Hiroshima Math. J.
35 (2005), 425–451

Parametrizations of infinite biconvex sets in affine root systems

Ken Ito

(Received June 22, 2004)
(Revised January 31, 2005)

Abstract. We investigate relationships between the set $\mathfrak{B}^{\infty}$ of all infinite "biconvex" sets in the positive root system Δ_+ of an arbitrary untwisted affine Lie algebra $\mathfrak{g}$ and the set $\mathscr{W}^{\infty}$ of all infinite "reduced word" of the Weyl group of $\mathfrak{g}$. The study is applied to the classification of "convex orders" on Δ_+ ([5]), which is indispensable to construct "convex bases" of Poincaré-Birkhoff-Witt type of the strictly upper triangular subalgebra U_q^+ of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. We construct a set $\mathscr{P}$ by using data of the underlying finite-dimensional simple Lie algebra, and bijective mappings $\nabla : \mathscr{P} \to \mathfrak{B}^{\infty}$ and $\chi : \mathscr{P} \to W^{\infty}$ such that $\nabla = \Phi^{\infty} \circ \chi$, where W^{∞} is a quotient set of $\mathscr{W}^{\infty}$ and $\Phi^{\infty} : W^{\infty} \to \mathfrak{B}^{\infty}$ is a natural injective mapping.

1. Introduction

Let Δ be the root system of a Kac-Moody Lie algebra $\mathfrak{g}$, Δ_+ (resp. Δ_-) the set of all positive (resp. negative) roots relative to the root basis $\Pi = \{\alpha_i \mid i \in \mathbf{I}\}$, and $W = \langle s_i \mid i \in \mathbf{I}\rangle$ the Weyl group of $\mathfrak{g}$, where s_i is the reflection associated with α_i. Then (W, S) is a Coxeter system with $S = \{s_i \mid i \in \mathbf{I}\}$ ([6]). We call an infinite sequence $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbf{N}} \in S^{\mathbf{N}}$ an *infinite reduced word* of (W, S) if the length of the element $[\boldsymbol{s}|_p] := \boldsymbol{s}(1) \cdots \boldsymbol{s}(p) \in W$ is p for each $p \in \mathbf{N}$, and call a subset $B \subset \Delta_+$ a *biconvex set* if it satisfies the following conditions:

C(i) $\beta, \gamma \in B,\ \beta + \gamma \in \Delta_+ \Rightarrow \beta + \gamma \in B$;

C(ii) $\beta, \gamma \in \Delta_+ \backslash B,\ \beta + \gamma \in \Delta_+ \Rightarrow \beta + \gamma \in \Delta_+ \backslash B$.

If, in addition, B is a subset of the set Δ_+^{re} of all positive real roots, then B is called a *real biconvex set*. The purpose of this article is to investigate in detail relationships between infinite reduced words and infinite real biconvex sets in the case where $\mathfrak{g}$ is an arbitrary untwisted affine Lie algebra.

Before explaining the detail of our work, we will explain the background of the theory of infinite reduced words and infinite real biconvex sets. The motive of this study is related to the construction of *convex bases* of the strictly upper triangular subalgebra U_q^+ of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Convex bases are Poincaré-Birkhoff-Witt type bases with a convex

2000 *Mathematics Subject Classification*. 20F55, 17B37.

Key words and phrases. affine root system, biconvex set, Weyl group, reduced word, quantum algebra, convex basis, convex order.

property concerning the "q-commutator" of two "q-root vectors" of U_q^+. The convex property is useful for calculating values of the standard Hopf pairing between U_q^+ and the strictly lower triangular subalgebra U_q^-, and is applied to explicit calculations of the universal R-matrix of $U_q(\mathfrak{g})$ ([7], [8]). By the way, each convex basis of U_q^+ is formed by monomials in certain q-root vectors of U_q^+ multiplied in a predetermined total order with a convex property on Δ_+. Such a total order on Δ_+ is called a *convex order*.

In the case where $\mathfrak{g}$ is an arbitrary finite-dimensional simple Lie algebra, it is known that there exists a natural bijective mapping from the set of all convex orders on Δ_+ to the set of all reduced expressions of the longest element of W ([10]), and G. Lusztig constructed convex bases of U_q^+ associated with all reduced expressions of the longest element of W by using a braid group action on $U_q(\mathfrak{g})$ ([9]). Therefore all convex bases of U_q^+ was constructed in the finite case.

In the case where $\mathfrak{g}$ is an arbitrary untwisted affine Lie algebra, convex orders on Δ_+ are closely related to infinite reduced words of (W, S). More precisely, each infinite reduced word naturally corresponds to a "*1-row type*" convex order on an infinite real biconvex set. In [5], we showed that each convex order on Δ_+ is made from each couple of "*maximal*" (infinite) real biconvex sets with convex orders which divides Δ_+^{re} into two parts. To analyze convex orders on maximal real biconvex sets, it is important to consider the following two problems: (1) classify all infinite real biconvex sets; (2) describe in detail relationships between the set of all infinite reduced words and the set of all infinite real biconvex sets. In this article, we concentrate on the two problems above for the untwisted affine case. Applying results in this article to [5], we classified all convex orders on Δ_+, and then gave a general method of constructing convex orders on Δ_+ for the untwisted affine case. On the other hand, in [1], J. Beck constructed convex bases of U_q^+ associated with convex orders on Δ_+ arisen from a certain couple of maximal real biconvex sets with 1-row type convex orders which divides Δ_+^{re} into two parts. However, we seem that it is possible to generalize Beck's construction, since we find in [5] that there exist several types of convex orders called "*n-row types*" on each maximal real biconvex set which are not used in Beck's construction. We are preparing an article concerning to construct all convex bases of U_q^+ associated with all convex order on Δ_+ by generalizing Beck's construction for the untwisted affine Lie algebra.

This paper is organized as follows. In Section 2, we first give the definition of biconvex sets for a class of root systems with Coxeter group actions, and then state several fundamental results. We next define infinite reduced words for each Coxeter system (W, S) and an equivalence relation $\sim$ on the set $\mathscr{W}^\infty$ of all infinite reduced words, and then define W^∞ to be the quotient set of

$\mathscr{W}^{\infty}$ relative to $\sim$. We next define an injective mapping $\Phi^{\infty}: W^{\infty} \to \mathfrak{B}^{\infty}$, where $\mathfrak{B}^{\infty}$ is the set of all infinite real biconvex sets. At the end of Section 2, we define a left action of W on W^{∞} which plays an important role in the proof of the main theorem. In Section 3, we introduce some notation for the untwisted affine cases. In Section 4, we give preliminary results for classical root systems. From Section 5 to Section 7, we treat only the untwisted affine case. In Section 5, we give several methods of constructing biconvex sets. In Section 6, we give a parametrization of real biconvex sets. In Section 7, we give the following main results.

Main Theorem. *If* $\mathfrak{g}$ *is an arbitrary untwisted affine Lie algebra, there exist parametrizations* (bijective mappings) $\nabla: \mathscr{P} \to \mathfrak{B}^{\infty}$ *and* $\chi: \mathscr{P} \to W^{\infty}$ *such that the following diagram is commutative:*

$$\begin{tikzcd} & \mathfrak{B}^{\infty} & \\ W^{\infty} \arrow[ur, "\Phi^{\infty}"] & & \mathscr{P}, \arrow[ul, "\nabla"'] \arrow[ll, "\chi"] \end{tikzcd}$$

where the set $\mathscr{P}$ *is defined by using data of the underlying finite-dimensional simple Lie algebra* $\mathring{\mathfrak{g}}$ (see Definition 6.5). *In particular,* Φ^{∞} *is bijective. Moreover,* W^{∞} *decomposes into the direct finite sum of orbits relative to a left action of* W.

Note that P. Cellini and P. Papi showed in [11] that if B is an infinite real biconvex set, then there exist $v, t \in W$ such that t is a translation, $\ell(vt) = \ell(v) + \ell(t)$, and $B = \bigcup_{k \geq 0} \Phi(vt^k)$, where $\Phi(z) = \{\beta \in \Delta_+ \mid z^{-1}(\beta) \in \Delta_-\}$.

2. Definitions and several results

Let $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, and $\mathbf{N}$ be the set of the real numbers, the rational numbers, the integers, and the positive integers, respectively. We denote by $\mathbf{N}_n$ the set $\{m \in \mathbf{N} \mid m \leq n\}$ for each $n \in \mathbf{N}$, and set $\mathbf{N}_{\infty} := \mathbf{N}$ and $\mathbf{N}_* := \mathbf{N} \amalg \{\infty\}$, where ∞ is a symbol. We extend the usual order $\leq$ on $\mathbf{N}$ to a total order on $\mathbf{N}_*$ by setting $n < \infty$ for each $n \in \mathbf{N}$. We also set $\infty + n = n + \infty = \infty n = n \infty = \infty$ for each $n \in \mathbf{N}_*$. We denote by $\#U$ the cardinality of a set U, and write $\#U = \infty$ if U is an infinite set. When A and B are subsets of U, we write $A \dot{\subset} B$ or $B \dot{\supset} A$ if $\#(A \backslash B) < \infty$, and write $A \doteq B$ if both $A \dot{\subset} B$ and $A \dot{\supset} B$. Then $\doteq$ is an equivalence relation on the power set of U. For each $F \subset \mathbf{R}$ and $a \in \mathbf{R}$, we set $F_{\geq a} := \{b \in F \mid b \geq a\}$ and $F_{>a} := \{b \in F \mid b > a\}$.

Let $\mathbf{F}$ be a subfield of $\mathbf{R}$, W a group generated by a set S of involutive generators (i.e., $s \neq 1$, $s^2 = 1$, $\forall s \in S$), and (V, Δ, Π) a triplet satisfying the following conditions ${}_{\mathbf{F}}\mathrm{R}(\mathrm{i})$–${}_{\mathbf{F}}\mathrm{R}(\mathrm{iv})$.

${}_{\mathbf{F}}\mathrm{R}$(i) It consists of a representation space V of W over $\mathbf{F}$, a W-invariant subset $\Delta \subset V \backslash \{0\}$ which is symmetric (i.e., $\Delta = -\Delta$), and a subset $\Pi = \{\alpha_s \mid s \in S\} \subset \Delta$.

${}_{\mathbf{F}}\mathrm{R}$(ii) Each element of Δ can be written as $\sum_{s \in S} a_s \alpha_s$ with either $a_s \in \mathbf{F}_{\geq 0}$ for all $s \in S$ or $a_s \in \mathbf{F}_{\leq 0}$ for all $s \in S$, but not in both ways. Accordingly, we write $\alpha > 0$ or $\alpha < 0$, and set $\Delta_+ := \{\alpha \in \Delta \mid \alpha > 0\}$ and $\Delta_- := \{\alpha \in \Delta \mid \alpha < 0\}$.

${}_{\mathbf{F}}\mathrm{R}$(iii) For each $s \in S$, $s(\alpha_s) = -\alpha_s$ and $s(\Delta_+ \backslash \{\alpha_s\}) = \Delta_+ \backslash \{\alpha_s\}$.

${}_{\mathbf{F}}\mathrm{R}$(iv) If $w \in W$ and $s, s' \in S$ satisfy $w(\alpha_{s'}) = \alpha_s$, then $ws'w^{-1} = s$.

Definition 2.1. Define subsets Δ^{re}, Δ^{im}, $\Delta^{re}_{\pm}$, and $\Delta^{im}_{\pm}$ of Δ by setting

$$\Delta^{re} := \{w(\alpha_s) \mid w \in W, s \in S\}, \qquad \Delta^{im} := \Delta \backslash \Delta^{re},$$

$$\Delta^{re}_{\pm} := \Delta^{re} \cap \Delta_{\pm}, \qquad \Delta^{im}_{\pm} := \Delta^{im} \cap \Delta_{\pm}.$$

Note that W stabilizes Δ^{re} and Δ^{im}_{+}. For each $y \in W$, we set

$$\Phi(y) := \{\beta \in \Delta_+ \mid y^{-1}(\beta) < 0\}.$$

Note that $\Phi(y) \subset \Delta^{re}_{+}$.

Theorem 2.2 ([4]). *The pair (W, S) is a Coxeter system, i.e., it satisfies the exchange condition. Moreover, if $y = s_1 s_2 \cdots s_n$ with $n \in \mathbf{N}$ and $s_1, s_2, \ldots, s_n \in S$ is a reduced expression of an element $y \in W \backslash \{1\}$, then*

$$\Phi(y) = \{\alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \cdots s_{n-1}(\alpha_{s_n})\}$$

and the elements of $\Phi(y)$ displayed above are distinct from each other. In particular, $\#\Phi(y) = \ell(y)$, where $\ell : W \to \mathbf{Z}_{\geq 0}$ is the length function of (W, S).

Remarks. (1) The action of W on V is faithful. Indeed, if $y = id_V$ for $y \in W$, then $\Phi(y) = \emptyset$, and hence $y = 1$. Therefore we may regards W as a subgroup of $\mathrm{GL}(V)$.

(2) For each Coxeter system (W, S), a triplet (V, Δ, Π) is called a *root system* of (W, S) over $\mathbf{F}$ if it satisfies the conditions ${}_{\mathbf{F}}\mathrm{R}$(i)–${}_{\mathbf{F}}\mathrm{R}$(iv).

(3) Let $\sigma : W \to \mathrm{GL}(V)$ be the geometric representation of a Coxeter system (W, S) (cf. [2]), where V is a real vector space with a basis $\Pi = \{\alpha_s \mid s \in S\}$. Then (V, Δ, Π) is a root system of (W, S) over $\mathbf{R}$ (cf. [4]), where $\Delta = \{\sigma(w)(\alpha_s) \mid w \in W, s \in S\}$. We call it the root system associated with the geometric representation.

(4) Let $\mathfrak{g}$ be a Kac-Moody Lie algebra over $\mathbf{Q}$ with $\mathfrak{h}$ the Cartan subalgebra, $\Delta \subset \mathfrak{h}^* \backslash \{0\}$ the root system of $\mathfrak{g}$, $\Pi = \{\alpha_i \mid i \in \mathbf{I}\}$ a root basis of Δ, and $W = \langle s_i \mid i \in \mathbf{I} \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the Weyl group of $\mathfrak{g}$, where $\mathfrak{h}^*$ is the dual vector space of $\mathfrak{h}$ and s_i is the simple reflection associated with α_i (cf. [6]). Set $\mathfrak{h}^{*\prime} := \mathrm{span}_{\mathbf{Q}} \Pi \subset \mathfrak{h}^*$ and $S := \{s_i \mid i \in \mathbf{I}\}$. Then (W, S) is a Coxeter system and $(\mathfrak{h}^{*\prime}, \Delta, \Pi)$ is a root system of (W, S) over $\mathbf{Q}$.

