \mathbf{R}} \|v + c\|_{W_0^{m-\ell,p}(\mathbf{R}^n)} \lesssim |v|_{W_0^{m-\ell,p}(\mathbf{R}^n)}, \tag{4.6}$$

$$\forall v \in W_0^{m-\ell-1,p}(\mathbf{R}^n), \qquad \|v\|_{W_0^{m-\ell-1,p}(\mathbf{R}^n)} \lesssim |v|_{W_0^{m-\ell-1,p}(\mathbf{R}^n)}, \tag{4.7}$$

$$\forall v \in W_{\ell-m}^{\ell,p}(\mathbf{R}^n), \qquad \inf_{q \in \mathbf{P}_{\ell-1}} \|v - q\|_{W_{\ell-m}^{\ell,p}(\mathbf{R}^n)} \lesssim |v|_{W_{\ell-m}^{\ell,p}(\mathbf{R}^n)}, \tag{4.8}$$

Now, let $\mathbf{P}'_k$, $k$ being an integer, be the space of all the polynomials of degree less than or equal to $\ell$ and depending only on $x_2, \dots, x_n$. Namely, if $k \geq 0$, then

$$\mathbf{P}'_k = \mathbf{H}'_0 + \cdots + \mathbf{H}'_k.$$

If $k < 0$, $\mathbf{P}'_k = \{0\}$. We shall use the following Lemma.

LEMMA 4.11. *Let $m$, $\ell$ and $p$ be as in Proposition 4.9. Then,*

$$\inf_{q \in \mathbf{P}'_\ell} \|u + q\|_{W_0^{m,p}(\mathbf{R}^n)} \lesssim |u|_{W_0^{m,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W_0^{m-1,p}(\mathbf{R}^n)}$$

*for each $u \in W_0^{m,p}(\mathbf{R}^n)$.*

PROOF [of lemma 4.11]. For each $k \geq 0$, we set

$$\Lambda_k = \{\mu = (0, \mu_2, \dots, \mu_n), |\mu| = \mu_2 + \cdots + \mu_n = k\}.$$

The proof of Lemma 4.11 rests on an identification between the space $\mathbf{H}_k$, $k \geq 0$, and $\mathbf{R}^{\mathrm{card}(\Lambda_k)}$ by means of the mapping

$$q \in \mathbf{H}_k \to (D^\mu q)_{\mu \in \Lambda_k} \in \mathbf{R}^{\mathrm{card}(\Lambda_k)}.$$

Next, from Lemma 4.10, we can write

$$\inf_{q \in \mathbf{P}'_\ell} \|u + q\|_{W_0^{m,p}(\mathbf{R}^n)} = \inf_{p_\ell \in \mathbf{H}'_\ell} \inf_{p_{\ell-1} \in \mathbf{H}'_{\ell-1}} \dots \inf_{p_0 \in \mathbf{H}'_0} \|u - (p_\ell + p_{\ell-1} + \cdots + p_0)\|_{W_0^{m,p}(\mathbf{R}^n)}$$

$$\lesssim \inf_{p_\ell \in \mathbf{H}'_\ell} \dots \inf_{p_1 \in \mathbf{H}'_1} \left\{ \sum_{\mu \in \Lambda_1} \|D^\mu u - D^\mu(p_\ell + \cdots + p_1)\|_{W_0^{m-1,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W_0^{m-1}(\mathbf{R}^n)} \right\}.$$

Since the polynomial $p_1 \in \mathbf{H}'_1$ can be identified to the constants $(D^\mu p_1, |\mu| = 1)$, it follows that

$$\inf_{p_\ell \in \mathbf{H}'_\ell} \dots \inf_{p_1 \in \mathbf{H}'_1} \left\{ \sum_{\mu \in \Lambda_1} \|D^\mu u - D^\mu (p_\ell + \dots + p_1)\|_{W_0^{m-1,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W_0^{m-1}(\mathbf{R}^n)} \right\}$$

$$\lesssim \inf_{p_\ell \in \mathbf{H}'_\ell} \dots \inf_{p_2 \in \mathbf{H}'_2} \left\{ \sum_{\mu \in \Lambda_2} \|D^\mu u - D^\mu (p_\ell + \dots + p_2)\|_{W_0^{m-1,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W_0^{m-1}(\mathbf{R}^n)} \right\}.$$

More generally each polynomial function $p_k \in \mathbf{H}'_k$ can be identified to the constants $(D^\mu p_k, |\mu| = k)$. Then, repeating the argument gives

$$\inf_{q \in \mathbf{P}'_\ell} \|u + q\|_{W_0^{m,p}(\mathbf{R}^n)} \lesssim |u|_{W_0^{m,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W_0^{m-1,p}(\mathbf{R}^n)},$$

which completes the proof of Lemma 4.11.

Now, since the space $\mathbf{Q}_\ell$ can be identified to the product $\mathbf{P}'_\ell \times x_1 \mathbf{P}_{\ell-2}$, for each $u \in \tilde{W}_0^{m,p}(\mathbf{R}^n)$ and for each $p = p_1 + p_2$, with $(p_1, p_2) \in \mathbf{P}'_\ell \times x_1 \mathbf{P}_{\ell-2}$, one can write

$$\begin{aligned} &\|u - p\|_{\tilde{W}_0^{m,p}(\mathbf{R}^n)} \\ &\quad = \|u - (p_1 + p_2)\|_{\tilde{W}_0^{m,p}(\mathbf{R}^n)} \\ &\quad = \|u - (p_1 + p_2)\|_{W_0^{m,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} - \frac{\partial p_2}{\partial x_1} \right\|_{W_0^{m-2,p}(\mathbf{R}^n)}. \end{aligned} \tag{4.9}$$

Next, since the degree of $p_2$ is lower than $\ell - 1$, it is clear that $p_2 \in W_0^{m,p}(\mathbf{R}^n)$. Hence, using Lemma 4.11, we have

$$\inf_{p_1 \in \mathbf{P}'_\ell} \|(u - p_2) - p_1\|_{W_0^{m,p}(\mathbf{R}^n)} \lesssim |u|_{W_0^{m,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} - \frac{\partial p_2}{\partial x_1} \right\|_{W_0^{m-1,p}(\mathbf{R}^n)}. \tag{4.10}$$

Now, combining (4.9), (4.10) and using the fact that the mapping $p \to \dfrac{\partial p}{\partial x_1}$ is one to one from $x_1 \mathbf{P}_{\ell-2}$ into $\mathbf{P}_{\ell-2}$, we get

$$\begin{aligned} &\inf_{p \in \mathbf{Q}_\ell} \|u - p\|_{\tilde{W}_0^{m,p}(\mathbf{R}^n)} \\ &\quad = \inf_{(p_1, p_2) \in \mathbf{P}'_\ell \times x_1 \mathbf{P}_{\ell-2}} \|u - (p_1 + p_2)\|_{\tilde{W}_0^{m,p}(\mathbf{R}^n)} \\ &\quad \lesssim \left( |u|_{W_0^{m,p}(\mathbf{R}^n)} + \inf_{p_2 \in x_1 \mathbf{P}_{\ell-2}} \left\{ \left\| \frac{\partial u}{\partial x_1} - \frac{\partial p_2}{\partial x_1} \right\|_{W_0^{m-1,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} - \frac{\partial p_2}{\partial x_1} \right\|_{W_0^{m-2,p}(\mathbf{R}^n)} \right\} \right) \\ &\quad \lesssim \left( |u|_{W_0^{m,p}(\mathbf{R}^n)} + \inf_{p \in \mathbf{P}_{\ell-2}} \left\{ \left\| \frac{\partial u}{\partial x_1} - p \right\|_{W_0^{m-1,p}(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial x_1} - p \right\|_{W_0^{m-2,p}(\mathbf{R}^n)} \right\} \right). \end{aligned}$$

Observe that

$$\left\|\frac{\partial u}{\partial x_1} - p\right\|_{W_0^{m-1,p}(\mathbf{R}^n)} = \left\|\frac{\partial u}{\partial x_1} - p\right\|_{W_{-1}^{m-2,p}(\mathbf{R}^n)} + \left|\frac{\partial u}{\partial x_1}\right|_{W_0^{m-1,p}(\mathbf{R}^n)}$$
$$\lesssim \left\|\frac{\partial u}{\partial x_1} - p\right\|_{W_0^{m-2,p}(\mathbf{R}^n)} + \left|\frac{\partial u}{\partial x_1}\right|_{W_0^{m-1,p}(\mathbf{R}^n)} \tag{4.11}$$

Together withe inequality (see Lemma 4.10)

$$\inf_{p \in \mathbf{P}_{\ell-2}} \left\|\frac{\partial u}{\partial x_1} - p\right\|_{W_0^{m-2,p}(\mathbf{R}^n)} \lesssim \left|\frac{\partial u}{\partial x_1}\right|_{W_0^{m-2,p}(\mathbf{R}^n)},$$

this yields therefore

$$\inf_{q \in \mathbf{Q}_\ell} \|u + q\|_{\tilde{W}_0^{m,p}(\mathbf{R}^n)} \lesssim |u|_{W_0^{m,p}(\mathbf{R}^n)} + \left|\frac{\partial u}{\partial x_1}\right|_{W_0^{m-2,p}(\mathbf{R}^n)},$$

completing the proof. ■

**Proof of Theorem 4.5**

$T$ is clearly continuous from $\tilde{W}_0^{1,p}(\mathbf{R}^n)/\mathbf{P}_{[1-n/p]}$ into $W_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$. It is also injective, thanks to Lemma 4.4. Let us prove that it is onto. Let $f \in W_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$. According to Proposition 4.1 of [2] and since $n/p' \neq 1$, there exists a vector function $\boldsymbol{w} = (w_1, \ldots, w_n) \in L^p(\mathbf{R}^n)^n$ such that $\operatorname{div} \boldsymbol{w} = f$. We set

$$z = \mathscr{F}^{-1}((|\xi|^2 + i\xi_1)^{-1}\mathscr{F}f) = \sum_{k=1}^{n} \mathscr{F}^{-1}\left(\frac{i\xi_k}{|\xi|^2 + 2i\xi_1}\mathscr{F}w_k\right).$$

We need the following multiplier theorem due to Lizorkin [18] (see also [11], Lemma 4.2 Ch. VII)

LEMMA 4.12. *Let $j, k \in \{1, \ldots, n\}$. The operators*

$$h \to \mathscr{F}^{-1}\left(\frac{\xi_k\xi_j}{|\xi|^2 + 2i\xi_1}\mathscr{F}h\right), \qquad h \to \mathscr{F}^{-1}\left(\frac{\xi_1}{|\xi|^2 + 2i\xi_1}\mathscr{F}h\right),$$

*are continuous from $L^p(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$, $1 < p < +\infty$.*

Hence, for each $j \leq n$, we have

$$\frac{\partial z}{\partial x_j} = -\sum_{k=1}^{n} \mathscr{F}^{-1}\left(\frac{\xi_j\xi_k}{|\xi|^2 + 2i\xi_1}\mathscr{F}w_k\right) \in L^p(\mathbf{R}^n),$$

and

$$\|\nabla z\|_{L^p(\mathbf{R}^n)} \lesssim \|\boldsymbol{w}\|_{L^p(\mathbf{R}^n)} \lesssim \|f\|_{W_0^{-1,p}(\mathbf{R}^n)}.$$

Lemma 4.13 (see [2]). *Let $h \in \mathscr{D}'(\mathbf{R}^n)$ such that $\nabla h \in L^p(\mathbf{R}^n)$, with $1 < p < +\infty$ and $p \neq n$. Then, there exists a constant $K$ such that $h + K \in W_0^{1,p}(\mathbf{R}^n)$ and*

$$\|h + K\|_{W_0^{1,p}(\mathbf{R}^n)} \lesssim \|\nabla h\|_{L^p(\mathbf{R}^n)}$$

From this lemma, it follows that there exists a constant $K$, such that $z + K \in W_0^{1,p}(\mathbf{R}^n)$ and

$$\|z + K\|_{W_0^{1,p}(\mathbf{R}^n)} \lesssim \|\nabla z\|_{L^p(\mathbf{R}^n)} \lesssim \|f\|_{W_0^{-1,p}(\mathbf{R}^n)}.$$

We set $u = z + K$. Then, $u \in W_0^{1,p}(\mathbf{R}^n)$ and satisfies

$$-\Delta u + 2\frac{\partial u}{\partial x_1} = -\Delta z + 2\frac{\partial z}{\partial x_1} = f.$$

In addition,

$$2\frac{\partial u}{\partial x_1} = f + \Delta u \in W_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']},$$

since the range of the Laplacian $\Delta : W_0^{1,p}(\mathbf{R}^n) \to W_0^{-1,p}(\mathbf{R}^n)$ is nothing but $W_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$. ■

**Proof of Theorem 4.6**

We start with the lemma

Lemma 4.14. *The operator $T : \mathbf{Q}_{\ell+2} \to \mathbf{P}_\ell$ is onto.*

Proof. If $\ell = 0$ and $p = c \in \mathbf{P}_\ell = \mathbf{R}$, then $p = T\left(\frac{1}{2}cx_1\right)$. Suppose that $\ell \geq 1$ and let $p \in \mathbf{P}_\ell$. Set

$$q = \frac{1}{2}\int_0^{x_1} p(t, x_2, \ldots, x_n)dt.$$

Then, $p - Tq \in \mathbf{P}_{\ell-1}$. The proof is completed by applying the hypothesis of induction. ■