LEMMA 2.3. *Let y_1 and y_2 be elements of W.*

(1) *We have $\Phi(y_1y_2)\backslash\Phi(y_1) \subset y_1\Phi(y_2)$.*

(2) *If $y_1\Phi(y_2) \subset \Delta_+$, then $\Phi(y_1) \amalg y_1\Phi(y_2) = \Phi(y_1y_2)$.*

(3) *If $\Phi(y_1) \subset \Phi(y_2)$, then $\Phi(y_2) = \Phi(y_1) \amalg y_1\Phi(y_1^{-1}y_2)$.*

(4) *The following two conditions are equivalent:*

$$\text{(i)}\quad \ell(y_2) - \ell(y_1) = \ell(y_1^{-1}y_2); \qquad \text{(ii)}\quad \Phi(y_1) \subset \Phi(y_2).$$

PROOF. (1) Suppose that $\beta \in \Phi(y_1y_2)\backslash\Phi(y_1)$. Then we have $y_1^{-1}(\beta) > 0$ and $y_2^{-1}(y_1^{-1}(\beta)) < 0$. Thus we get $y_1^{-1}(\beta) \in \Phi(y_2)$ or $\beta \in y_1\Phi(y_2)$.

(2) If $\beta \in y_1\Phi(y_2)$ then $y_1^{-1}(\beta) > 0$, and hence $\beta \notin \Phi(y_1)$. Thus we get $\Phi(y_1) \cap y_1\Phi(y_2) = \emptyset$. Hence, by (1) we have $\Phi(y_1y_2) \subset \Phi(y_1) \amalg y_1\Phi(y_2)$. We next prove that $\Phi(y_1) \subset \Phi(y_1y_2)$. Suppose that $\beta \in \Phi(y_1)$ satisfies $\beta \notin \Phi(y_1y_2)$. Then we have $y_1^{-1}(\beta) < 0$ and $y_2^{-1}(y_1^{-1}(\beta)) > 0$, which imply $-y_1^{-1}(\beta) \in \Phi(y_2)$. This contradicts the assumption. Thus we get $\Phi(y_1) \subset \Phi(y_1y_2)$. We next prove that $y_1\Phi(y_2) \subset \Phi(y_1y_2)$. If $\beta \in y_1\Phi(y_2)$ then $y_1^{-1}(\beta) \in \Phi(y_2)$, and hence $y_2^{-1}(y_1^{-1}(\beta)) < 0$. Thus we get $\beta \in \Phi(y_1y_2)$. Therefore $\Phi(y_1) \amalg y_1\Phi(y_2) \subset \Phi(y_1y_2)$.

(3) We first prove that $y_1\Phi(y_1^{-1}y_2) \subset \Delta_+$. Suppose that $\beta \in \Phi(y_1^{-1}y_2)$ satisfies $y_1(\beta) < 0$. Then we have $-y_1(\beta) \in \Phi(y_1) \subset \Phi(y_2)$, which implies $y_2^{-1}y_1(\beta) > 0$. This contradicts $\beta \in \Phi(y_1^{-1}y_2)$. Thus we get $y_1\Phi(y_1^{-1}y_2) \subset \Delta_+$, and hence $\Phi(y_2) = \Phi(y_1) \amalg y_1\Phi(y_1^{-1}y_2)$ by (2).

(4)(i) $\Rightarrow$ (ii) By Theorem 2.2, we have

$$\begin{aligned}\ell(y_2) - \ell(y_1) &\leq \# y_1^{-1}\{\Phi(y_2)\backslash\Phi(y_1)\} \\ &\leq \#\Phi(y_1^{-1}y_2) = \ell(y_1^{-1}y_2) = \ell(y_2) - \ell(y_1),\end{aligned}$$

where the second inequality follows from (1). Thus we get $\# y_1^{-1}\{\Phi(y_2)\backslash\Phi(y_1)\} = \ell(y_2) - \ell(y_1)$, and hence $\Phi(y_1) \subset \Phi(y_2)$.

(ii) $\Rightarrow$ (i) By (3) and Theorem 2.2, we get $\ell(y_2) = \ell(y_1) + \ell(y_1^{-1}y_2)$. □

DEFINITION 2.4. For subsets $A, B \subset \Delta_+$ satisfying $B \subset A$, we call B a *convex set in A* if it satisfies the following condition:

$C(i)_A$ $\quad \beta, \gamma \in B,\ \beta + \gamma \in A \Rightarrow \beta + \gamma \in B.$

We also call B a *coconvex set in A* if it satisfies the following condition:

$C(ii)_A$ $\quad \beta, \gamma \in A\backslash B,\ \beta + \gamma \in A \Rightarrow \beta + \gamma \in A\backslash B.$

Note that B is a coconvex set in A if and only if $A\backslash B$ is a convex set in A. Furthermore, we call B a *biconvex set in A* if B is both a convex set in A and a coconvex set in A. If, in addition, $B \subset \Delta_+^{re}$, then B is said to be a *real convex set in A*, a *real coconvex set in A*, or a *real biconvex set in A* if B is a convex set in A, a coconvex set in A, or a biconvex set in A, respectively.

We will say simply that B is a *convex set*, a *real convex set*, a *coconvex set*, a *real coconvex set*, a *biconvex set*, or a *real biconvex set* if B is a convex set in Δ_+, a real convex set in Δ_+, a coconvex set in Δ_+, a real coconvex set in Δ_+, a biconvex set in Δ_+, or a real biconvex set in Δ_+, respectively. We denote $\mathrm{C(i)}_{\Delta_+}$ and $\mathrm{C(ii)}_{\Delta_+}$ simply by C(i) and C(ii), respectively. Let $\mathfrak{B}$ be the set of all finite biconvex sets and $\mathfrak{B}^\infty$ the set of all infinite real biconvex sets. We set $\mathfrak{B}^* := \mathfrak{B} \amalg \mathfrak{B}^\infty$.

Remark. The condition $\mathrm{C(ii)}_A$ is equivalent to the following condition:

$$\beta, \gamma \in A,\ \beta + \gamma \in B \Rightarrow \beta \in B \ \text{ or } \ \gamma \in B.$$

For each couple of subsets $A, B \subset \Delta$, we set

$$A \dotplus B := \{\alpha + \beta \,|\, \alpha \in A, \beta \in B\} \cap \Delta.$$

Lemma 2.5. *Let A, B, and C be subsets of Δ_+ satisfying $B, C \subset A$, and $\{B_\lambda\}_{\lambda \in \Lambda}$ a family of subsets of A.*

(1) *If B is a biconvex set in A, then $A \backslash B$ is biconvex in A.*

(2) *If B is a biconvex set in A, then $B \cap C$ is a biconvex set in C.*

(3) *Suppose that $B \subset C$ and C is a convex set in A. Then B is a convex set in C if and only if B is a convex set in A.*

(4) *If $(B_\lambda \dotplus B_{\lambda'}) \cap A \subset \bigcup_{\lambda \in \Lambda} B_\lambda$ for each pair $\lambda, \lambda' \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} B_\lambda$ is a convex set in A.*

(5) *If B_λ is a convex set in A for each $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} B_\lambda$ is a convex set in A.*

(6) *If B_λ is a biconvex set in A for each λ and $\preceq$ is a total order on Λ such that $B_\lambda \subseteq B_{\lambda'}$ for each $\lambda \preceq \lambda'$, then both $\bigcup_{\lambda \in \Lambda} B_\lambda$ and $\bigcap_{\lambda \in \Lambda} B_\lambda$ are biconvex sets in A.*

Proof. (1)–(5) They are obvious.

(6) Set $B_1 := \bigcup_{\lambda \in \Lambda} B_\lambda$. By the assumption on the total order $\preceq$, the family $\{B_\lambda\}_{\lambda \in \Lambda}$ satisfies the sufficient condition in (4). Hence, B_1 is a convex set in A. On the other hand, since $A \backslash B_1 = \bigcap_{\lambda \in \Lambda} (A \backslash B_\lambda)$, $A \backslash B_1$ is a convex set in A by (1) and (5). Thus B_1 is a biconvex set in A. Set $B_2 := \bigcap_{\lambda \in \Lambda} B_\lambda$. Let $\preceq^{op}$ be the opposite order of $\preceq$. Then $A \backslash B_\lambda \subseteq A \backslash B_{\lambda'}$ if $\lambda \preceq^{op} \lambda'$. Hence, $B_3 := \bigcup_{\lambda \in \Lambda} (A \backslash B_\lambda)$ is a biconvex set in A. Thus B_2 is a biconvex set in A since $B_2 = A \backslash B_3$. $\square$

Theorem 2.6 ([10]). *The assignment $y \mapsto \Phi(y)$ defines an injective mapping from W to $\mathfrak{B}$. Moreover, if the root system (V, Δ, Π) over $\mathbf{F}$ satisfies the following two conditions, then Φ is surjective:*

${}_{\mathbf{F}}\mathrm{R(v)}$ *each $\alpha \in \Delta_+ \backslash \Pi$ can be written as $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Delta_+$;*

${}_{\mathbf{F}}\mathrm{R(vi)}$ *there exists a mapping $\mathrm{ht} : \Delta_+ \to \mathbf{F}_{>0}$ such that $\mathrm{ht}(\beta + \gamma) = \mathrm{ht}(\beta) + \mathrm{ht}(\gamma)$ for all $\beta, \gamma \in \Delta_+$ satisfying $\beta + \gamma \in \Delta_+$.*

Remarks. (1) The surjectivity of the mapping follows from the fact that if C is a non-empty finite coconvex set then $C \cap \Pi \neq \emptyset$. The conditions ${}_{\mathbf{F}}\mathrm{R(v)}$ and ${}_{\mathbf{F}}\mathrm{R(vi)}$ are used to prove the fact.

(2) Suppose that the root system (V, Δ, Π) over $\mathbf{F}$ satisfies the following two condition instead of ${}_{\mathbf{F}}\mathrm{R(v)}$ and ${}_{\mathbf{F}}\mathrm{R(vi)}$:

${}_{\mathbf{F}}\mathrm{R(v)}'$ each $\alpha \in \Delta_+ \backslash \Pi$ can be written as $\alpha = b\beta + c\gamma$ with $b, c \in \mathbf{F}_{\geq 1}$ and $\beta, \gamma \in \Delta_+$;

${}_{\mathbf{F}}\mathrm{R(vi)}'$ there exists a mapping $\mathrm{ht} : \Delta_+ \to \mathbf{F}_{>0}$ such that $\mathrm{ht}(b\beta + c\gamma) = b\,\mathrm{ht}(\beta) + c\,\mathrm{ht}(\gamma)$ for all $b, c \in \mathbf{F}_{>0}$ and $\beta, \gamma \in \Delta_+$ satisfying $b\beta + c\gamma \in \Delta_+$.

Then Φ is still surjective if $\mathfrak{B}$ is replaced by the set of all finite subsets $B \subset \Delta_+$ satisfying the following two conditions:

${}_{\mathbf{F}}\mathrm{C(i)}'$ $\beta, \gamma \in B,\ b, c \in \mathbf{F}_{>0},\ b\beta + c\gamma \in \Delta_+ \Rightarrow b\beta + c\gamma \in B$;

${}_{\mathbf{F}}\mathrm{C(ii)}'$ $\beta, \gamma \in \Delta_+ \backslash B,\ b, c \in \mathbf{F}_{>0},\ b\beta + c\gamma \in \Delta_+ \Rightarrow b\beta + c\gamma \in \Delta_+ \backslash B$.

(3) Let (W, S) be a Coxeter system, and (V, Δ, Π) the root system of (W, S) over $\mathbf{R}$ associated with the geometric representation. Then (V, Δ, Π) satisfies ${}_{\mathbf{R}}\mathrm{R(v)}'$ and ${}_{\mathbf{R}}\mathrm{R(vi)}'$. The condition ${}_{\mathbf{R}}\mathrm{R(v)}'$ is easily checked by reforming the proof of Proposition 2.1 in [3]. Since Π is linearly independent, we can define a mapping $\mathrm{ht} : \Delta_+ \to \mathbf{R}_{>0}$ by setting $\mathrm{ht}(\alpha) := \sum_{s \in S} a_s$ for each $\alpha \in \Delta_+$, where a_s's are non-negative real numbers such that $\alpha = \sum_{s \in S} a_s \alpha_s$. Then the mapping ht satisfies the required property in ${}_{\mathbf{R}}\mathrm{R(vi)}'$.

Definition 2.7. For each $n \in \mathbf{N}_*$, we denote by $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbf{N}_n}$ a sequence consisting of elements $\boldsymbol{s}(p) \in S$ for $p \in \mathbf{N}_n$, and denote by $S^{\mathbf{N}_n}$ the set of such sequences. We also denote by $(\boldsymbol{s}(1), \boldsymbol{s}(2), \dots, \boldsymbol{s}(n))$ a sequence $\boldsymbol{s} \in S^{\mathbf{N}_n}$ with $n < \infty$. For each $\boldsymbol{s} \in S^{\mathbf{N}_n}$ and $m \in \mathbf{N}_n$, we define a sequence $\boldsymbol{s}|_m \in S^{\mathbf{N}_m}$ by setting $\boldsymbol{s}|_m(p) := \boldsymbol{s}(p)$ for each $p \in \mathbf{N}_m$, and call the sequence $\boldsymbol{s}|_m$ the *initial* m-*section* of $\boldsymbol{s}$. Let $\{\boldsymbol{s}_p\}_{p \in \mathbf{N}}$ be a family of finite sequences of elements of S such that $\boldsymbol{s}_p$ is the initial m_p-section of $\boldsymbol{s}_{p+1}$ with $m_p < m_{p+1}$ for each $p \in \mathbf{N}$. Then we see that there exists a unique infinite sequence $\boldsymbol{s}_\infty$ of elements of S such that $\boldsymbol{s}_p$ is the initial m_p-section of $\boldsymbol{s}_\infty$ for each $p \in \mathbf{N}$, and denote by $\lim_{p \to \infty} \boldsymbol{s}_p$ the infinite sequence $\boldsymbol{s}_\infty$. For each $\boldsymbol{s} \in S^{\mathbf{N}_n}$ and $\boldsymbol{s}' \in S^{\mathbf{N}_{n'}}$ with $n < \infty$ and $n' \in \mathbf{N}_*$, we define a sequence $\boldsymbol{s}\boldsymbol{s}' = (\boldsymbol{s}\boldsymbol{s}'(p))_{p \in \mathbf{N}_{n+n'}} \in S^{\mathbf{N}_{n+n'}}$ by setting

$$\boldsymbol{s}\boldsymbol{s}'(p) := \begin{cases} \boldsymbol{s}(p) & \text{for } p \leq n, \\ \boldsymbol{s}'(p-n) & \text{for } n+1 \leq p. \end{cases}$$

The product $\boldsymbol{s}\boldsymbol{s}'$ satisfies the associative law: $(\boldsymbol{s}\boldsymbol{s}')\boldsymbol{s}'' = \boldsymbol{s}(\boldsymbol{s}'\boldsymbol{s}'')$ for $\boldsymbol{s} \in S^{\mathbf{N}_n}$, $\boldsymbol{s}' \in S^{\mathbf{N}_{n'}}$, $\boldsymbol{s}'' \in S^{\mathbf{N}_{n''}}$ with $n, n' < \infty$. Therefore, the product $\boldsymbol{s}_1 \cdots \boldsymbol{s}_{p-1}\boldsymbol{s}_p$ is defined naturally for each family $\{\boldsymbol{s}_1, \dots, \boldsymbol{s}_{p-1}, \boldsymbol{s}_p\}$ of sequences of elements of S such that $\boldsymbol{s}_i$ for $i \in \mathbf{N}_{p-1}$ are finite sequences. For each finite sequence $\boldsymbol{s} \in S^{\mathbf{N}_n}$ and $p \in \mathbf{N}_*$, we define $\boldsymbol{s}^p \in S^{\mathbf{N}_{np}}$ by setting

$$s^p := \overbrace{s \cdots s}^{p} \quad \text{for } p < \infty, \qquad s^\infty := \lim_{p\to\infty} s^p.$$

For each $s \in S^{\mathbf{N}_n}$ with $n < \infty$, we define an element $[s] \in W$ by setting

$$[s] := s(1)s(2)\cdots s(n).$$

For each $s \in S^{\mathbf{N}_n}$ with $n \in \mathbf{N}_*$, we define a mapping $\phi_s : \mathbf{N} \to \Delta^{re}$ by setting

$$\phi_s(p) := [s|_{p-1}](\alpha_{s(p)})$$

for each $p \in \mathbf{N}_n$, where $[s|_0] := 1$. For each $s \in S^{\mathbf{N}}$, we define a mapping $\tilde{\Phi}^\infty$ from $S^{\mathbf{N}}$ to the power set of Δ^{re}_+ by setting

$$\tilde{\Phi}^\infty(s) := \bigcup_{p\in\mathbf{N}} \Phi([s|_p]).$$

We call an element $s \in S^{\mathbf{N}}$ an *infinite reduced word* of (W,S) if $\ell([s|_p]) = p$ for all $p \in \mathbf{N}$, and denote by $\mathscr{W}^\infty$ the subset of $S^{\mathbf{N}}$ of all infinite reduced words of (W,S).