Lemma 4.15. *Let $m \geq 2$ and and set $\ell = [m - n/p]$. Let $T^*$ be the adjoint operator of $T : \tilde{W}_0^{m,p}(\mathbf{R}^n)/\mathbf{Q}_\ell \to W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}$. Then, $T^*$ is injective.*

Proof. The adjoint operator $T^*$ is defined from $W_0^{-m+2,p'}(\mathbf{R}^n) \perp \mathbf{P}_{\ell-2}$ into $\tilde{\mathscr{W}}_0^{-m,p'}(\mathbf{R}^n) \perp \mathbf{Q}_\ell$ as follows:

$$\forall u \in W_0^{-m+2,p'}(\mathbf{R}^n) \perp \mathbf{P}_{\ell-2}, \qquad \forall v \in \tilde{W}_0^{m,p}(\mathbf{R}^n) \quad \langle T^*u, v\rangle = \langle u, Tv\rangle.$$

If $T^*u = 0$ then $-\Delta u - 2\dfrac{\partial u}{\partial x_1} = 0$. Since $u$ is a tempered distribution, and using the same argument of lemma 4.4, we deduce that $u$ is polynomial. Further, $u = 0$ since $W_0^{-m+2,p'}(\mathbf{R}^n)$ contains only the trivial polynomial function.

Lemma 4.16. *Let $m \geq 2$ be an integer and set $\ell = [m - n/p]$. Suppose that $n/p \neq \{1, \ldots, m\}$. Then, the operator $T : \tilde{W}_0^{m,p}(\mathbf{R}^n)/\mathbf{Q}_\ell \to W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}$ is an isomorphism.*

Proof. The linear mapping $T$ is clearly bounded. It is also injective; indeed, let $u \in \tilde{W}_0^{m,p}(\mathbf{R}^n)$ such that $Tu \in \mathbf{P}_{\ell-2}$. According to Lemma 4.14, there exists $\theta \in \mathbf{Q}_\ell$ such that $T\theta = Tu$. Hence, $T(\theta - u) = 0$. Thus, by virtue of Lemma 4.4, $\theta - u$ belongs to $\mathbf{Q}_\ell$ and $u \in \mathbf{Q}_\ell$.

Let us prove that the range of $T$ is a closed subspace of $W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}$. Let $\alpha$ be an arbitrary multi-index such that $|\alpha| = m - 2$. Lemma 4.12 yields

$$\begin{aligned}
\left\|\frac{\partial^2(D^\alpha u)}{\partial x_i \partial x_j}\right\|_{L^p(\mathbf{R}^n)} &= \|\mathscr{F}^{-1}(\xi_i\xi_j \mathscr{F} D^\alpha u)\|_{L^p(\mathbf{R}^n)} \\
&= \left\|\mathscr{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2 + 2i\xi_1}\mathscr{F} TD^\alpha u\right)\right\|_{L^p(\mathbf{R}^n)} \\
&\lesssim \|D^\alpha Tu\|_{L^p(\mathbf{R}^n)} \\
&\lesssim |Tu|_{W_0^{m-2,p}(\mathbf{R}^n)} \\
&\lesssim \|Tu\|_{W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}}
\end{aligned}$$

$$\begin{aligned}
\left\|\frac{\partial(D^\alpha u)}{\partial x_1}\right\|_{L^p(\mathbf{R}^n)} &= \|\mathscr{F}^{-1}(\xi_1 \mathscr{F} D^\alpha u)\|_{L^p(\mathbf{R}^n)} \\
&= \left\|\mathscr{F}^{-1}\left(\frac{\xi_1}{|\xi|^2 + 2i\xi_1}\mathscr{F} D^\alpha Tu\right)\right\|_{L^p(\mathbf{R}^n)} \\
&\lesssim \|D^\alpha Tu\|_{L^p(\mathbf{R}^n)} \\
&\lesssim \|Tu\|_{W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}}
\end{aligned}$$

Hence,

$$|u|_{\tilde{W}_0^{m,p}(\mathbf{R}^n)} \lesssim \|Tu\|_{W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}}$$

Combining with Proposition 4.9 gives

$$\|u\|_{W_0^{m,p}(\mathbf{R}^n)/\mathbf{Q}_\ell} \lesssim \|Tu\|_{W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}}.$$

We conclude that the range of $T$ is a closed subspace of $W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}$. By means of Banach's Closed range theorem, and since the adjoint of $T$ is injective, we deduce that this range is nothing but the whole space $W_0^{m-2,p}(\mathbf{R}^n)/\mathbf{P}_{\ell-2}$. ∎

Theorem 4.6 stems directly from Lemma 4.14 and 4.16.

**Proof of Theorem 4.7**

Firstly, let $u \in \tilde{W}_0^{-m+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[m-n/p']}$. Then, for each $p \in \mathbf{P}_{[m-n/p']}$, we have

$$\begin{aligned} \langle Tu, p\rangle &= -\langle \Delta u, p\rangle + 2\left\langle \frac{\partial u}{\partial x_1}, p\right\rangle \\ &= -\langle u, \Delta p\rangle + 2\left\langle \frac{\partial u}{\partial x_1}, p\right\rangle \\ &= 0. \end{aligned}$$

Hence, $Tu \in W_0^{-m,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m-n/p']}$. On the other hand, Theorem 2.1 asserts that the operator

$$\Delta : W_0^{m,p'}(\mathbf{R}^n)/\mathbf{P}_{[m-n/p']} \to W_0^{m-2,p'}(\mathbf{R}^n)/\mathbf{P}_{[m-2-n/p']},$$

is an isomorphism if $m \geq 2$ and $n/p' \notin \{1, \dots, m\}$. It follows that $\Delta^k$ is an isomorphism between $W_0^{2k,p'}(\mathbf{R}^n)/\mathbf{P}_{[2k-n/p']}$ and $L^{p'}(\mathbf{R}^n)$ for $k \geq 1$ and $n/p' \notin \{1, \dots, 2k\}$. By duality and transposition $\Delta^k$ is also an isomorphism between $L^p(\mathbf{R}^n)$ and $W_0^{-2k,p}(\mathbf{R}^n) \perp \mathbf{P}_{[2k-n/p']}$. Let $\Delta^{-k}$ be its inverse. From Theorem 4.6, we know that $T$ is an isomorphism between $\tilde{W}_0^{2,p}(\mathbf{R}^n)/\mathbf{Q}_{[2-n/p]}$ and $L^p(\mathbf{R}^n)$ if $n/p \notin \{1, 2\}$. Moreover, it is quite clear that $\Delta^k$ is a continuous operator from $\tilde{W}_0^{2,p}(\mathbf{R}^n)$ into $\tilde{W}_0^{-2k+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[2k-2-n/p']}$. The operator $\Delta^k \circ T^{-1} \circ \Delta^{-k}$ is well defined and continuous from $W_0^{-2k,p}(\mathbf{R}^n) \perp \mathbf{P}_{[2k-n/p']}$ into $\tilde{W}_0^{-2k+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[2k-2-n/p']}$ (here $T^{-1}$ is from $L^p(\mathbf{R}^n)$ into $\tilde{W}_0^{2,p}(\mathbf{R}^n)/\mathbf{Q}_{[2-n/p]}$). Moreover, $T \circ (\Delta^k \circ T^{-1} \circ \Delta^{-k}) = I$, we deduce that $T$, considered as an operator from $\tilde{W}_0^{-2k+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[2k-n/p']}$ into $W_0^{-2k,p}(\mathbf{R}^n) \perp \mathbf{P}_{[2k-n/p']}$, is onto. It is also injective. It follows that is an isomorphism, thanks to Banach Theorem. ∎

## 5. The Oseen system in $\mathbf{R}^n$

In this section, we consider the *nonhomogeneous* Oseen problem: Given a vector field $\boldsymbol{f}$ and a function $h$, we look for a pair $(\boldsymbol{u}, \pi)$ satisfying

$$\begin{aligned} -\Delta \boldsymbol{u} + 2\frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f} \qquad &\text{in } \mathbf{R}^n, \\ \operatorname{div} \boldsymbol{u} = h \qquad &\text{in } \mathbf{R}^n. \end{aligned} \tag{5.1}$$

We start with a characterization of the kernel of the operator $(\boldsymbol{u}, \mu) \mapsto (T\boldsymbol{u} + \nabla \mu, \operatorname{div} \boldsymbol{u})$.

Proposition 5.1. *Let $m \geq 1$ be an integer, and set $\ell = -[n/p - m] - 1$. Then, $(\boldsymbol{u}, \pi) \in \tilde{\boldsymbol{W}}_0^{m,p}(\mathbf{R}^n) \times W_0^{m-1,p}(\mathbf{R}^n)$ is a solution of (5.1) if and only if $(\boldsymbol{u}, \pi) \in \mathcal{N}_\ell$, where*

$$\mathcal{N}_\ell = \left\{ (\boldsymbol{\lambda}, \mu) \in (\mathbf{Q}_\ell)^n \times \mathbf{P}_{\ell-1}^{\Delta}; -\Delta \boldsymbol{\lambda} + \frac{\partial \boldsymbol{\lambda}}{\partial x_1} + \nabla \mu = \mathbf{0}, \operatorname{div} \boldsymbol{\lambda} = 0 \right\}.$$

*Moreover, a pair $(\boldsymbol{\lambda}, \mu)$ belongs to $\mathcal{N}_\ell$ if and only if there exists a vector function $\boldsymbol{\Phi} = (\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_n) \in (\mathbf{P}_{\ell+2})^n$ such that $\operatorname{div} \boldsymbol{\Phi} \in \mathbf{Q}_{\ell+1}$, $(\Delta \circ T)\boldsymbol{\Phi}_i = 0$, $i = 1, \ldots, n$, and*

$$\begin{aligned} \boldsymbol{\lambda} &= \Delta \boldsymbol{\Phi} - \nabla(\operatorname{div} \boldsymbol{\Phi}) \\ \mu &= T(\operatorname{div} \boldsymbol{\Phi}). \end{aligned} \tag{5.2}$$

**Proof of Proposition 5.1**

Let $(\boldsymbol{u}, \pi) \in \mathcal{S}'(\mathbf{R}^n) \times \mathcal{S}'(\mathbf{R}^n)$ be a solution of (5.1), with $\boldsymbol{f} = \mathbf{0}$ and $h = 0$. Then taking the divergence of the first equation of (5.1), we obtain

$$\Delta \pi = 0.$$

Thus, $\pi$ is a harmonic polynomial function. Now, we have

$$\Delta\left( -\Delta \boldsymbol{u} + 2\frac{\partial \boldsymbol{u}}{\partial x_1} \right) = -\Delta(\nabla \pi) = 0.$$

It follows that

$$|\xi|^2(|\xi|^2 - i\xi_1)\hat{\boldsymbol{u}}(\xi) = 0.$$

Hence the support of $\hat{\boldsymbol{u}}$ is included in $\{0\}$ and consequently $\boldsymbol{u}$ is a polynomial function. If in addition $\boldsymbol{u} \in \tilde{W}_0^{m,p}(\mathbf{R}^n)^n$ and $\pi \in W_0^{m-1,p}(\mathbf{R}^n)$, then necessarily $\boldsymbol{u} \in (\mathbf{Q}_\ell)^n$ and $\pi \in \mathbf{P}_{\ell-1}^{\Delta}$. This completes the proof of the first assertion of Proposition 5.1. Now, according to the Lemma 4.14 there exists a function $r \in \mathbf{Q}_{\ell+1}$ such that $Tr = \pi$. The vector function $\boldsymbol{u} + \nabla r$ belongs to $(\mathbf{P}_\ell)^n$.

Hence, there exists a vectorial function $\boldsymbol{\varphi} \in (\mathbf{P}_{\ell+2})^n$ such that $\Delta\boldsymbol{\varphi} = \boldsymbol{u} + \nabla r$ (since $\Delta\mathbf{P}_{\ell+2} = \mathbf{P}_\ell$). Furthermore, by applying the divergence operator to this identity one deduces that the function $s = \operatorname{div}\boldsymbol{\varphi} - r$ is harmonic, and consequently belongs to $\mathbf{P}^{\Delta}_{\ell+1}$. Since $\operatorname{div}(\mathbf{P}^{\Delta}_{k+1}) = \mathbf{P}^{\Delta}_k$ (see [12] or [1]), there exists a function $\boldsymbol{\theta} \in (\mathbf{P}^{\Delta}_{\ell+2})^n$ such that $\operatorname{div}\boldsymbol{\theta} = -s$. Set $\boldsymbol{\Phi} = \boldsymbol{\varphi} + \boldsymbol{\theta} \in (\mathbf{P}_{\ell+2})^n$. Then, $\operatorname{div}\boldsymbol{\Phi} = r$, $\Delta\boldsymbol{\Phi} = \Delta\boldsymbol{\varphi}$. It follows that $\boldsymbol{u} = \Delta\boldsymbol{\Phi} - \nabla(\operatorname{div}\boldsymbol{\Phi})$ and $\pi = T(\operatorname{div}\boldsymbol{\Phi})$. Since $Tu_i + \partial_i\pi = 0$, we deduce that $\Delta(T\Phi_i) = 0$, $i = 1, \dots, n$ which completes the proof of (5.2). The converse is straightforward. Indeed, let $\boldsymbol{\Phi} = (\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_n) \in (\mathbf{P}_{\ell+2})^n$ such that $\operatorname{div}\boldsymbol{\Phi} \in \mathbf{Q}_{\ell+1}$ and $(\Delta \circ T)\boldsymbol{\Phi}_i = 0$, $i = 1, \dots, n$, and consider the pair $(\boldsymbol{\lambda}, \mu)$ given by (5.2). Obviously $\mu \in \mathbf{P}^{\Delta}_{\ell-1}$ since $T\mathbf{Q}_{\ell+1} = \mathbf{P}_\ell$. Moreover, $\Delta\boldsymbol{\Phi} \in (\mathbf{Q}_\ell)^n$ since $\Delta\boldsymbol{\Phi} \in (\mathbf{P}_\ell)^n$ and

$$2\frac{\partial(\Delta\boldsymbol{\Phi})}{\partial x_1} = T\Delta\boldsymbol{\Phi} + \Delta^2\boldsymbol{\Phi} = \Delta^2\boldsymbol{\Phi} \in (\mathbf{P}_{\ell-2})^n.$$