Lemma 2.8. *For a pair (s, s') of elements of $\mathscr{W}^\infty$, we write $s \sim s'$ if for each $(p,q) \in \mathbf{N}^2$ there exists $(p_0, q_0) \in \mathbf{Z}_{\geq p} \times \mathbf{Z}_{\geq q}$ such that*

$$\ell([s|_p]^{-1}[s'|_{p_0}]) = p_0 - p, \qquad \ell([s'|_q]^{-1}[s_{q_0}]) = q_0 - q.$$

Then $\sim$ is an equivalence relation on $\mathscr{W}^\infty$.

Proof. The reflexive law and the symmetric law are obvious. To prove the transitive law, suppose that $s \sim s'$, $s' \sim s''$ for some $s, s', s'' \in \mathscr{W}^\infty$. For each $p \in \mathbf{N}$, choose $p_0 \geq p$ and $p_1 \geq p_0$ satisfying $\ell([s|_p]^{-1}[s'|_{p_0}]) = p_0 - p$ and $\ell([s'|_{p_0}]^{-1}[s''|_{p_1}]) = p_1 - p_0$. Then we have

$$\begin{aligned} p_1 - p = |\ell([s|_p]^{-1}) - \ell([s''|_{p_1}])| &\leq \ell([s|_p]^{-1}[s''|_{p_1}]) \\ &\leq \ell([s|_p]^{-1}[s'|_{p_0}]) + \ell([s'|_{p_0}]^{-1}[s''|_{p_1}]) \\ &= (p_0 - p) + (p_1 - p_0) = p_1 - p. \end{aligned}$$

Thus we get $\ell([s|_p]^{-1}[s''|_{p_1}]) = p_1 - p$. Similarly, we see that for each $q \in \mathbf{N}$ there exists $q_1 \in \mathbf{Z}_{\geq q}$ such that $\ell([s''|_q]^{-1}[s|_{q_1}]) = q_1 - q$. Therefore we get $s \sim s''$. □

Definition 2.9. We denote by W^∞ the quotient set of $\mathscr{W}^\infty$ relative to the equivalence relation $\sim$, and by $[s]$ the coset containing $s \in \mathscr{W}^\infty$.

Proposition 2.10. *Let s and s' be elements of $S^{\mathbf{N}}$.*

(1) *We have* $\boldsymbol{s} \in \mathscr{W}^{\infty}$ *if and only if* $\phi_{\boldsymbol{s}}(p) > 0$ *for all* $p \in \mathbf{N}$.

(2) *If* $\boldsymbol{s} \in \mathscr{W}^{\infty}$, *then* $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) = \{\phi_{\boldsymbol{s}}(p) \mid p \in \mathbf{N}\}$ *and all the elements* $\phi_{\boldsymbol{s}}(p)$ *of* $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s})$ *are distinct from each other.*

(3) *If* $\boldsymbol{s} \in \mathscr{W}^{\infty}$, *then* $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) \in \mathfrak{B}^{\infty}$.

(4) *Suppose that* $\boldsymbol{s}, \boldsymbol{s}' \in \mathscr{W}^{\infty}$. *Then* $\boldsymbol{s} \sim \boldsymbol{s}'$ *if and only if* $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) = \tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}')$.

Proof. (1) We see that $\boldsymbol{s} \in \mathscr{W}^{\infty}$ if and only if $\ell([\boldsymbol{s}|_{p-1}]\boldsymbol{s}(p)) > \ell([\boldsymbol{s}|_{p-1}])$ for all $p \in \mathbf{N}$. Hence the assertion follows from the fact that $z(\alpha_s) > 0$ if and only if $\ell(zs) > \ell(z)$ for $z \in W$ and $s \in S$.

(2) This follows from Theorem 2.2.

(3) We see that $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s})$ is an infinite set by (2). For each $p \leq q$, we have $\Phi([\boldsymbol{s}|_p]) \subseteq \Phi([\boldsymbol{s}|_q])$. Thus we get $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) \in \mathfrak{B}^{\infty}$ by Lemma 2.5(6) and Theorem 2.6.

(4) By Lemma 2.3(4), we see that the condition $\boldsymbol{s} \sim \boldsymbol{s}'$ is equivalent to the condition that for each $(p,q) \in \mathbf{N}^2$ there exists $(p_0, q_0) \in \mathbf{Z}_{\geq p} \times \mathbf{Z}_{\geq q}$ such that $\Phi([\boldsymbol{s}|_p]) \subset \Phi([\boldsymbol{s}'|_{p_0}])$ and $\Phi([\boldsymbol{s}'|_q]) \subset \Phi([\boldsymbol{s}|_{q_0}])$. Thus $\boldsymbol{s} \sim \boldsymbol{s}'$ if and only if $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) = \tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}')$. □

Definition 2.11. Thanks to Proposition 2.10(3)(4), we have an injective mapping

$$\Phi^{\infty} : W^{\infty} \to \mathfrak{B}^{\infty}, \qquad [\boldsymbol{s}] \mapsto \Phi^{\infty}([\boldsymbol{s}]) := \tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}).$$

We define a left action of W on W^{∞}.

Definition 2.12. For each $x \in W$ and $\boldsymbol{s} \in S^{\mathbf{N}}$, we set

$$\tilde{\boldsymbol{\Phi}}^{\infty}(x, \boldsymbol{s}) := \{\beta \in \Delta_{+}^{re} \mid \exists p_0 \in \mathbf{N}; \forall p \geq p_0, (x[\boldsymbol{s}|_p])^{-1}(\beta) < 0\}.$$

Lemma 2.13. (1) *If* $\boldsymbol{s} \in \mathscr{W}^{\infty}$, *then* $\tilde{\boldsymbol{\Phi}}^{\infty}(1, \boldsymbol{s}) = \tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s})$.

(2) *If* $x \in W$ *and* $\boldsymbol{s} \in \mathscr{W}^{\infty}$, *then there exists an element* $\boldsymbol{s}' \in \mathscr{W}^{\infty}$ *such that* $\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}') = \tilde{\boldsymbol{\Phi}}^{\infty}(x, \boldsymbol{s})$. *More precisely, a required* $\boldsymbol{s}'$ *can be constructed by applying the following procedure* Steps 1–3.

Step 1. *Choose a non-negative integer* p_0 *such that*

$$\Phi(x^{-1}) \cap \tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) \subset \Phi([\boldsymbol{s}|_{p_0}]). \tag{2.1}$$

Step 2. *In the case where* $x[\boldsymbol{s}|_{p_0}] = 1$, *set* $\boldsymbol{s}'(p) := \boldsymbol{s}(p_0 + p)$ *for each* $p \in \mathbf{N}$. *In the case where* $x[\boldsymbol{s}|_{p_0}] \neq 1$, *choose a reduced expression* $x[\boldsymbol{s}|_{p_0}] = \boldsymbol{s}'(1) \cdots \boldsymbol{s}'(l_0)$ *with* $l_0 \in \mathbf{N}$, *and set* $\boldsymbol{s}'(l_0 + p) := \boldsymbol{s}(p_0 + p)$ *for each* $p \in \mathbf{N}$.

Step 3. *Set* $\boldsymbol{s}' := (\boldsymbol{s}'(p))_{p \in \mathbf{N}}$.

(3) *If* $x \in W$ *and* $\boldsymbol{s} \in \mathscr{W}^{\infty}$, *then*

$$\tilde{\boldsymbol{\Phi}}^{\infty}(x, \boldsymbol{s}) = \{\Phi(x) \backslash (-\Lambda)\} \amalg \{x\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) \backslash \Lambda\},$$

where $\Lambda := x\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) \cap \Delta_{-}^{re}$. *In particular, if* $x\tilde{\boldsymbol{\Phi}}^{\infty}(\boldsymbol{s}) \subset \Delta_{+}^{re}$ *then*

$$\tilde{\Phi}^\infty(x, s) = \Phi(x) \amalg x\tilde{\Phi}^\infty(s).$$

(4) *Suppose that* $s, s' \in \mathscr{W}^\infty$ *and* $x, y \in W$.

(i) *If* $\tilde{\Phi}^\infty(s) = \tilde{\Phi}^\infty(s')$, *then* $\tilde{\Phi}^\infty(x, s) = \tilde{\Phi}^\infty(x, s')$.

(ii) *If* $\tilde{\Phi}^\infty(y, s) = \tilde{\Phi}^\infty(s')$, *then* $\tilde{\Phi}^\infty(xy, s) = \tilde{\Phi}^\infty(x, s')$.

Proof. (1) Suppose that $\beta \in \tilde{\Phi}^\infty(1, s)$. Since $[s|_{p_0}]^{-1}(\beta) < 0$ for some $p_0 \in \mathbf{N}$, we have $\beta \in \tilde{\Phi}^\infty([s|_{p_0}]) \subset \tilde{\Phi}^\infty(s)$. Thus we get $\tilde{\Phi}^\infty(1, s) \subset \tilde{\Phi}^\infty(s)$. On the other hand, for each $p < q$, we have

$$[s|_q]^{-1}(\phi_s(p)) = -s(q) \cdots s(p+1)(\alpha_{s(p)}) < 0,$$

and hence $\phi_s(p) \in \tilde{\Phi}^\infty(1, s)$ for each $p \in \mathbf{N}$. Thus we get $\tilde{\Phi}^\infty(s) \subset \tilde{\Phi}^\infty(1, s)$ by Proposition 2.10(2).

(2) Let s' be an element of $S^{\mathbf{N}}$ constructed as in (Step 1)–(Step 3). By the construction, we have $x[s|_{p_0}] = [s'|_{l_0}]$ for some unique $l_0 \in \mathbf{Z}_{\geq 0}$. Since $s(p_0 + p) = s'(l_0 + p)$ for each $p \in \mathbf{N}$, we have

$$x[s|_{p_0+p}] = [s'|_{l_0+p}], \tag{2.2}$$

$$x\phi_s(p_0 + p) = \phi_{s'}(l_0 + p). \tag{2.3}$$

By the condition (2.1) and the equality (2.3), we have $\phi_{s'}(l_0 + p) > 0$ for each $p \in \mathbf{N}$ since $\phi_s(p_0 + p) \notin \Phi([s|_{p_0}])$. In addition, by Theorem 2.2 we have $\phi_{s'}(p) > 0$ for each $1 \leq p \leq l_0$. Thus we get $s' \in \mathscr{W}^\infty$ by Proposition 2.10(1). Moreover, by (1) and the equality (2.2), we get $\tilde{\Phi}^\infty(x, s) = \tilde{\Phi}^\infty(s')$.

(3) Since $-\Lambda = \Phi(x) \cap (-x\tilde{\Phi}^\infty(s))$, we have

$$\Phi(x) \backslash (-\Lambda) = \{\beta \in \Delta_+^{re} \mid \beta \in \Phi(x), -x^{-1}(\beta) \in \Delta_+^{re} \backslash \tilde{\Phi}^\infty(s)\}.$$

On the other hand, since $x\tilde{\Phi}^\infty(s) \backslash \Lambda = x\tilde{\Phi}^\infty(s) \cap \Delta_+^{re}$, we have

$$x\tilde{\Phi}^\infty(s) \backslash \Lambda = \{\beta \in \Delta_+^{re} \mid \beta \notin \Phi(x), x^{-1}(\beta) \in \tilde{\Phi}^\infty(s)\}.$$

Therefore, by (1) we get $\tilde{\Phi}^\infty(x, s) = \{\Phi(x) \backslash (-\Lambda)\} \amalg \{x\tilde{\Phi}^\infty(s) \backslash \Lambda\}$.

(4)(i) This is straightforward from (3).

(4)(ii) By the argument in the proof of (2), there exist an element $\tilde{s} \in \mathscr{W}^\infty$ and $(p_0, l_0) \in (\mathbf{Z}_{\geq 0})^2$ satisfying $y[s|_{p_0+p}] = [\tilde{s}|_{l_0+p}]$ for all $p \in \mathbf{N}$. Then we have $\tilde{\Phi}^\infty(\tilde{s}) = \tilde{\Phi}^\infty(y, s) = \tilde{\Phi}^\infty(s')$. Hence, by (4)(i) we have $\tilde{\Phi}^\infty(x, \tilde{s}) = \tilde{\Phi}^\infty(x, s')$. Moreover, since $xy[s|_{p_0+p}] = x[\tilde{s}|_{l_0+p}]$ for all $p \in \mathbf{N}$, we have $\tilde{\Phi}^\infty(xy, s) = \tilde{\Phi}^\infty(x, \tilde{s})$. Thus we get $\tilde{\Phi}^\infty(xy, s) = \tilde{\Phi}^\infty(x, \tilde{s}) = \tilde{\Phi}^\infty(x, s')$. □

Definition 2.14. Thanks to Proposition 2.10(4) and Lemma 2.13(1)(2)(4), we have a left action of W on W^∞ such that $x.[s] = [s']$ if $x \in W$ and $s, s' \in \mathscr{W}^\infty$ satisfy $\tilde{\Phi}^\infty(x, s) = \tilde{\Phi}^\infty(s')$.