Thus, the pair $(\boldsymbol{\lambda}, \mu)$ belongs to $\mathcal{N}_\ell$. ■

Let us notice that $\mathcal{N}_\ell = \{(\mathbf{0}, 0)\}$ if $\ell < 0$, $\mathcal{N}_0 = \mathbf{R} \times \{0\}$ and $\mathcal{N}_1 = \mathbf{Q}_1^+ \times \mathbf{R}$. Our first existence result is for $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbf{R}^n)$ and $g \in L^p(\mathbf{R}^n)$. Note that a different proof of the next theorem, in the particular case $n = 3$, is given in [3].

Theorem 5.2. *Assume $n/p \neq 1$ and $n/p' \neq 1$.*
*Let $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$ and $h \in \tilde{W}_0^{0,p}(\mathbf{R}^n)$ satisfying*

$$\forall q \in \mathbf{P}_{[2-n/p']}, \qquad \left\langle \frac{\partial h}{\partial x_1}, q \right\rangle = 0 \tag{5.3}$$

*Then the Oseen system* (5.1) *has a unique solution $(\boldsymbol{u}, \pi) \in (\tilde{\boldsymbol{W}}_0^{1,p}(\mathbf{R}^n) \times L^p(\mathbf{R}^n))/\mathcal{N}_{[1-n/p]}$. Moreover, the following estimate holds*

$$\inf_{\boldsymbol{\lambda} \in \mathbf{P}_{[1-n/p]}} \|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\tilde{W}_0^{1,p}(\mathbf{R}^n)} + \|\pi\|_{L^p(\mathbf{R}^n)} \lesssim (\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbf{R}^n)} + \|h\|_{\tilde{W}_0^{0,p}(\mathbf{R}^n)}). \tag{5.4}$$

**Proof of Theorem 5.2**

1) Consider first $(\boldsymbol{u}, \pi) \in \tilde{\boldsymbol{W}}_0^{1,p}(\mathbf{R}^n) \times L^p(\mathbf{R}^n)$. Then $-\Delta\boldsymbol{u} + 2\dfrac{\partial\boldsymbol{u}}{\partial x_1} + \nabla\pi \in \mathbf{W}_0^{-1,p}(\mathbf{R}^n)$. Thus, due to the density of $\mathscr{D}(\mathbf{R}^n)$ in $\tilde{W}_0^{1,p}(\mathbf{R}^n)$, for any $\boldsymbol{\lambda} \in \tilde{\boldsymbol{W}}_0^{1,p'}(\mathbf{R}^n)$, we have

$$\left\langle -\Delta\boldsymbol{u} + 2\frac{\partial\boldsymbol{u}}{\partial x_1} + \nabla\pi, \lambda \right\rangle_{\mathbf{W}_0^{-1,p}(\mathbf{R}^n) \times \mathbf{W}_0^{1,p'}(\mathbf{R}^n)} = \left\langle \boldsymbol{u}, -\Delta\lambda - 2\frac{\partial\lambda}{\partial x_1} \right\rangle_{\mathbf{W}_0^{-1,p}(\mathbf{R}^n) \times \mathbf{W}_0^{1,p'}(\mathbf{R}^n)}$$
$$- \langle \pi, \operatorname{div}\boldsymbol{\lambda} \rangle_{L^p(\mathbf{R}^n) \times L^{p'}(\mathbf{R}^n)}.$$

Thus, necessarily $-\Delta\boldsymbol{u} + 2\dfrac{\partial\boldsymbol{u}}{\partial x_1} + \nabla\pi \in \mathbf{W}_0^{-1,p}(\mathbf{R}^3) \perp \mathbf{P}_{[1-n/p']}$.

2) The uniqueness is a straightforward consequence of Proposition 5.1.

3) Let us prove existence. Given $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\mathbf{R}^3) \perp \mathbf{P}_{[1-n/p']}$ and $h \in \tilde{W}_0^{0,p}(\mathbf{R}^n)$ satisfying (5.3), it is easy to see that $\operatorname{div} \boldsymbol{f} - Th \in W_0^{-2,p}(\mathbf{R}^n) \perp \mathbf{P}_{[2-n/p']}$. From Theorem 2.1 (applied with $m = \ell = -1$), we know that the Laplace operator defined by

$$\Delta : L^p(\mathbf{R}^n) \to W_0^{-2,p}(\mathbf{R}^n) \perp \mathbf{P}_{[2-n/p']}$$

is an isomorphism. Thus there exists a unique function $\pi \in L^p(\mathbf{R}^n)$ such that

$$\Delta\pi = \operatorname{div} \boldsymbol{f} - Th$$

satisfying the estimate

$$\begin{aligned} \|\pi\|_{L^p(\mathbf{R}^n)} &\lesssim \left\| \operatorname{div} \boldsymbol{f} + \Delta h - 2\frac{\partial h}{\partial x_1} \right\|_{W_0^{-2,p}(\mathbf{R}^n)} \\ &\lesssim (\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbf{R}^n)} + \|h\|_{\tilde{W}_0^{0,p}(\mathbf{R}^n)}). \end{aligned} \tag{5.5}$$

Hence, $\boldsymbol{f} - \nabla\pi \in \mathbf{W}_0^{-1,p}(\mathbf{R}^n)$. Furthermore, since the elements of $\mathbf{P}_{[1-n/p']}$ are at most constants, for any $\boldsymbol{\lambda} \in \mathbf{P}_{[1-n/p']}$, we can write

$$\langle \nabla\pi, \boldsymbol{\lambda} \rangle_{\mathbf{W}_0^{-1,p}(\mathbf{R}^n) \times \mathbf{W}_0^{1,p'}(\mathbf{R}^n)} = \langle \pi, \operatorname{div} \boldsymbol{\lambda} \rangle_{L^p(\mathbf{R}^n) \times L^{p'}(\mathbf{R}^n)} = 0.$$

We deduce that $\boldsymbol{f} - \nabla\pi \in \mathbf{W}_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$. By virtue of Theorem 4.5, there exists a vector field $\boldsymbol{u} \in \tilde{\boldsymbol{W}}_0^{1,p}(\mathbf{R}^n)$ such that

$$-\Delta\boldsymbol{u} + 2\frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{f} - \nabla\pi,$$

with the estimate

$$\inf_{\boldsymbol{\lambda} \in \mathbf{P}_{[1-n/p]}} \|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\tilde{\boldsymbol{W}}_0^{1,p}(\mathbf{R}^n)} \lesssim (\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbf{R}^n)} + \|\nabla\pi\|_{\mathbf{W}_0^{-1,p}(\mathbf{R}^n)}). \tag{5.6}$$

From (5.5) and (5.6), we obtain (5.4). It remains to prove that $\operatorname{div} \boldsymbol{u} = h$. Let us observe that $\operatorname{div} \boldsymbol{u} - h$ $(\in L^p(\mathbf{R}^n))$ satisfies

$$T(\operatorname{div} \boldsymbol{u} - h) = 0.$$

Combining with Lemma 4.4, we deduce that $\operatorname{div} \boldsymbol{u} - h$ is a polynomial function of $L^p(\mathbf{R}^n)$, which implies that $\operatorname{div} \boldsymbol{u} = h$, completing the proof. ∎

For our next existence result, we need to prove a preliminary result on polynomial functions that belong to $\mathbf{Q}_k^+$. To that end, we first begin with the following lemma.

Lemma 5.3. *Let $p$ be a function in $\mathbf{P}_\ell$ not depending on the variable $x_i$*

*for some i,* $2 \le i \le n$. *Then there exists a function* $q \in \mathbf{Q}_{\ell+2}$, *not depending on* $x_i$ *such that*

$$p = -\Delta q + 2\frac{\partial q}{\partial x_1}.$$

PROOF. The proof is analogous to that of Lemma 4.14. ■

PROPOSITION 5.4. *Let* $\ell \ge 0$ *be an integer. Then we have*

$$\mathbf{Q}_\ell^+ = \mathrm{div}(\mathbf{Q}_{\ell+1}^+).$$

PROOF. Let us begin with $\ell = 0$, and $p = c \in \mathbf{R}$. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbf{P}_1$, such that $\lambda_1 = 0$ and for any integer $i \ge 2$, $\lambda_i = \dfrac{1}{n-1} c x_i$. Then we easily see that $\boldsymbol{\lambda} \in \mathbf{Q}_1^+$ and

$$\mathrm{div}\, \boldsymbol{\lambda} = c.$$

Suppose now $\ell \ge 1$ and $p \in \mathbf{Q}_\ell^+$. Then, for any integer $i \ge 2$, $\dfrac{\partial p}{\partial x_i}$ $(x_i = 0)$ is a polynomial of $\mathbf{P}_{\ell-1}$ not depending on $x_i$. Hence, by virtue of Lemma 5.3, there exists a polynomial function $h_i \in \mathbf{Q}_{\ell+1}$ not depending on $x_i$, such that

$$-\Delta h_i + 2\frac{\partial h_i}{\partial x_1} = \frac{\partial p}{\partial x_i} \qquad (x_i = 0).$$

Next, define the vector field $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ as:

$$\lambda_1 = 0 \qquad \text{and} \qquad \lambda_i = \frac{1}{n-1}\int_0^{x_i} p(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)dt + h_i \quad \text{for } i \ge 2.$$

One can verifies that $\boldsymbol{\lambda} \in \mathbf{Q}_{\ell+1}^+$ and satisfies

$$\mathrm{div}\, \boldsymbol{\lambda} = p,$$

which completes the proof of the proposition. ■

We are now in a position to prove our next result.

THEOREM 5.5. *Let* $m \ge 2$ *be an integer and suppose that* $n/p \ne \{1, \dots, m\}$ *and* $n/p' \ne 1$ *if* $m = 2$. *Let* $\boldsymbol{f} \in \mathbf{W}_0^{m-2,p}(\mathbf{R}^n)$ *and* $h \in \tilde{W}_0^{m-1,p}(\mathbf{R}^n)$. *Then the Oseen system* (5.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in (\tilde{\boldsymbol{W}}_0^{m,p}(\mathbf{R}^n) \times W_0^{m-1,p}(\mathbf{R}^n))/\mathscr{N}_{[m-n/p]}$. *Moreover, the following estimate holds*

$$\inf_{(\boldsymbol{\lambda},\mu) \in \mathscr{N}_{[m-n/p]}} (\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\tilde{\boldsymbol{W}}_0^{m,p}(\mathbf{R}^n)} + \|\pi + \mu\|_{W_0^{m-1,p}(\mathbf{R}^n)})$$
$$\lesssim (\|\boldsymbol{f}\|_{\mathbf{W}_0^{m-2,p}(\mathbf{R}^n)} + \|h\|_{\tilde{W}_0^{m-1,p}(\mathbf{R}^n)}). \tag{5.7}$$

**Proof of Theorem 5.5**

1) If $(\boldsymbol{u}, \pi) \in \tilde{\boldsymbol{W}}_0^{m,p}(\mathbf{R}^n) \times W_0^{m-1,p}(\mathbf{R}^n)$ satisfies (5.1) with $\boldsymbol{f} = \mathbf{0}$ and $h = 0$, then $(\boldsymbol{u}, \pi) \in \mathcal{N}_{[m-n/p]}$, thanks to Proposition 5.1.

2) The beginning of the proof of existence is similar to that of the preceding theorem. Given $\boldsymbol{f} \in \mathbf{W}_0^{m-2,p}(\mathbf{R}^n)$ and $h \in \tilde{W}_0^{m-1,p}(\mathbf{R}^n)$, we have $\operatorname{div} \boldsymbol{f} - Th \in W_0^{m-3,p}(\mathbf{R}^n)$. Considering first the case $m \geq 3$, and using Theorem 2.1 1), we deduce the existence of a function $\pi \in W_0^{m-1,p}(\mathbf{R}^n)$ such that

$$\Delta\pi = \operatorname{div} \boldsymbol{f} - Th.$$

If $m = 2$, then, we easily see that $\operatorname{div} \boldsymbol{f} - Th \in W_0^{-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[1-n/p']}$. Again from Theorem 2.1, there exists a unique function $\pi \in W_0^{1,p}(\mathbf{R}^n)$ satisfying the previous equality. Thus summarizing, we conclude that for $m \geq 2$, there exists a function $\pi \in W_0^{m-1,p}(\mathbf{R}^n)$ satisfying the previous Laplace equation. Next, we see that $\boldsymbol{f} - \nabla\pi \in \mathbf{W}_0^{m-2,p}(\mathbf{R}^n)$. Thanks to Theorem 4.6, there exists a vector field $\boldsymbol{u} \in \tilde{\boldsymbol{W}}_0^{m,p}(\mathbf{R}^n)$ satisfying

$$-\Delta\boldsymbol{u} + 2\frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{f} - \nabla\pi.$$