Proposition 2.15. *If $x \in W$ and $\boldsymbol{s} \in \mathscr{W}^{\infty}$, then*

$$\Phi^{\infty}(x.[\boldsymbol{s}]) = \{\Phi(x)\backslash(-\Lambda)\} \amalg \{x\Phi^{\infty}([\boldsymbol{s}])\backslash\Lambda\},$$

where $\Lambda := x\Phi^{\infty}([\boldsymbol{s}]) \cap \Delta_{-}^{re}$. In particular, if $x\Phi^{\infty}([\boldsymbol{s}]) \subset \Delta_{+}^{re}$ then

$$\Phi^{\infty}(x.[\boldsymbol{s}]) = \Phi(x) \amalg x\Phi^{\infty}([\boldsymbol{s}]).$$

Proof. This follows from Lemma 2.13(3). □

3. Notation for the untwisted affine cases

In this section, we prepare some notation for the untwisted affine cases referring to the book [6]. Let $\mathrm{A} = [a_{ij}]_{i,j \in \mathbf{I}}$ be a generalized Cartan matrix of the affine type $X_r^{(1)}$ with $\mathbf{I} = \{0, 1, \dots, r\}$, where $X = A, B, C, D, E, F, G$. We set $\mathring{\mathbf{I}} = \{1, \dots, r\}$. Then we may assume that $[a_{ij}]_{i,j \in \mathring{\mathbf{I}}}$ is the Cartan matrix of the finite type X_r. Let $(\mathfrak{h}, \Pi, \Pi^{\vee})$ be a minimal realization of A over $\mathbf{Q}$, that is, a triplet consisting of a $(r+2)$-dimensional vector space $\mathfrak{h}$ over $\mathbf{Q}$ and linearly independent subsets $\Pi = \{\alpha_i \mid i \in \mathbf{I}\} \subset \mathfrak{h}^*$ and $\Pi^{\vee} = \{\alpha_i^{\vee} \mid i \in \mathbf{I}\} \subset \mathfrak{h}$ satisfying $\langle \alpha_i^{\vee}, \alpha_j \rangle = a_{ij}$ for each $i, j \in \mathbf{I}$, where $\mathfrak{h}^*$ is the dual vector space of $\mathfrak{h}$ and $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbf{Q}$ is the canonical pairing. Let $\mathfrak{g}$ be the affine Kac-Moody Lie algebra over $\mathbf{Q}$ associated with $(\mathfrak{h}, \Pi, \Pi^{\vee})$, $\Delta \subset \mathfrak{h}^*\backslash\{0\}$ the root system of $\mathfrak{g}$, Δ^{re} (resp. Δ^{im}) the set of all real (resp. imaginary) roots, and $W = \langle s_i \mid i \in \mathbf{I} \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the Weyl group of $\mathfrak{g}$, where $s_i = s_{\alpha_i}$ is the reflection associated with α_i. Let Δ_+ (resp. Δ_-) be the set of all positive (resp. negative) roots relative to Π, Δ_+^{re} (resp. Δ_-^{re}) the set of all positive (resp. negative) real roots, Δ_+^{im} (resp. Δ_-^{im}) the set of all positive (resp. negative) imaginary roots, and $\mathrm{ht} : \Delta_+ \to \mathbf{N}$ the height function on Δ_+. Set

$$\mathring{\Pi} := \{\alpha_i \mid i \in \mathring{\mathbf{I}}\}, \qquad \mathring{\mathfrak{h}}^* := \mathrm{span}_{\mathbf{Q}}\, \mathring{\Pi}, \qquad \mathring{W} := \langle s_i \mid i \in \mathring{\mathbf{I}} \rangle, \qquad \mathring{\Delta} := \mathring{W}(\mathring{\Pi}).$$

Note that $\mathring{\Delta}$ is a root system in $\mathring{\mathfrak{h}}^*$ of the finite type X_r with $\mathring{\Pi}$ a root basis and $\mathring{W}$ the Weyl group. Let $\mathring{\Delta}_+$ (resp. $\mathring{\Delta}_-$) be the set of all positive (resp. negative) roots relative to $\mathring{\Pi}$. Denote by θ the highest root of $\mathring{\Delta}$ and set $\delta := \alpha_0 + \theta$. Then

$$\Delta^{re} = \{m\delta + \varepsilon \mid m \in \mathbf{Z}, \varepsilon \in \mathring{\Delta}\}, \qquad \Delta^{im} = \{m\delta \mid m \in \mathbf{Z}\backslash\{0\}\},$$

$$\Delta_+^{re} = \mathring{\Delta}_+ \amalg \{n\delta + \varepsilon \mid n \in \mathbf{N}, \varepsilon \in \mathring{\Delta}\}, \qquad \Delta_+^{im} = \{n\delta \mid n \in \mathbf{N}\}.$$

Let $(d_i)_{i \in \mathbf{I}}$ be relatively prime positive integers such that $[d_i a_{ij}]_{i,j \in \mathbf{I}}$ is a symmetric matrix, and λ_0 a non-zero element of $\mathfrak{h}^*$ such that $\mathfrak{h}^* = \mathfrak{h}^{*\prime} \oplus \mathbf{Q}\lambda_0$ and $\langle \alpha_i^{\vee}, \lambda_0 \rangle = \delta_{i0}$, where $\mathfrak{h}^{*\prime} := \mathrm{span}_{\mathbf{Q}}\, \Pi$. Define a non-degenerate symmetric bilinear form $(\cdot|\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbf{Q}$ by setting

$$(\alpha_i|\lambda) := d_i\langle\alpha_i^\vee, \lambda\rangle \quad (i \in \mathbf{I}, \lambda \in \mathfrak{h}^*), \qquad (\lambda_0|\lambda_0) := 0.$$

In particular, $(\alpha_i|\alpha_j) = d_i a_{ij}$. Note that $(\delta|\delta) = 0$ and the following direct sum decomposition is an orthogonal decomposition:

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus (\mathbf{Q}\delta + \mathbf{Q}\lambda_0).$$

For each $\lambda \in \mathfrak{h}^*$, we denote by $\bar{\lambda}$ the image of λ by the orthogonal projection onto $\mathring{\mathfrak{h}}^*$. Each $\beta \in \Delta$ can be uniquely written as $m\delta + \bar{\beta}$ with $m \in \mathbf{Z}$ and $\bar{\beta} \in \mathring{\Delta} \amalg \{0\}$.

For each $\alpha \in \Delta^{re}$, we denote by s_α the reflection with respect to α. For each $\lambda \in \mathfrak{h}^*$, we define an element $t_\lambda \in \mathrm{GL}(\mathfrak{h}^*)$ called a translation by setting

$$t_\lambda(\mu) := \mu + (\mu|\delta)\lambda - \{(\mu|\lambda) + \tfrac{1}{2}(\lambda|\lambda)(\mu|\delta)\}\delta$$

for each $\mu \in \mathfrak{h}^*$. In particular, $t_\lambda(\mu) = \mu - (\mu|\lambda)\delta$ for each $\mu \in \mathfrak{h}^{*\prime}$.

Lemma 3.1 ([6]). *Set $\check{\alpha}_i := \frac{2\alpha_i}{(\alpha_i|\alpha_i)}$ for each $i \in \mathring{\mathbf{I}}$, and set $\mathring{Q}^\vee := \bigoplus_{i\in\mathring{\mathbf{I}}} \mathbf{Z}\check{\alpha}_i$ and $T := \{t_\lambda \mid \lambda \in \mathring{Q}^\vee\}$. Then T is a normal subgroup of W such that $W = \mathring{W} \ltimes T$.*

Each element $x \in W$ can be uniquely written as $x = \bar{x}t_x$ with $\bar{x} \in \mathring{W}$ and $t_x \in T$. The mapping $\bar{\cdot} : W \to \mathring{W}$, $x \mapsto \bar{x}$, is a group homomorphism, which satisfies that $\overline{x(\lambda)} = \bar{x}(\bar{\lambda})$ and $\overline{s_\alpha} = s_{\bar{\alpha}}$ for each $x \in W$, $\lambda \in \mathfrak{h}^{*\prime}$, and $\alpha \in \Delta^{re}$.

4. Preliminary results for classical root systems

In this section, we give preliminary results for classical root systems. We use the notation introduced in Section 3. For each subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\mathring{\Pi}_\mathbf{J} := \{\alpha_j \mid j \in \mathbf{J}\}, \qquad \mathring{\mathfrak{h}}_\mathbf{J}^* := \mathrm{span}_\mathbf{Q}\,\mathring{\Pi}_\mathbf{J},$$

$$\mathring{W}_\mathbf{J} := \langle s_j \mid j \in \mathbf{J}\rangle \subset \mathring{W}, \qquad \mathring{\Delta}_\mathbf{J} := \mathring{W}_\mathbf{J}(\mathring{\Pi}_\mathbf{J}) \subset \mathring{\Delta}.$$

Note that $\mathring{\Delta}_\mathbf{J}$ is a root system in $\mathring{\mathfrak{h}}_\mathbf{J}^*$ with $\mathring{\Pi}_\mathbf{J}$ a root basis and $\mathring{W}_\mathbf{J}$ the Weyl group if $\mathbf{J} \neq \varnothing$. Let $\mathring{\Delta}_{\mathbf{J}+}$ (resp. $\mathring{\Delta}_{\mathbf{J}-}$) be the set of all positive (resp. negative) roots relative to $\mathring{\Pi}_\mathbf{J}$. For each $\mathbf{K} \subset \mathbf{J}$, we denote by $\mathring{W}_\mathbf{J}^\mathbf{K}$ the minimal coset representatives of the set $\mathring{W}_\mathbf{J}/\mathring{W}_\mathbf{K}$ of all right cosets. If $\mathbf{J} = \mathring{\mathbf{I}}$ we denote it simply by $\mathring{W}^\mathbf{K}$. Note that each element $w \in \mathring{W}_\mathbf{J}$ can be uniquely written as $w^\mathbf{K} w_\mathbf{K}$ with $w^\mathbf{K} \in \mathring{W}_\mathbf{J}^\mathbf{K}$ and $w_\mathbf{K} \in \mathring{W}_\mathbf{K}$, where $w^\mathbf{K}$ is a unique element of the smallest length in the right coset $w\mathring{W}_\mathbf{K}$. Moreover, we have

$$\mathring{W}_\mathbf{J}^\mathbf{K} = \{w \in \mathring{W}_\mathbf{J} \mid w(\alpha_j) > 0 \text{ for all } j \in \mathbf{K}\},$$

and $\mathring{W}_\mathbf{J}^\mathbf{K}\mathring{W}_\mathbf{K}^\mathbf{L} = \mathring{W}_\mathbf{J}^\mathbf{L}$ if $\mathbf{L} \subset \mathbf{K} \subset \mathbf{J}$. In addition, we set

$$\mathring{\Delta}_\mathbf{J}^\mathbf{K} := \mathring{\Delta}_\mathbf{J} \backslash \mathring{\Delta}_\mathbf{K}, \qquad \mathring{\Delta}_{\mathbf{J}\pm}^\mathbf{K} := \mathring{\Delta}_\mathbf{J}^\mathbf{K} \cap \mathring{\Delta}_\pm.$$

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we remove $\mathbf{J}$ from the symbols above.

Lemma 4.1. (1) *The following equality holds:*

$$\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+} = \left\{\varepsilon = \sum\nolimits_{j\in\mathbf{J}} m_j\alpha_j \in \mathring{\Delta}_{\mathbf{J}+}(m_j \in \mathbf{Z}_{\geq 0}) \,|\, m_j > 0 \textit{ for some } j \in \mathbf{J}\backslash\mathbf{K}\right\}.$$

(2) *We have* $\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm} \dotplus \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm} \subset \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm}$ *and* $\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm} \dotplus \mathring{\Delta}_{\mathbf{K}} \subset \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm}$.

(3) *For each* $v \in \mathring{W}_{\mathbf{K}}$, *we have* $v\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm} = \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm}$.

(4) *Let* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be subsets of* $\mathbf{J}$, *and let* w_1 *and* w_2 *be elements of* $\mathring{W}_{\mathbf{J}}$. *Then the following two conditions are equivalent:*

$$\text{(i)}\quad w_1\mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}\pm} \subset w_2\mathring{\Delta}^{\mathbf{K}_2}_{\mathbf{J}\pm}; \qquad \text{(ii)}\quad \mathbf{K}_1 \supset \mathbf{K}_2, \quad w_1 \in w_2\mathring{W}_{\mathbf{K}_1}.$$

Proof. (1) This is straightforward from the definition.

(2) This follows immediately from (1).

(3) Let ε be an element of $\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+}$, and write $\varepsilon = \sum_{j\in\mathbf{J}} m_j\alpha_j$ with $m_j \in \mathbf{Z}_{\geq 0}$ for all $j \in \mathbf{J}$ and $m_{j_*} > 0$ for some $j_* \in \mathbf{J}\backslash\mathbf{K}$. Since $v(\alpha_j) \in \alpha_j + \sum_{k\in\mathbf{K}} \mathbf{Z}\alpha_k$ for each $j \in \mathbf{J}\backslash\mathbf{K}$, we have $v(\varepsilon) = \sum_{j\in\mathbf{J}\backslash\mathbf{K}} m_j\alpha_j + \sum_{k\in\mathbf{K}} m'_k\alpha_k \in \mathring{\Delta}$ with $m'_k \in \mathbf{Z}$, which implies that $v(\varepsilon) \in \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+}$ since $m_{j_*} > 0$. Thus $v\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+} \subset \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+}$ for each $v \in \mathring{W}_{\mathbf{K}}$, and hence $v\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+} = \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+}$.

(4) Suppose that $\mathbf{K}_1 \supset \mathbf{K}_2$ and $w_1 = w_2v$ with $v \in \mathring{W}_{\mathbf{K}_1}$. Then, by (3) we have $w_1\mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}+} = w_2\mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}+} \subset w_2\mathring{\Delta}^{\mathbf{K}_2}_{\mathbf{J}+}$. Conversely, suppose that $w\mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}+} \subset \mathring{\Delta}^{\mathbf{K}_2}_{\mathbf{J}+}$ with $w = w_2^{-1}w_1$. Then we have $w^{\mathbf{K}_1}\mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}+} \subset \mathring{\Delta}^{\mathbf{K}_2}_{\mathbf{J}+}$ by (3), and hence $w^{\mathbf{K}_1}(\alpha_j) > 0$ for all $j \in \mathbf{J}\backslash\mathbf{K}_1$ since $\mathring{\Pi}_{\mathbf{J}\backslash\mathbf{K}_1} \subset \mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}+}$. Moreover, $w^{\mathbf{K}_1}(\alpha_k) > 0$ for all $k \in \mathbf{K}_1$ since $w^{\mathbf{K}_1} \in \mathring{W}^{\mathbf{K}_1}_{\mathbf{J}}$. Thus $w^{\mathbf{K}_1}(\alpha_j) > 0$ for all $j \in \mathbf{J}$, and hence $w^{\mathbf{K}_1} = 1$ and $w = w_{\mathbf{K}_1} \in \mathring{W}_{\mathbf{K}_1}$. Therefore $\mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}+} = w\mathring{\Delta}^{\mathbf{K}_1}_{\mathbf{J}+} \subset \mathring{\Delta}^{\mathbf{K}_2}_{\mathbf{J}+}$, which implies that $\mathbf{K}_1 \supset \mathbf{K}_2$. $\square$

Definition 4.2. Let $\mathbf{J}$ be a non-empty subset of $\mathring{\mathbf{I}}$.