It follows that $\operatorname{div} \boldsymbol{u} - h \in \mathbf{W}_0^{m-1,p}(\mathbf{R}^n)$ verifies

$$T(\operatorname{div} \boldsymbol{u} - h) = 0.$$

Therefore, $\operatorname{div} \boldsymbol{u} - h = q \in \mathbf{Q}^+_{[m-1-n/p]}$. Proposition 5.4 implies that there exists a polynomial $\boldsymbol{\lambda} \in \mathbf{Q}^+_{[m-n/p]} \subset \tilde{\boldsymbol{W}}_0^{m,p}(\mathbf{R}^n)$ such that

$$\operatorname{div} \boldsymbol{\lambda} = q.$$

Thus, $(\boldsymbol{u} - \boldsymbol{\lambda}, \pi) \in \tilde{\boldsymbol{W}}_0^{m,p}(\mathbf{R}^n) \times W_0^{m-1,p}(\mathbf{R}^n)$ is a solution of the system (5.1). ■

THEOREM 5.6. *Suppose that* $m \geq 2$, $n/p' \neq \{1, \dots, m\}$ *and* $n/p \neq \{1, 2[m/2] + 2 - m\}$. *Let* $\boldsymbol{f} \in \mathbf{W}_0^{-m,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m-n/p']}$ *and* $h \in \tilde{W}_0^{-m+1,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[m+1-n/p']}$. *Then the Oseen system* (5.1) *has a unique solution* $(\boldsymbol{u}, \pi)$ *in* $(\tilde{\boldsymbol{W}}_0^{-m+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[m-n/p']}) \times W_0^{-m+1,p}(\mathbf{R}^n)$. *Moreover, the following estimate holds*

$$\|\boldsymbol{u}\|_{\tilde{\boldsymbol{W}}_0^{-m+2,p}(\mathbf{R}^n)} + \|\pi\|_{W_0^{-m+1,p}(\mathbf{R}^n)} \lesssim (\|\boldsymbol{f}\|_{\mathbf{W}_0^{-m,p}(\mathbf{R}^n)} + \|h\|_{\tilde{W}_0^{-m+1,p}(\mathbf{R}^n)}). \tag{5.8}$$

**Proof of Theorem 5.6**

The proof is similar to the previous ones. Let $\boldsymbol{f} \in \mathbf{W}_0^{-m,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m-n/p']}$ and $h \in \tilde{W}_0^{-m+1,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[m+1-n/p']}$. Then, $\operatorname{div} \boldsymbol{f} - Th \in \mathbf{W}_0^{-m-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m+1-n/p']}$. Now, since the operator

$$\Delta : W_0^{-m+1,p}(\mathbf{R}^n) \to W_0^{-m-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m+1-n/p']}$$

is an isomorphism, there exists a unique function $\pi \in W_0^{-m+1,p}(\mathbf{R}^n)$ such that

$$\Delta\pi = \operatorname{div} \boldsymbol{f} - Th.$$

Now since $\Delta\pi \in W_0^{-m-1,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m+1-n/p']}$, we deduce that $\nabla\pi \in \mathbf{W}_0^{-m,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m-n/p']}$ which implies that $\boldsymbol{f} - \nabla\pi \in \mathbf{W}_0^{-m,p}(\mathbf{R}^n) \perp \mathbf{P}_{[m-n/p']}$. Hence, using Theorem 4.7, there exists a unique vector field $\boldsymbol{u} \in \tilde{\boldsymbol{W}}_0^{-m+2,p}(\mathbf{R}^n) \perp\perp \mathbf{P}_{[m-n/p']}$ such that

$$-\Delta\boldsymbol{u} + 2\frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{f} - \nabla\pi.$$

Finally, since the space $\mathbf{W}_0^{-m+1,p}(\mathbf{R}^n)$ does not contain non trivial polynomial functions, we easily deduce that $\operatorname{div} \boldsymbol{u} = h$. ■

## Appendix A. Proof of Proposition 3.2

Suppose that $n \geq 3$

(a) We have

$$\int_{\mathbf{R}^n} |\mathcal{O}(\boldsymbol{x})|^p d\boldsymbol{x} = \int_{|x| \geq 2x_1} |\mathcal{O}(\boldsymbol{x})|^p d\boldsymbol{x} + \int_{|x| \leq 2x_1} |\mathcal{O}(\boldsymbol{x})|^p d\boldsymbol{x}$$

If $|x| \geq 2x_1$, then (2.1) gives

$$|\mathcal{O}(\boldsymbol{x})| \leq c\frac{(1+|x|)^{(n-3)/2}}{|x|^{n-2}} e^{x_1-|x|} \leq c\frac{(1+|x|)^{(n-3)/2}}{|x|^{n-2}} e^{-|x|/2},$$

Thus,

$$\int_{|x| \geq 2x_1} |\mathcal{O}(\boldsymbol{x})|^p d\boldsymbol{x} \leq \int_{\mathbf{R}^n} \frac{(1+|x|)^{(n-3)/2}}{|x|^{p(n-2)}} e^{-p|x|/2}\, dx < +\infty,$$

if $p(n-2) < n$. Now, let $\alpha$ be a real, $0 < \alpha < 1$. Using (2.1) in each region $\{\boldsymbol{x}; (1+\alpha^{k+1})x_1 \leq |x| \leq (1+\alpha^k)x_1\}$ gives

$$\int_{|x| \leq 2x_1} |\mathcal{O}(\boldsymbol{x})|^p d\boldsymbol{x} = \sum_{k=0}^{+\infty} \int_{(1+\alpha^{k+1})x_1 \leq |x| \leq (1+\alpha^k)x_1} |\mathcal{O}(\boldsymbol{x})|^p dx.$$

$$\leq c_2 \sum_{k=0}^{+\infty} \int_0^{+\infty} \left( \int_{[(1+\alpha^{k+1})^2-1]^{1/2}x_1}^{[(1+\alpha^k)^2-1]^{1/2}x_1} \rho_1^{n-2}\, d\rho_1 \right) \frac{(1+x_1)^{(n-3)/2} e^{-p\alpha^{k+1}x_1}}{x_1^{(n-2)p}}\, dx_1,$$

$$\leq c_3(\alpha) \sum_{k=0}^{+\infty} \alpha^k \int_0^{+\infty} \frac{(1+x_1)^{(n-3)/2} e^{-p\alpha^{k+1}x_1}}{x_1^{(n-2)p-n+1}} \, dx_1,$$

$$\leq c_4(\alpha) \sum_{k=0}^{+\infty} \alpha^{[(n-1)p/2-n+1](k+1)} \int_0^{+\infty} \frac{(1+t)^{(n-3)/2} e^{-pt}}{t^{(n-2)p-n+1}} \, dt,$$

where $\rho_1 = (x_2^2 + \cdots + x_n^2)^{1/2}$. Hence,

$$\int_{|x| \leq 2x_1} |\mathcal{O}(\boldsymbol{x})|^p d\boldsymbol{x} < +\infty$$

if $2 < p < n/(n-2)$. This condition is possible only if $n = 3$, and this proves part (a).