(1) A subset $P \subset \mathring{\Delta}$ is called a *closed* set if it satisfies the condition that if $\varepsilon, \eta \in P$, $\varepsilon + \eta \in \mathring{\Delta}$ then $\varepsilon + \eta \in P$ (cf. [2, §1.7]). We call a subset $P \subset \mathring{\Delta}_{\mathbf{J}}$ a *coclosed set in* $\mathring{\Delta}_{\mathbf{J}}$ if $\mathring{\Delta}_{\mathbf{J}}\backslash P$ is a closed set, and call a subset $P \subset \mathring{\Delta}_{\mathbf{J}}$ a *biclosed set in* $\mathring{\Delta}_{\mathbf{J}}$ if both P and $\mathring{\Delta}_{\mathbf{J}}\backslash P$ are closed sets.

(2) We call a subset $P \subset \mathring{\Delta}_{\mathbf{J}}$ a *parabolic set in* $\mathring{\Delta}_{\mathbf{J}}$ if P is a closed set such that $P \cup (-P) = \mathring{\Delta}_{\mathbf{J}}$ (cf. [2]).

(3) A subset $P \subset \mathring{\Delta}$ is called a *symmetric* set if $P = -P$ (cf. [2]).

(4) We call a subset $P \subset \mathring{\Delta}$ a *pointed* set if $P \cap (-P) = \emptyset$.

Proposition 4.3 ([2]). *The following three conditions are equivalent:*

(i) P *is a parabolic set in* $\mathring{\Delta}_{\mathbf{J}}$;

(ii) P *is a closed subset of* $\mathring{\Delta}_{\mathbf{J}}$ *such that* $P \supset w\mathring{\Delta}_{\mathbf{J}+}$ *for some* $w \in \mathring{W}_{\mathbf{J}}$;

(iii) $P = w(\mathring{\Delta}_{\mathbf{J}+} \amalg \mathring{\Delta}_{\mathbf{K}-})$ *for some* $\mathbf{K} \subset \mathbf{J}$ *and* $w \in \mathring{W}_{\mathbf{J}}$.

Proposition 4.4 ([2]). *If* P *is a pointed closed subset of* $\mathring{\Delta}_{\mathbf{J}}$, *then there exists an element* $w \in \mathring{W}_{\mathbf{J}}$ *such that* $wP \subset \mathring{\Delta}_{\mathbf{J}-}$.

Proposition 4.5. *Let P be a subset of $\mathring{\Delta}$. Then there exist a unique symmetric subset $P_s \subset P$ and a unique pointed subset $P_p \subset P$ such that $P = P_p \amalg P_s$. Moreover, if P is closed then both P_s and P_p are closed sets satisfying*

$$P_p \dotplus P_s \subset P_p. \tag{4.1}$$

Proof. Suppose that there exist a symmetric subset $P_s \subset P$ and a pointed subset $P_p \subset P$ such that $P = P_p \amalg P_s$. Then we have

$$P_s = \{\varepsilon \in P \mid -\varepsilon \in P\}, \tag{4.2}$$

$$P_p = \{\varepsilon \in P \mid -\varepsilon \in \mathring{\Delta} \backslash P\}. \tag{4.3}$$

This proves the uniqueness of the decomposition. On the other hand, it is easy to see that the above subsets give the desired decomposition of P.

In addition, we suppose that P is closed. Let ε and η be elements of P_s such that $\varepsilon + \eta \in \mathring{\Delta}$. Then we have $\varepsilon + \eta \in P$ and $-\varepsilon, -\eta \in P$. Thus we get $-(\varepsilon + \eta) \in P$, and hence $\varepsilon + \eta \in P_s$. Therefore P_s is closed.

We next prove (4.1). Suppose that $\varepsilon + \eta \in P_s$ for some $\varepsilon \in P_p$ and $\eta \in P_s$. Then $\varepsilon = (\varepsilon + \eta) + (-\eta) \in P_s$, since P_s is closed and $-\eta \in P_s$. This is a contradiction. Hence, (4.1) is valid.

Suppose that $\varepsilon + \eta \in P_s$ for some $\varepsilon, \eta \in P_p$. Then, since $-\varepsilon - \eta \in P_s$ we have $-\varepsilon = \eta + (-\varepsilon - \eta) \in P_p$ by (4.1). This contradicts $P_p \cap (-P_p) = \varnothing$. Thus we get $\varepsilon + \eta \in P_p$ for each $\varepsilon, \eta \in P_p$ satisfying $\varepsilon + \eta \in \mathring{\Delta}$. Therefore P_p is closed. □

Proposition 4.6. *The following four conditions are equivalent:*

(i) *P is a pointed biclosed set in $\mathring{\Delta}_{\mathbf{J}}$;*
(ii) *P is a pointed coclosed set in $\mathring{\Delta}_{\mathbf{J}}$;*
(iii) *P is a subset of $\mathring{\Delta}_{\mathbf{J}}$ such that $\mathring{\Delta}_{\mathbf{J}} \backslash P$ is a parabolic set in $\mathring{\Delta}_{\mathbf{J}}$;*
(iv) *$P = u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$ for some unique $\mathbf{K} \subset \mathbf{J}$ and unique $u \in \mathring{W}^{\mathbf{K}}_{\mathbf{J}}$.*

Proof. (i) $\Rightarrow$ (ii) It is Clear.

(ii) $\Rightarrow$ (iii) It is clear that $P \subset \mathring{\Delta}_{\mathbf{J}}$. By Proposition 4.5, we have

$$\mathring{\Delta}_{\mathbf{J}} = P_p \amalg P_s \amalg (\mathring{\Delta}_{\mathbf{J}} \backslash P)_p \amalg (\mathring{\Delta}_{\mathbf{J}} \backslash P)_s, \tag{4.4}$$

where P_s (resp. $(\mathring{\Delta}_{\mathbf{J}} \backslash P)_s$) is the symmetric part of P (resp. $\mathring{\Delta}_{\mathbf{J}} \backslash P$) and P_p (resp. $(\mathring{\Delta}_{\mathbf{J}} \backslash P)_p$) is the pointed part of P (resp. $\mathring{\Delta}_{\mathbf{J}} \backslash P$). Then we have

$$-P_p = (\mathring{\Delta}_{\mathbf{J}} \backslash P)_p. \tag{4.5}$$

Indeed, if $\varepsilon \in P_p$ then we have $-\varepsilon \in \mathring{\Delta}_{\mathbf{J}} \backslash P$ and $-(-\varepsilon) \in P$ by (4.2), and hence $-\varepsilon \in (\mathring{\Delta}_{\mathbf{J}} \backslash P)_p$ by (4.3). Thus $-P_p \subset (\mathring{\Delta}_{\mathbf{J}} \backslash P)_p$. Similarly we have $-(\mathring{\Delta}_{\mathbf{J}} \backslash P)_p \subset P_p$.

By (4.5), we have $-(\mathring{\Delta}_{\mathbf{J}}\backslash P) = P_p \amalg (\mathring{\Delta}_{\mathbf{J}}\backslash P)_s$. Moreover, we have $P_s = \varnothing$ since P is pointed. Thus we get $\mathring{\Delta}_{\mathbf{J}} = -(\mathring{\Delta}_{\mathbf{J}}\backslash P) \cup (\mathring{\Delta}_{\mathbf{J}}\backslash P)$ by (4.4), and hence $\mathring{\Delta}_{\mathbf{J}}\backslash P$ is a parabolic set in $\mathring{\Delta}_{\mathbf{J}}$.

(iii) $\Rightarrow$ (iv) By Proposition 4.3, there exist a subset $\mathbf{K} \subset \mathbf{J}$ and an element $w \in \mathring{W}_{\mathbf{J}}$ such that $\mathring{\Delta}_{\mathbf{J}}\backslash P = w(\mathring{\Delta}_{\mathbf{J}+} \amalg \mathring{\Delta}_{\mathbf{K}-})$. Then $P = w\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$ since $P \subset \mathring{\Delta}_{\mathbf{J}}$, and hence $P = w^{\mathbf{K}}\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$ by Lemma 4.1(3). The uniqueness follows from Lemma 4.1(4).

(iv) $\Rightarrow$ (i) It is clear that $u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$ is pointed. By Lemma 4.1(2), we have $u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-} \dotplus u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-} \subset u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$, and hence $u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$ is closed. Moreover, by Lemma 4.1(2) we have $u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+} \dotplus u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+} \subset u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+}$ and $u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+} \dotplus u\mathring{\Delta}_{\mathbf{K}} \subset u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+}$. In addition, $u\mathring{\Delta}_{\mathbf{K}}$ is closed. Thus $\mathring{\Delta}_{\mathbf{J}}\backslash u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$ is closed, since $\mathring{\Delta}_{\mathbf{J}}\backslash u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-} = u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+} \amalg u\mathring{\Delta}_{\mathbf{K}}$. $\square$

5. The construction of biconvex sets

In this section, we give several methods of constructing biconvex sets for the root system of an arbitrary untwisted affine Lie algebra.

Definition 5.1. For each $\varepsilon \in \mathring{\Delta}$ and $P \subset \mathring{\Delta}$, we define subsets $\langle\varepsilon\rangle, \langle P\rangle \subset \Delta^{re}_{+}$ by setting

$$\langle\varepsilon\rangle := \{m\delta + \varepsilon \,|\, m \in \mathbf{Z}_{\geq 0}\} \cap \Delta^{re}_{+}, \qquad \langle P\rangle := \coprod\nolimits_{\varepsilon \in P} \langle\varepsilon\rangle.$$

Lemma 5.2. (1) *Let P be a subset of $\mathring{\Delta}$, and x an element of W. Then*

(i) $\langle P\rangle = \{\beta \in \Delta_{+} \,|\, \bar{\beta} \in P\}$; (ii) $\overline{x\langle P\rangle} \subset \bar{x}P$; (iii) $x\langle P\rangle \doteq \langle \bar{x}P\rangle$.

(2) *For subsets $P, P' \subset \mathring{\Delta}$, the following three conditions are equivalent:*

(i) $P \subset P'$; (ii) $\langle P\rangle \subset \langle P'\rangle$; (iii) $\langle P\rangle \mathrel{\dot{\subset}} \langle P'\rangle$.

Proof. (1) The (i) is straightforward from the definition. To prove (ii), suppose that $\beta \in \langle P\rangle$. Then $\bar{\beta} \in P$ by (i), hence $\overline{x(\beta)} = \bar{x}(\bar{\beta}) \in \bar{x}P$. Thus (ii) is valid. We prove (iii). Write $x = t_{\lambda}\bar{x}$ with $\lambda \in \mathring{Q}^{\vee}$. Then we have $x(m\delta + \varepsilon) = (m - (\bar{x}\varepsilon \,|\, \lambda))\delta + \bar{x}\varepsilon$ for each $m \in \mathbf{Z}_{\geq 0}$ and $\varepsilon \in P$. Thus we get $x(m\delta + \varepsilon) \in \langle\bar{x}\varepsilon\rangle$ for all $m > (\bar{x}\varepsilon \,|\, \lambda)$, and hence $x\langle\varepsilon\rangle \doteq \langle\bar{x}\varepsilon\rangle$. Thus (iii) is valid.

(2) It is obvious. $\square$

Definition 5.3. For each subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\Delta^{re}_{\mathbf{J}} := \langle\mathring{\Delta}_{\mathbf{J}}\rangle, \qquad \Delta_{\mathbf{J}} := \Delta^{re}_{\mathbf{J}} \amalg \Delta^{im},$$

$$\Delta^{re}_{\mathbf{J}\pm} := \Delta^{re}_{\mathbf{J}} \cap \Delta_{\pm}, \qquad \Delta_{\mathbf{J}\pm} := \Delta_{\mathbf{J}} \cap \Delta_{\pm}.$$

We define $W_{\mathbf{J}}$ to be the subgroup of W generated by the set $\{s_\alpha \mid \alpha \in \Delta_{\mathbf{J}}^{re}\}$. Let $\mathfrak{B}_{\mathbf{J}}$ be the set of all finite biconvex sets in $\Delta_{\mathbf{J}+}$ and $\mathfrak{B}_{\mathbf{J}}^{\infty}$ the set of all infinite real biconvex sets in $\Delta_{\mathbf{J}+}$. We set $\mathfrak{B}_{\mathbf{J}}^{*} := \mathfrak{B}_{\mathbf{J}} \amalg \mathfrak{B}_{\mathbf{J}}^{\infty}$.

For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, let

$$\mathring{\Delta}_{\mathbf{J}} = \coprod_{c=1}^{\mathrm{C}(\mathbf{J})} \mathring{\Delta}_{\mathbf{J}_c}$$

be the irreducible decomposition of $\mathring{\Delta}_{\mathbf{J}}$ with $\mathrm{C}(\mathbf{J})$ the number of the irreducible components. For each $c = 1, \dots, \mathrm{C}(\mathbf{J})$, we denote by $\theta_{\mathbf{J}_c}$ the highest root of $\mathring{\Delta}_{\mathbf{J}_c}$ relative to the root basis $\mathring{\Pi}_{\mathbf{J}_c}$, and set

$$\Pi_{\mathbf{J}_c} := \mathring{\Pi}_{\mathbf{J}_c} \amalg \{\delta - \theta_{\mathbf{J}_c}\}, \qquad \Pi_{\mathbf{J}} := \coprod_{c=1}^{\mathrm{C}(\mathbf{J})} \Pi_{\mathbf{J}_c},$$

$$\mathfrak{h}_{\mathbf{J}}^{*\prime} := \mathrm{span}_{\mathbf{Q}} \Pi_{\mathbf{J}}, \qquad S_{\mathbf{J}} := \{s_\alpha \mid \alpha \in \Pi_{\mathbf{J}}\}.$$

For each $s \in S_{\mathbf{J}}$, we denote by α_s a unique element of $\Pi_{\mathbf{J}}$ such that $s = s_{\alpha_s}$. Note that

$$\Delta_{\mathbf{J}}^{re} = \coprod_{c=1}^{\mathrm{C}(\mathbf{J})} \Delta_{\mathbf{J}_c}^{re}, \qquad W_{\mathbf{J}} = \prod_{c=1}^{\mathrm{C}(\mathbf{J})} W_{\mathbf{J}_c}, \qquad \mathfrak{h}_{\mathbf{J}}^{*\prime} = \mathring{\mathfrak{h}}_{\mathbf{J}}^{*} \oplus \mathbf{Q}\delta.$$

We set $\mathring{Q}_{\mathbf{J}}^{\vee} := \bigoplus_{j \in \mathbf{J}} \mathbf{Z}\check{\alpha}_j$ and $T_{\mathbf{J}} := \{t_\lambda \mid \lambda \in \mathring{Q}_{\mathbf{J}}^{\vee}\}$. Then $W_{\mathbf{J}} = \mathring{W}_{\mathbf{J}} \ltimes T_{\mathbf{J}}$ (see Lemma 3.1). For the sake of notational convenience, we also set $\Pi_{\varnothing} := \varnothing$, $S_{\varnothing} := \varnothing$, $W_{\varnothing} := \{1\} \subset W$, and $\mathfrak{h}_{\varnothing}^{*\prime} := \{0\} \subset \mathfrak{h}^{*}$.

Proposition 5.4. *For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, the pair $(W_{\mathbf{J}}, S_{\mathbf{J}})$ is a Coxeter system and the triplet $(\mathfrak{h}_{\mathbf{J}}^{*\prime}, \Delta_{\mathbf{J}}, \Pi_{\mathbf{J}})$ is a root system of $(W_{\mathbf{J}}, S_{\mathbf{J}})$ over $\mathbf{Q}$ with the properties ${}_{\mathbf{Q}}\mathrm{R(v)}$ and ${}_{\mathbf{Q}}\mathrm{R(vi)}$.*

Proof. Thanks to Theorem 2.2, it suffices to show that the triplet $(\mathfrak{h}_{\mathbf{J}}^{*\prime}, \Delta_{\mathbf{J}}, \Pi_{\mathbf{J}})$ satisfies the six conditions ${}_{\mathbf{Q}}\mathrm{R(i)}$–${}_{\mathbf{Q}}\mathrm{R(vi)}$. The conditions ${}_{\mathbf{Q}}\mathrm{R(i)}$, ${}_{\mathbf{Q}}\mathrm{R(iv)}$, and the first equality in ${}_{\mathbf{Q}}\mathrm{R(iii)}$ are obvious. By definition, we have

$$\Delta_{\mathbf{J}} = \Delta_{\mathbf{J}-} \amalg \Delta_{\mathbf{J}+}, \qquad \Delta_{\mathbf{J}-} = -\Delta_{\mathbf{J}+}, \qquad \Delta_{\mathbf{J}+} = \bigcup_{c=1}^{\mathrm{C}(\mathbf{J})} \Delta_{\mathbf{J}_c+}.$$

Hence, to check the condition ${}_{\mathbf{Q}}\mathrm{R(ii)}$, it suffices to show that each element of $\Delta_{\mathbf{J}_c+}$ can be written as $\sum_{\alpha \in \Pi_{\mathbf{J}_c}} x_\alpha \alpha$ with $x_\alpha \in \mathbf{Z}_{\geq 0}$ for all $\alpha \in \Pi_{\mathbf{J}_c}$. We have

$$m\delta - \theta_{\mathbf{J}_c} = (\delta - \theta_{\mathbf{J}_c}) + (m-1)\delta \tag{5.1}$$

for each $m \in \mathbf{Z}_{\geq 2}$, and

$$m\delta - \varepsilon = \{m\delta - (\varepsilon + \alpha_j)\} + \alpha_j \tag{5.2}$$

for each $m \in \mathbf{Z}_{\geq 1}$ and $\varepsilon \in \mathring{\Delta}_{\mathbf{J}_c+} \backslash \{\theta_{\mathbf{J}_c}\}$, where $j \in \mathbf{J}_c$ such that $\varepsilon + \alpha_j \in \mathring{\Delta}_{\mathbf{J}_c+}$. If $\beta \in \Delta_{\mathbf{J}_c+} \backslash \Pi_{\mathbf{J}_c}$ satisfies $\bar{\beta} \in \mathring{\Delta}_{\mathbf{J}_c+}$, then we have either

$$\beta = m\delta + \bar{\beta} \text{ with } m \geq 1 \qquad \text{or} \qquad \beta = \bar{\beta} = \varepsilon + \eta \text{ with } \varepsilon, \eta \in \mathring{\Delta}_{\mathbf{J}_c+}. \tag{5.3}$$

In addition, we have

$$\delta = (\delta - \theta_{\mathbf{J}_c}) + \theta_{\mathbf{J}_c} \qquad \text{and} \qquad m\delta = (m-1)\delta + \delta \tag{5.4}$$

for each $m \in \mathbf{Z}_{\geq 2}$. By (5.1)–(5.4), we see that each element of $\Delta_{\mathbf{J}_c+} \backslash \Pi_{\mathbf{J}_c}$ can be written as $\beta + \gamma$ with $\beta, \gamma \in \Delta_{\mathbf{J}_c+}$. Hence, by induction on values of elements of $\Delta_{\mathbf{J}_c+}$ by the height function $\mathrm{ht} : \Delta_+ \to \mathbf{N}$, we see that each element of $\Delta_{\mathbf{J}_c+}$ can be written as a $\mathbf{Z}_{\geq 0}$-linear combination of $\Pi_{\mathbf{J}_c}$. Thus ${}_{\mathbf{Q}}\mathrm{R(ii)}$ and ${}_{\mathbf{Q}}\mathrm{R(v)}$ are satisfied, and ${}_{\mathbf{Q}}\mathrm{R(vi)}$ is clear since $\mathrm{ht}|_{\Delta_{\mathbf{J}+}}$ satisfies the required property in ${}_{\mathbf{Q}}\mathrm{R(vi)}$. Finally, we check the second equality in ${}_{\mathbf{Q}}\mathrm{R(iii)}$. Suppose that $\alpha_{s_0} \in \Pi_{\mathbf{J}_c}$ with $s_0 \in S_{\mathbf{J}_c}$. Since $\mathbf{N}\alpha_{s_0} \cap \Delta_{\mathbf{J}_c+} = \{\alpha_{s_0}\}$, each element of $\Delta_{\mathbf{J}_c+} \backslash \{\alpha_{s_0}\}$ can be written as $\sum_{s \in S_{\mathbf{J}_c}} x_s \alpha_s$ with $x_s \in \mathbf{Z}_{\geq 0}$ for all $s \in S_{\mathbf{J}_c}$ and $x_{s_1} > 0$ for some $s_1 \neq s_0$. This fact implies that $s_0(\Delta_{\mathbf{J}_c+} \backslash \{\alpha_{s_0}\}) = \Delta_{\mathbf{J}_c+} \backslash \{\alpha_{s_0}\}$. Thus the second equality in ${}_{\mathbf{Q}}\mathrm{R(iii)}$ is valid, since s_0 fixes pointwise $\Delta_{\mathbf{J}+} \backslash \Delta_{\mathbf{J}_c+}$. □

Corollary 5.5. *Let* $\mathbf{J}$ *be an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$.

(1) *The assignment* $y \mapsto \Phi_{\mathbf{J}}(y) := \Phi(y) \cap \Delta_{\mathbf{J}+}$ *defines a bijective mapping from* $W_{\mathbf{J}}$ *to* $\mathfrak{B}_{\mathbf{J}}$.

(2) *Suppose that* $y = s_1 s_2 \cdots s_n$ *with* $n \in \mathbf{N}$ *and* $s_1, s_2, \ldots, s_n \in S_{\mathbf{J}}$ *is a reduced expression of an element* $y \in W_{\mathbf{J}} \backslash \{1\}$. *Then the following equality holds:*

$$\Phi_{\mathbf{J}}(y) = \{\alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \cdots s_{n-1}(\alpha_{s_n})\},$$

where the elements of $\Phi_{\mathbf{J}}(y)$ *displayed above are distinct from each other. In particular,* $\#\Phi_{\mathbf{J}}(y) = \ell_{\mathbf{J}}(y)$, *where* $\ell_{\mathbf{J}} : W_{\mathbf{J}} \to \mathbf{Z}_{\geq 0}$ *is the length function of* $(W_{\mathbf{J}}, S_{\mathbf{J}})$.

Proof. Since $\Phi_{\mathbf{J}}(y) = \{\beta \in \Delta_{\mathbf{J}+} \mid y^{-1}(\beta) < 0\}$, the part (1) follows from Theorem 2.6 and Proposition 5.4. The part (2) follows from Theorem 2.2 and Proposition 5.4. □

Remarks. (1) An assertion similar to the part (1) of Corollary 5.5 was stated by P. Cellini and P. Papi in the proof of Theorem 3.12 in [11] with an outline of the proof. However, it seems that the detailed proof was not given in the paper.

(2) By Corollary 5.5(2) and the remark (1) below Theorem 2.2, the action of $W_{\mathbf{J}}$ on $\mathfrak{h}_{\mathbf{J}}^{*\prime}$ is faithful, and hence we may regards $W_{\mathbf{J}}$ as a subgroup of $\mathrm{GL}(\mathfrak{h}_{\mathbf{J}}^{*\prime})$.

Definition 5.6. For each $w \in \mathring{W}_{\mathbf{J}}$ and $\mathbf{K} \subset \mathbf{J}$, we set

$$\Delta_{\mathbf{J}}^{\mathbf{K}}(w, \pm) := \langle w \mathring{\Delta}_{\mathbf{J}\pm}^{\mathbf{K}} \rangle.$$

We denote it simply by $\Delta_{\mathbf{J}}(w,\pm)$ if $\mathbf{K}=\varnothing$, by $\Delta^{\mathbf{K}}(w,\pm)$ if $\mathbf{J}=\mathring{\mathbf{I}}$, and by $\Delta(w,\pm)$ if $\mathbf{K}=\varnothing$ and $\mathbf{J}=\mathring{\mathbf{I}}$.

Lemma 5.7. (1) *The set $\Delta^{\mathbf{K}}_{\mathbf{J}}(w,\pm)$ is an infinite set if and only if $\mathbf{K}\subsetneq\mathbf{J}$.*
(2) *For each $u\in\mathring{W}^{\mathbf{K}}_{\mathbf{J}}$ and $v\in\mathring{W}_{\mathbf{K}}$, we have*

$$\Delta^{\mathbf{K}}_{\mathbf{J}}(uv,\pm)=\Delta^{\mathbf{K}}_{\mathbf{J}}(u,\pm), \tag{5.5}$$

$$\Delta^{\mathbf{K}}_{\mathbf{J}}(u,-)=\Phi(u)\amalg u\Delta^{\mathbf{K}}_{\mathbf{J}}(1,-), \tag{5.6}$$

$$\Delta^{re}_{\mathbf{J}+}=\Delta^{\mathbf{K}}_{\mathbf{J}}(u,-)\amalg u\Delta^{re}_{\mathbf{K}+}\amalg\Delta^{\mathbf{K}}_{\mathbf{J}}(u,+). \tag{5.7}$$

Proof. (1) This follows from the fact that $\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm}$ is not empty if and only if $\mathbf{K}\subsetneq\mathbf{J}$.

(2) By Lemma 4.1(3), we have $uv\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm}=u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm}$, which implies (5.5). By definition, we have

$$\begin{aligned}\Delta^{\mathbf{K}}_{\mathbf{J}}(u,-)&=(\mathring{\Delta}_{\mathbf{J}+}\cap u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-})\amalg\{m\delta+\varepsilon\,|\,m\in\mathbf{N},\varepsilon\in u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}\}\\&=(\mathring{\Delta}_{\mathbf{J}+}\cap u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-})\amalg u\Delta^{\mathbf{K}}_{\mathbf{J}}(1,-).\end{aligned}$$

Moreover, since $u\in\mathring{W}^{\mathbf{K}}_{\mathbf{J}}$ we have $u\mathring{\Delta}_{\mathbf{K}-}\subset\mathring{\Delta}_{-}$, and hence $\Phi(u)=\mathring{\Delta}_{\mathbf{J}+}\cap u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}$. Thus (5.6) is valid. By definition, we have

$$\Delta^{re}_{\mathbf{J}+}=\langle\mathring{\Delta}_{\mathbf{J}}\rangle,\qquad\Delta^{\mathbf{K}}_{\mathbf{J}}(u,\pm)=\langle u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm}\rangle,\qquad u\Delta^{re}_{\mathbf{K}+}=\langle u\mathring{\Delta}_{\mathbf{K}}\rangle.$$

Thus (5.7) is valid, since $\mathring{\Delta}_{\mathbf{J}}=u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}-}\amalg u\mathring{\Delta}_{\mathbf{K}}\amalg u\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}+}$. □

Note that $\Delta_{\mathbf{J}+}$ is a convex set and that a subset $B\subset\Delta_{\mathbf{J}+}$ is a convex set in $\Delta_{\mathbf{J}+}$ if and only if B is a convex set (see Lemma 2.5(3)).

Proposition 5.8. *Let P be a subset of $\mathring{\Delta}$, and $\mathbf{J}$ a non-empty subset of $\mathring{\mathbf{I}}$.*
(1) *If P is a closed set, then $\langle P\rangle\amalg\Delta^{im}_{+}$ is a convex set.*
(2) *If P is a pointed closed set, then $\langle P\rangle$ is a real convex set.*
(3) *If P is a pointed biclosed set in $\mathring{\Delta}_{\mathbf{J}}$, then $\langle P\rangle$ is a real biconvex set in $\Delta_{\mathbf{J}+}$.*

Proof. (1) Suppose that $\beta+\gamma\in\Delta_{+}$ with $\beta,\gamma\in\langle P\rangle\amalg\Delta^{im}_{+}$. Then $\bar{\beta},\bar{\gamma}\in P\amalg\{0\}$ and $\bar{\beta}+\bar{\gamma}\in\mathring{\Delta}\amalg\{0\}$. Since P is closed, we have $\bar{\beta}+\bar{\gamma}\in P\amalg\{0\}$, and hence $\beta+\gamma\in\langle P\rangle\amalg\Delta^{im}_{+}$ by (i) of Lemma 5.2(1). Thus $\langle P\rangle\amalg\Delta^{im}_{+}$ is a convex set.

(2) It is clear that $\langle P\rangle\subset\Delta^{re}_{+}$. Suppose that $\beta+\gamma\in\Delta_{+}$ with $\beta,\gamma\in\langle P\rangle$. Then $\bar{\beta},\bar{\gamma}\in P$ and $\bar{\beta}+\bar{\gamma}\in\mathring{\Delta}\amalg\{0\}$. If $\bar{\beta}+\bar{\gamma}=0$ then $\bar{\beta}=-\bar{\gamma}\in P\cap(-P)$. This contradicts $P\cap(-P)=\varnothing$. Thus we get $\bar{\beta}+\bar{\gamma}\in\mathring{\Delta}$. Since P is closed, we have $\bar{\beta}+\bar{\gamma}\in P$, and hence $\beta+\gamma\in\langle P\rangle$ by (i) of Lemma 5.2(1). Therefore $\langle P\rangle$ is a real convex set.

(3) Since P is a pointed closed set, it follows from (2) that $\langle P\rangle$ is a real convex set. Since P is a biclosed set in $\mathring{\Delta}_{\mathbf{J}}$, the set $\mathring{\Delta}_{\mathbf{J}}\backslash P$ is a closed set. Thus, by (1), we see that $\Delta_{\mathbf{J}+}\backslash\langle P\rangle$ is a convex set, since $\Delta_{\mathbf{J}+}\backslash\langle P\rangle = \langle\mathring{\Delta}_{\mathbf{J}}\backslash P\rangle \amalg \Delta_{+}^{im}$. $\square$

Corollary 5.9. *Let* $\mathbf{K}$ *be a subset of* $\mathbf{J}$, *and* u *an element of* $\mathring{W}_{\mathbf{J}}^{\mathbf{K}}$. *Then*

(i) $u\Delta_{\mathbf{K}+}$ *is a convex set*; (ii) $\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)$ *is a real biconvex set in* $\Delta_{\mathbf{J}+}$.