(b) If $|x| \geq 2x_1$, then (2.1) gives again

$$|\mathcal{O}(\boldsymbol{x}) - g_2(\boldsymbol{x})| \leq c \frac{(1+|x|)^{(n-3)/2}}{|x|^{n-2}} e^{x_1 - |x|} |1 - e^{-x_1}| \leq c \frac{(1+|x|)^{(n-3)/2}}{|x|^{n-3}} e^{-|x|/2},$$

Similarly, in each region $\{\boldsymbol{x}; (1+\alpha^{k+1})x_1 \leq |x| \leq (1+\alpha^k)x_1\}$, we have

$$|\mathcal{O}(\boldsymbol{x}) - g_2(\boldsymbol{x})| \leq c_1(\alpha) \frac{(1+x_1)^{(n-3)/2}}{x_1^{n-2}} e^{x_1 - |x|} |1 - e^{-x_1}|$$

$$\leq c_2(\alpha) \frac{(1+x_1)^{(n-5)/2}}{x_1^{n-3}} e^{-\alpha^{k+1}x_1},$$

where we used the inequality $(1+x_1)|1-e^{-x_1}| \leq cx_1$. The constants $c_1$ and $c_2$ do not depend on $k$. Similarly we prove that

$$\int_{\mathbf{R}^n} |\mathcal{O}(\boldsymbol{x}) - g_2|^p d\boldsymbol{x} < +\infty,$$

if $2 < p < n/(n-3)$, which is only possible if $3 \leq n \leq 5$. If $n = 3$, we have clearly $\mathcal{O}(\boldsymbol{x}) - g_2 \in L^\infty(\mathbf{R}^3)$.

## Appendix B. Proof of Corollary 3.4

Part (a) follows from Proposition 3.2 and Young's inequality

$$\|\mathcal{O} * f\|_{L^r(\mathbf{R}^3)} \lesssim \|\mathcal{O}\|_{L^\theta(\mathbf{R}^3)} \|f\|_{L^p(\mathbf{R}^3)},$$

with

$$\frac{1}{\theta} = 1 + \frac{1}{r} - \frac{1}{p} \qquad (\text{hence } 2 < \theta < 3).$$

When $r = p_2^*$, $1 < p < 3/2$, one can use the inequality $|\mathcal{O} * f| \le I_2(|f|)$, where $I_2(f)$ is the Riesz potential of $f$ defined by (cf. [22])

$$I_2(f)(\boldsymbol{x}) = \int_{\mathbf{R}^n} \frac{f(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{n-2}}. \tag{5.9}$$

Part (b) comes also from Proposition 3.2 and Young's inequality.

Part (c) follows from the following lemma combined with Marcinkiewicz interpolation theorem (see for instance [22], Appendix B).

LEMMA 5.7. *Suppose that* $1 < p < 3/2$ *and* $n = 3$. *If* $f \in L^p(\mathbf{R}^3)$ *then*

$$m(\{\boldsymbol{x}; |(\mathcal{O}(\boldsymbol{x}) - g_2) * f| > \lambda\} \le \left(A_p \frac{\|f\|_{L^p(\mathbf{R}^3)}}{\lambda}\right)^q,$$

*where* $m$ *denotes the Lebesgue measure and*

$$q = \frac{p(p+3)}{(3-2p)}.$$

PROOF. Following Stein (see [22], Chap. V.1.2 Theorem 1), we set

$$K_1(\boldsymbol{x}) = \mathcal{O}(\boldsymbol{x}) - g_2(\boldsymbol{x}) \quad \text{if } |x| \le \mu, \qquad K_1(\boldsymbol{x}) = 0 \quad \text{if } |x| > \mu$$

$$K_2(\boldsymbol{x}) = \mathcal{O}(\boldsymbol{x}) - g_2(\boldsymbol{x}) \quad \text{if } |x| > \mu, \qquad K_2(\boldsymbol{x}) = 0 \quad \text{if } |x| \le \mu.$$

Without loss of generality, suppose that $\|f\|_{L^p(\mathbf{R}^n)} = 1$. Then,

$$|K_1(\boldsymbol{x})| \le c \frac{|e^{x_1 - |x|} - e^{-|x|}|}{|x|} \le c \frac{|x_1|}{|x|} \le c,$$

and we deduce that $K_1 \in L^1(\mathbf{R}^3)$, Moreover,

$$\|K_1\|_{L^1(\mathbf{R}^3)} \le c \int_{|x| \le \mu} d\boldsymbol{x} = \mu^3.$$

On the other hand,

$$\|K_2 * f\|_{L^\infty(\mathbf{R}^3)} \le \|K_2\|_{L^{p'}(\mathbf{R}^3)} \|f\|_{L^p(\mathbf{R}^3)}.$$

We have also $|K_2(\boldsymbol{x})| \le \dfrac{c}{|x|}$. Thus,

$$\int_{\mathbf{R}^3} |K_2(\boldsymbol{x})|^{p'} d\boldsymbol{x} \le c \int_{|x| \ge \mu} \frac{1}{|x|^{p'}} \, dx = \mu^{3-p'}.$$

We choose $\mu = \lambda^{p'/(3-p')}$. Hence,

$$\|K_2 * f\|_{L^\infty(\mathbf{R}^3)} \le \|K_2\|_{L^{p'}(\mathbf{R}^3)} \|f\|_{L^p(\mathbf{R}^3)} \le \lambda,$$

and

$$m(\{\boldsymbol{x}; |K_2 * f| > \lambda\}) = 0.$$

Moreover,

$$\|K_1\|_{L^1(\mathbf{R}^3)} \leq \lambda^{3p'/(3-p')}.$$

We get

$$\frac{\|K_1 * f\|^p_{L^p(\mathbf{R}^2)}}{\lambda^p} \leq \lambda^{3qp/(3-q)-p} = \lambda^{-(p+3)p/(3-2p)}.$$

Thus,

$$\begin{aligned} m(\{\boldsymbol{x}; |(\mathcal{O}(\boldsymbol{x}) - g_2) * f| > \lambda\}) &\leq m(\{\boldsymbol{x}; |K_1 * f| > \lambda\}) + m(\{\boldsymbol{x}; |K_2 * f| > \lambda\}). \\ &\lesssim \lambda^{-p}\|K_1 * f\|^p_{L^p(\mathbf{R}^3)} \\ &\lesssim \lambda^{-p}\|K_1\|^p_{L^1(\mathbf{R}^3)}\|f\|^p_{L^p(\mathbf{R}^3)} \\ &\lesssim \lambda^{3p'p/(3-p')-p} = \lambda^{-(p+3)p/(3-2p)}. \quad \blacksquare \end{aligned}$$

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