Proof. We have $u\Delta_{\mathbf{K}+} = \langle u\mathring{\Delta}_{\mathbf{K}}\rangle \amalg \Delta_{+}^{im}$. Since $u\mathring{\Delta}_{\mathbf{K}}$ is a closed set, (i) follows from Proposition 5.8(1). It follows from Proposition 4.6 that $u\mathring{\Delta}_{\mathbf{J}\pm}^{\mathbf{K}}$ is a pointed biclosed set in $\mathring{\Delta}_{\mathbf{J}}$, hence (ii) follows from Proposition 5.8(3). $\square$

Lemma 5.10. *For* $\mathbf{K}\subset\mathbf{J}$ *and* $u\in\mathring{W}_{\mathbf{J}}^{\mathbf{K}}$, *we have* $\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)\dotplus u\Delta_{\mathbf{K}+}\subset\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)$.

Proof. Suppose that $\beta+\gamma\in\Delta_{+}$ with $\beta\in\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)$ and $\gamma\in u\Delta_{\mathbf{K}+}$. Then we have $\bar{\beta}\in u\mathring{\Delta}_{\mathbf{J}\pm}^{\mathbf{K}}$, $\bar{\gamma}\in u\mathring{\Delta}_{\mathbf{K}}\amalg\{0\}$, and $\bar{\beta}+\bar{\gamma}\in\mathring{\Delta}\amalg\{0\}$. Thus we get $\bar{\beta}+\bar{\gamma}\in u\mathring{\Delta}_{\mathbf{J}\pm}^{\mathbf{K}}$ by Lemma 4.1(2), and hence $\beta+\gamma\in\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)$ by (i) of Lemma 5.2(1). $\square$

Proposition 5.11. *Let* $\mathbf{K}$ *be a subset of* $\mathbf{J}$, *and* u *an element of* $\mathring{W}_{\mathbf{J}}^{\mathbf{K}}$.

(1) *If* C *is a convex set in* $u\Delta_{\mathbf{K}+}$, *then* $C\amalg\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)$ *is a convex set in* $\Delta_{\mathbf{J}+}$.

(2) *If* C *is a biconvex set in* $u\Delta_{\mathbf{K}+}$, *then* $C\amalg\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)$ *is a biconvex set in* $\Delta_{\mathbf{J}+}$.

Proof. (1) It follows from (ii) of Corollary 5.9 that $\Delta_{\mathbf{J}}^{\mathbf{K}}(u,\pm)$ is a convex set in $\Delta_{\mathbf{J}+}$. Thus the assertion follows from Lemma 2.5(4) and Lemma 5.10.

(2) By the equality (5.7), we have

$$\Delta_{\mathbf{J}+}\backslash\{C\amalg\Delta_{\mathbf{J}}^{\mathbf{K}}(u,-)\} = (u\Delta_{\mathbf{K}+}\backslash C)\amalg\Delta_{\mathbf{J}}^{\mathbf{K}}(u,+).$$

Since both C and $u\Delta_{\mathbf{K}+}\backslash C$ are convex sets in $u\Delta_{\mathbf{K}+}$, we see that both $C\amalg\Delta_{\mathbf{J}}^{\mathbf{K}}(u,-)$ and $(u\Delta_{\mathbf{K}+}\backslash C)\amalg\Delta_{\mathbf{J}}^{\mathbf{K}}(u,+)$ are convex sets in $\Delta_{\mathbf{J}}$ by (1), hence $C\amalg\Delta_{\mathbf{J}}^{\mathbf{K}}(u,-)$ is a biconvex set in $\Delta_{\mathbf{J}+}$. To prove of the assertion for $C\amalg\Delta^{\mathbf{J}}(u,+)$, it suffices to exchange the sign $\Delta^{\mathbf{J}}(u,-)$ for $\Delta^{\mathbf{J}}(u,+)$. $\square$

6. A parametrization of infinite real biconvex sets

In this section, we give a parametrization of the set $\mathfrak{B}_{\mathbf{J}}^{\infty}$ of all infinite real biconvex sets in $\Delta_{\mathbf{J}+}$ for each non-empty subset $\mathbf{J}\subset\mathring{\mathbf{I}}$.

Lemma 6.1. (1) *If* B *is a real coconvex set in* $\Delta_{\mathbf{J}+}$, *then for each* $\varepsilon\in\mathring{\Delta}_{\mathbf{J}}$ *we have either* $\langle\varepsilon\rangle\subset B$ *or* $\langle\varepsilon\rangle\dot{\subset}\Delta_{\mathbf{J}+}^{re}\backslash B$.

(2) *If* B *is a real biconvex set in* $\Delta_{\mathbf{J}+}$ *and a subset* $P\subset\mathring{\Delta}_{\mathbf{J}}$ *satisfies* $\langle P\rangle\dot{\subset}B$, *then we have* $\langle P\rangle\subset B$ *and* $\langle -P\rangle\cap B=\varnothing$.

Proof. (1) Suppose that there exists $m \in \mathbf{Z}_{\geq 0}$ such that $m\delta + \varepsilon \in \Delta^{re}_{\mathbf{J}+} \backslash B$. Since $B \subset \Delta^{re}_{+}$ we have $\Delta^{im}_{+} \subset \Delta_{\mathbf{J}+} \backslash B$. Thus we get $(m+l)\delta + \varepsilon \in \Delta^{re}_{\mathbf{J}+} \backslash B$ for all $l \in \mathbf{N}$ by the convexity of $\Delta_{\mathbf{J}+} \backslash B$, and hence $\langle \varepsilon \rangle \dot{\subset} \Delta^{re}_{\mathbf{J}+} \backslash B$.

(2) By (1), we have $\langle P \rangle \subset B$. Suppose that $\langle -P \rangle \cap B \neq \varnothing$. Then there exists an element $\varepsilon \in P$ such that $m\delta - \varepsilon \in B$ for some $m \in \mathbf{Z}_{\geq 0}$. Moreover, we have $\delta + \varepsilon \in B$ since $\langle \varepsilon \rangle \subset B$. By the convexity of B, we have $(m+1)\delta = (m\delta - \varepsilon) + (\delta + \varepsilon) \in B$. This contradicts $B \subset \Delta^{re}_{\mathbf{J}+}$. Hence we have $\langle -P \rangle \cap B = \varnothing$. □

Proposition 6.2. *Let B be a real convex set in $\Delta_{\mathbf{J}+}$, and set*

$$\bar{B} := \{\bar{\beta} \mid \beta \in B\}, \qquad P_B := \{\varepsilon \in \mathring{\Delta}_{\mathbf{J}} \mid \langle \varepsilon \rangle \dot{\subset} B\}.$$

Then both $\bar{B}$ and P_B are pointed closed subsets of $\mathring{\Delta}_{\mathbf{J}}$ such that $P_B \subset \bar{B}$. Moreover, if B is a real biconvex set in $\Delta_{\mathbf{J}+}$ then P_B is a pointed biclosed set in $\mathring{\Delta}_{\mathbf{J}}$.

Proof. It is clear that $\bar{B}, P_B \subset \mathring{\Delta}_{\mathbf{J}}$. Suppose that $\varepsilon + \eta \in \mathring{\Delta}$ with $\varepsilon, \eta \in \bar{B}$. By definition, there exist $\beta, \gamma \in B$ such that $\bar{\beta} = \varepsilon$, $\bar{\gamma} = \eta$. By the convexity of B, we have $\beta + \gamma \in B$, and hence $\varepsilon + \eta = \overline{\beta + \gamma} \in \bar{B}$. Thus $\bar{B}$ is a closed set. Suppose that $\varepsilon + \eta \in \mathring{\Delta}$ with $\varepsilon, \eta \in P_B$. By definition, we have $\langle \varepsilon \rangle, \langle \eta \rangle \dot{\subset} B$, and hence there exist $m, n \in \mathbf{Z}_{\geq 0}$ such that $(m+k)\delta + \varepsilon \in B$ and $(n+k)\delta + \eta \in B$ for all $k \in \mathbf{Z}_{\geq 0}$. By the convexity of B, we have $(m+n+k)\delta + \varepsilon + \eta \in B$ for all $k \in \mathbf{Z}_{\geq 0}$. Thus we get $\langle \varepsilon + \eta \rangle \dot{\subset} B$, and hence $\varepsilon + \eta \in P_B$. Therefore P_B is a closed set. Suppose that $\varepsilon \in \bar{B} \cap (-\bar{B})$. Then we have $\varepsilon, -\varepsilon \in \bar{B}$. Hence we may assume that $\varepsilon \in \bar{B} \cap \mathring{\Delta}_{\mathbf{J}+}$. Then there exist $m \in \mathbf{Z}_{\geq 0}$ and $n \in \mathbf{N}$ such that $m\delta + \varepsilon$, $n\delta - \varepsilon \in B$. By the convexity of B, we have $(m+n)\delta = (m\delta + \varepsilon) + (n\delta - \varepsilon) \in B$. This contradicts $B \subset \Delta^{re}_{\mathbf{J}+}$. Thus we get $\bar{B} \cap (-\bar{B}) = \varnothing$. Moreover, by definition, we have $P_B \subset \bar{B}$, and hence $P_B \cap (-P_B) = \varnothing$.

Next we prove the second assertion. It suffices to show that P_B is a coclosed set in $\mathring{\Delta}_{\mathbf{J}}$. By the definition of P_B and Lemma 6.1(1), we see that

$$P_B = \{\varepsilon \in \mathring{\Delta}_{\mathbf{J}} \mid \langle \varepsilon \rangle \subset B\}, \qquad \mathring{\Delta}_{\mathbf{J}} \backslash P_B = \{\varepsilon \in \mathring{\Delta}_{\mathbf{J}} \mid \langle \varepsilon \rangle \dot{\subset} \Delta^{re}_{\mathbf{J}+} \backslash B\}. \tag{6.1}$$

Suppose that $\varepsilon + \eta \in \mathring{\Delta}$ with $\varepsilon, \eta \in \mathring{\Delta}_{\mathbf{J}} \backslash P_B$. Then $\langle \varepsilon \rangle, \langle \eta \rangle \dot{\subset} \Delta^{re}_{\mathbf{J}+} \backslash B$ by (6.1). By the convexity of $\Delta_{\mathbf{J}+} \backslash B$, we have $\langle \varepsilon + \eta \rangle \dot{\subset} \Delta^{re}_{\mathbf{J}+} \backslash B$, and hence $\varepsilon + \eta \in \mathring{\Delta}_{\mathbf{J}} \backslash P_B$. Thus P_B is a coclosed set in $\mathring{\Delta}_{\mathbf{J}}$. □

Proposition 6.3. *Let $\mathbf{J}$ be an arbitrary non-empty subset of $\mathring{\mathbf{I}}$.*

(1) *If B is a real convex set in $\Delta_{\mathbf{J}+}$, then there exists an element $w \in \mathring{W}_{\mathbf{J}}$ such that $B \subset \Delta_{\mathbf{J}}(w, -)$.*

(2) *The assignment $w \mapsto \Delta_{\mathbf{J}}(w, -)$ defines a bijective mapping from $\mathring{W}_{\mathbf{J}}$ to the set $\mathfrak{M}$ of all maximal real convex sets in $\Delta_{\mathbf{J}+}$ (relative to the inclusion relation). Moreover, $\mathfrak{M}$ coincides with the set of all maximal real biconvex sets in $\Delta_{\mathbf{J}+}$.*

Proof. (1) It follows from Proposition 6.2 that $\bar{B}$ is a pointed closed subset of $\mathring{\Delta}_{\mathbf{J}}$. Hence, by Proposition 4.4, there exists an element $w \in \mathring{W}_{\mathbf{J}}$ such that $\bar{B} \subset w\mathring{\Delta}_{\mathbf{J}-}$. Then the following inclusion relation holds:

$$B \subset \langle \bar{B} \rangle \subset \langle w\mathring{\Delta}_{\mathbf{J}-} \rangle = \Delta_{\mathbf{J}}(w, -).$$

(2) It follows from Corollary 5.9 that $\Delta_{\mathbf{J}}(w, -)$ is a real biconvex set in $\Delta_{\mathbf{J}+}$ for each $w \in \mathring{W}_{\mathbf{J}}$. In particular, $\Delta_{\mathbf{J}}(w, -)$ is a real convex set in $\Delta_{\mathbf{J}+}$. To prove the maximality of $\Delta_{\mathbf{J}}(w, -)$, suppose that $\Delta_{\mathbf{J}}(w, -) \subset B$ for some real convex set B in $\Delta_{\mathbf{J}+}$. By (1), there exists an element $w' \in \mathring{W}_{\mathbf{J}}$ such that $B \subset \Delta_{\mathbf{J}}(w', -)$. Since $\Delta_{\mathbf{J}}(w, -) \subset \Delta_{\mathbf{J}}(w', -)$, we see that $w\mathring{\Delta}_{\mathbf{J}-} \subset w'\mathring{\Delta}_{\mathbf{J}-}$, which implies that $w = w'$, and hence $\Delta_{\mathbf{J}}(w, -) = B$. Therefore $\Delta_{\mathbf{J}}(w, -)$ is a maximal real convex set in $\Delta_{\mathbf{J}+}$. Moreover, by the argument above, the injectivity of the mapping is obvious. Finally, we prove the surjectivity of the mapping. Let B be a maximal real convex set in $\Delta_{\mathbf{J}+}$. By (1), there exists an element $w \in \mathring{W}_{\mathbf{J}}$ such that $B \subset \Delta_{\mathbf{J}}(w, -)$. The maximality of B implies that $B = \Delta_{\mathbf{J}}(w, -)$. □

Proposition 6.4. *Let* $\mathbf{J}$ *be an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$, *and* B *a real biconvex set in* $\Delta_{\mathbf{J}+}$. *Then there exist a unique subset* $\mathbf{K} \subset \mathbf{J}$ *and a unique element* $u \in \mathring{W}_{\mathbf{J}}^{\mathbf{K}}$ *such that* $\Delta_{\mathbf{J}}^{\mathbf{K}}(u, -) \subset B$ *and* $B \doteq \Delta_{\mathbf{J}}^{\mathbf{K}}(u, -)$. *Moreover,* B *is an infinite set if and only if* $\mathbf{K} \subsetneq \mathbf{J}$.

Proof. It follows from Proposition 6.2 that P_B is a pointed biclosed subset of $\mathring{\Delta}_{\mathbf{J}}$. Hence, by Proposition 4.6, there exist a unique subset $\mathbf{K} \subset \mathbf{J}$ and a unique element $u \in \mathring{W}_{\mathbf{J}}^{\mathbf{K}}$ such that $P_B = u\mathring{\Delta}_{\mathbf{J}-}^{\mathbf{K}}$. By (6.1), we see that $\langle \varepsilon \rangle \subset B$ for each $\varepsilon \in u\mathring{\Delta}_{\mathbf{J}-}^{\mathbf{K}}$ and that $\langle \varepsilon \rangle \doteq \Delta_{+}^{re} \backslash B$ for each $\varepsilon \in \mathring{\Delta}_{\mathbf{J}} \backslash u\mathring{\Delta}_{\mathbf{J}-}^{\mathbf{K}}$. Thus we get $\Delta_{\mathbf{J}}^{\mathbf{K}}(u, -) \subset B$ and $B \doteq \Delta_{\mathbf{J}}^{\mathbf{K}}(u, -)$. The second assertion follows from Lemma 5.7(1). □

Definition 6.5. For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\tilde{\mathscr{P}}_{\mathbf{J}} := \{(\mathbf{K}, u, y) \mid \mathbf{K} \subset \mathbf{J}, u \in \mathring{W}_{\mathbf{J}}^{\mathbf{K}}, y \in W_{\mathbf{K}}\},$$

$$\mathscr{P}_{\mathbf{J}} := \{(\mathbf{K}, u, y) \in \tilde{\mathscr{P}}_{\mathbf{J}} \mid \mathbf{K} \subsetneq \mathbf{J}\},$$

where $W_{\mathbf{K}}$ is the subgroup of W defined in Definition 5.3. For each $(\mathbf{K}, u, y) \in \tilde{\mathscr{P}}_{\mathbf{J}}$, we define a subset $\nabla_{\mathbf{J}}(\mathbf{K}, u, y) \subset \Delta_{\mathbf{J}+}^{re}$ by setting

$$\nabla_{\mathbf{J}}(\mathbf{K}, u, y) := \Delta_{\mathbf{J}}^{\mathbf{K}}(u, -) \amalg u\Phi_{\mathbf{K}}(y).$$

Note that $\nabla_{\mathbf{J}}(\mathbf{K}, u, y) = \Phi_{\mathbf{J}}(y)$ if $\mathbf{K} = \mathbf{J}$ and that $\nabla_{\mathbf{J}}(\mathbf{K}, u, y) = \Delta_{\mathbf{J}}(u, -)$ if $\mathbf{K} = \varnothing$. In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we remove $\mathbf{J}$ from the symbols above.

Lemma 6.6. (1) *For each* $(\mathbf{K}, u, y) \in \tilde{\mathscr{P}}_{\mathbf{J}}$, *the following equality holds:*

$$\nabla_{\mathbf{J}}(\mathbf{K}, u, y) = \Phi(u) \amalg u\nabla_{\mathbf{J}}(\mathbf{K}, 1, y). \tag{6.2}$$

Moreover, $\nabla_{\mathbf{J}}(\mathbf{K}, u, y)$ *is an infinite set if and only if* $(\mathbf{K}, u, y) \in \mathscr{P}_{\mathbf{J}}$.

(2) *Let* $(\mathbf{K}_1, u_1, y_1)$ *and* $(\mathbf{K}_2, u_2, y_2)$ *be elements of* $\tilde{\mathscr{P}}_{\mathbf{J}}$. *Then the following two conditions are equivalent:*

$$\text{(i)}\quad \nabla_{\mathbf{J}}(\mathbf{K}_1, u_1, y_1) \dot{\subset} \nabla_{\mathbf{J}}(\mathbf{K}_2, u_2, y_2); \qquad \text{(ii)}\quad \mathbf{K}_1 \supset \mathbf{K}_2, \quad u_1 \in u_2 \mathring{W}_{\mathbf{K}_1}.$$

Proof. (1) By the equality (5.6), we have

$$\Phi(u) \amalg u\nabla_{\mathbf{J}}(\mathbf{K}, 1, y) = \Phi(u) \amalg u\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -) \amalg u\Phi_{\mathbf{K}}(y)$$

$$= \Delta_{\mathbf{J}}^{\mathbf{K}}(u, -) \amalg u\Phi_{\mathbf{K}}(y) = \nabla_{\mathbf{J}}(\mathbf{K}, u, y).$$

The second assertion follows from Lemma 5.7(1).

(2) The assertion follows from Lemma 4.1(4) and Lemma 5.2(2), since (i) is equivalent to the condition: $\Delta_{\mathbf{J}}^{\mathbf{K}_1}(u_1, -) \dot{\subset} \Delta_{\mathbf{J}}^{\mathbf{K}_2}(u_2, -)$. □

Theorem 6.7. *The assignment* $(\mathbf{K}, u, y) \mapsto \nabla_{\mathbf{J}}(\mathbf{K}, u, y)$ *defines a bijective mapping from* $\tilde{\mathscr{P}}_{\mathbf{J}}$ *to* $\mathfrak{B}_{\mathbf{J}}^*$, *which maps* $\mathscr{P}_{\mathbf{J}}$ *onto* $\mathfrak{B}_{\mathbf{J}}^{\infty}$.

Proof. For each $(\mathbf{K}, u, y) \in \tilde{\mathscr{P}}_{\mathbf{J}}$, we see that $u\Phi_{\mathbf{K}}(y)$ is a biconvex set in $u\Delta_{\mathbf{K}+}$, and hence $\nabla_{\mathbf{J}}(\mathbf{K}, u, y)$ is a real biconvex set in $\Delta_{\mathbf{J}+}$ by Proposition 5.11(2). Thus the mapping $\nabla_{\mathbf{J}}$ is well-defined. Moreover, we have $\nabla_{\mathbf{J}}(\mathscr{P}_{\mathbf{J}}) \subset \mathfrak{B}_{\mathbf{J}}^{\infty}$ and $\nabla_{\mathbf{J}}(\tilde{\mathscr{P}}_{\mathbf{J}} \backslash \mathscr{P}_{\mathbf{J}}) \subset \mathfrak{B}_{\mathbf{J}}$ by the second assertion in Lemma 6.6(1). To prove the injectivity, suppose that $\nabla_{\mathbf{J}}(\mathbf{K}_1, u_1, y_1) = \nabla_{\mathbf{J}}(\mathbf{K}_2, u_2, y_2)$. By Lemma 6.6(2), we have $\mathbf{K}_1 = \mathbf{K}_2$ and $u_1 \in u_2 \mathring{W}_{\mathbf{K}_1}$, and hence $u_1 = u_2$ since $u_1, u_2 \in \mathring{W}_{\mathbf{J}}^{\mathbf{K}_1}$. Thus we get $\Delta_{\mathbf{J}}^{\mathbf{K}_1}(u_1, -) = \Delta_{\mathbf{J}}^{\mathbf{K}_2}(u_2, -)$ and $\Phi_{\mathbf{K}_1}(y_1) = \Phi_{\mathbf{K}_1}(y_2)$. By Corollary 5.5(1), we get $y_1 = y_2$ and $(\mathbf{K}_1, u_1, y_1) = (\mathbf{K}_2, u_2, y_2)$. Finally, we prove the surjectivity. Suppose that $B \in \mathfrak{B}_{\mathbf{J}} \amalg \mathfrak{B}_{\mathbf{J}}^{\infty}$. Then $B \subset \Delta_{\mathbf{J}+}^{re}$. By Proposition 6.4, there exist a subset $\mathbf{K} \subset \mathbf{J}$ and an element $u \in \mathring{W}_{\mathbf{J}}^{\mathbf{K}}$ such that $\Delta_{\mathbf{J}}^{\mathbf{K}}(u, -) \subset B$ and $B \dot{\subset} \Delta_{\mathbf{J}}^{\mathbf{K}}(u, -)$. Then $B \cap u\Delta_{\mathbf{K}+}^{re}$ is a finite biconvex set in $u\Delta_{\mathbf{K}+}$, since $B \cap u\Delta_{\mathbf{K}+}^{re} = B \cap u\Delta_{\mathbf{K}+}$. By Corollary 5.5(1), there exists an element $y \in W_{\mathbf{K}}$ such that $B \cap u\Delta_{\mathbf{K}+}^{re} = u\Phi_{\mathbf{K}}(y)$. Moreover, we have $B \cap \Delta_{\mathbf{J}}^{\mathbf{K}}(u, +) = \varnothing$ by Lemma 6.1(2). Thus we get $(\mathbf{K}, u, y) \in \tilde{\mathscr{P}}_{\mathbf{J}}$ and $B = \Delta_{\mathbf{J}}^{\mathbf{K}}(u, -) \amalg u\Phi_{\mathbf{K}}(y) = \nabla_{\mathbf{J}}(\mathbf{K}, u, y)$ by (5.7). □

7. Main theorem

In this section, we describe in detail relationships between the set $\mathscr{W}_{\mathbf{J}}^{\infty}$ of all infinite reduced words of the Coxeter system $(W_{\mathbf{J}}, S_{\mathbf{J}})$ and the set $\mathfrak{B}_{\mathbf{J}}^{\infty}$ of all infinite real biconvex sets in $\Delta_{\mathbf{J}+}$ for each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$. Let $W_{\mathbf{J}}^{\infty}$ be the quotient set of $\mathscr{W}_{\mathbf{J}}^{\infty}$ obtained by applying Definition 2.6 to the Coxeter system $(W_{\mathbf{J}}, S_{\mathbf{J}})$, and $\Phi_{\mathbf{J}}^{\infty} : W_{\mathbf{J}}^{\infty} \to \mathfrak{B}_{\mathbf{J}}^{\infty}$ the injective mapping obtained by applying Definition 2.7(1) to the root system $(\mathfrak{h}_{\mathbf{J}}^{*\prime}, \Delta_{\mathbf{J}}, \Pi_{\mathbf{J}})$ of the Coxeter system $(W_{\mathbf{J}}, S_{\mathbf{J}})$.

Proposition 7.1 ([1]). *Let* $\mathbf{K}$ *be a proper subset of* $\mathbf{J}$, *and* λ *an element of the lattice* $\mathring{Q}_{\mathbf{J}}^{\vee}$ (*see* Definition 5.3) *such that* $(\alpha_j|\lambda) > 0$ *for all* $j \in \mathbf{J} \backslash \mathbf{K}$ *and*

$(\alpha_k|\lambda) = 0$ *for all* $k \in \mathbf{K}$. *Choose a reduced expression* $t_\lambda = \boldsymbol{s}(1)\boldsymbol{s}(2)\cdots\boldsymbol{s}(n)$ *with* $n \in \mathbf{N}$ *and* $\boldsymbol{s}(1), \boldsymbol{s}(2), \ldots, \boldsymbol{s}(n) \in S_{\mathbf{J}}$, *and define an infinite sequence* $\boldsymbol{s} \in S_{\mathbf{J}}^{\mathbf{N}}$ *by setting* $\boldsymbol{s} := (\boldsymbol{s}(1), \boldsymbol{s}(2), \ldots, \boldsymbol{s}(n))^{\infty}$ (*see* Definition 2.7). *Then the infinite sequence* $\boldsymbol{s}$ *is an element of* $\mathscr{W}_{\mathbf{J}}^{\infty}$ *such that* $\Phi_{\mathbf{J}}^{\infty}([\boldsymbol{s}]) = \Delta_{\mathbf{J}}^{\mathbf{K}}(1, -)$.

Remark. In [1], J. Beck showed the previous proposition in the case where $\mathbf{J} = \mathring{\mathbf{I}}$ and $\mathbf{K} = \varnothing$.

Definition 7.2. For each proper subset $\mathbf{K}$ of $\mathbf{J}$, we denote by $z_{\mathbf{J}}^{\mathbf{K}}$ the unique element of $W_{\mathbf{J}}^{\infty}$ such that $\Phi_{\mathbf{J}}^{\infty}(z_{\mathbf{J}}^{\mathbf{K}}) = \Delta_{\mathbf{J}}^{\mathbf{K}}(1, -)$, and define a mapping $\chi_{\mathbf{J}} : \mathscr{P}_{\mathbf{J}} \to W_{\mathbf{J}}^{\infty}$ by setting for each $(\mathbf{K}, u, y) \in \mathscr{P}_{\mathbf{J}}$:

$$\chi_{\mathbf{J}}((\mathbf{K}, u, y)) := uy.z_{\mathbf{J}}^{\mathbf{K}}.$$

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we remove $\mathbf{J}$ from the symbols above.

Lemma 7.3. *For each* $\mathbf{K} \subset \mathbf{J}$ *and* $y \in \mathring{W}_{\mathbf{K}} T_{\mathbf{J}}$, *we have*

$$y\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -) \doteq \Delta_{\mathbf{J}}^{\mathbf{K}}(1, -), \tag{7.1}$$

$$y\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -)\backslash \Lambda \subset \Delta_{\mathbf{J}}^{\mathbf{K}}(1, -), \tag{7.2}$$

$$\{\Phi_{\mathbf{J}}(y)\backslash(-\Lambda)\} \cap \Delta_{\mathbf{K}+}^{re} = \Phi_{\mathbf{J}}(y) \cap \Delta_{\mathbf{K}+}^{re}, \tag{7.3}$$

where $\Lambda := y\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -) \cap \Delta_{\mathbf{J}-}^{re}$.

Proof. Since $\bar{y} \in \mathring{W}_{\mathbf{K}}$ we have $\bar{y}\mathring{\Delta}_{\mathbf{J}-}^{\mathbf{K}} = \mathring{\Delta}_{\mathbf{J}-}^{\mathbf{K}}$ by Lemma 4.1(3). Hence (7.1) follows from (iii) of Lemma 5.2(1). Moreover, by (ii) of Lemma 5.2(1), we have

$$\overline{y\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -)} \subset \mathring{\Delta}_{\mathbf{J}-}^{\mathbf{K}}. \tag{7.4}$$

By the definition of Λ, we have

$$y\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -)\backslash \Lambda = y\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -) \cap \Delta_{\mathbf{J}+}^{re}.$$

Thus (7.2) follows from (7.4). Moreover, by (7.4) we have

$$-\Lambda = (-y\Delta_{\mathbf{J}}^{\mathbf{K}}(1, -)) \cap \Delta_{\mathbf{J}+}^{re} \subset \Delta_{\mathbf{J}}^{\mathbf{K}}(1, +),$$

and hence $(-\Lambda) \cap \Delta_{\mathbf{K}+}^{re} = \varnothing$. Thus we get

$$\{\Phi_{\mathbf{J}}(y)\backslash(-\Lambda)\} \cap \Delta_{\mathbf{K}+}^{re} = \Phi_{\mathbf{J}}(y) \cap \Delta_{\mathbf{K}+}^{re}. \qquad \square$$

Theorem 7.4. *Let* $\mathbf{J}$ *be an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$.

(1) *For each* $x \in W_{\mathbf{J}}$ *and* $\mathbf{K} \subsetneq \mathbf{J}$, *we have the following equality:*

$$\Phi_{\mathbf{J}}^{\infty}(x.z_{\mathbf{J}}^{\mathbf{K}}) = \nabla_{\mathbf{J}}(\mathbf{K}, \bar{x}^{\mathbf{K}}, z_x) \tag{7.5}$$

with a unique element $z_x \in W_{\mathbf{K}}$ *such that*

$$\Phi_{\mathbf{J}}((\bar{x}^{\mathbf{K}})^{-1}x) \cap \Delta_{\mathbf{K}+} = \Phi_{\mathbf{K}}(z_x). \tag{7.6}$$

(2) *Both* $\Phi_{\mathbf{J}}^{\infty}$ *and* $\chi_{\mathbf{J}}$ *are bijective and the following diagram is commutative:*

$$\begin{tikzcd} & \mathfrak{B}_{\mathbf{J}}^{\infty} & \\ W_{\mathbf{J}}^{\infty} \arrow[ur, "\Phi_{\mathbf{J}}^{\infty}"] & & \mathscr{P}_{\mathbf{J}}. \arrow[ul, "\nabla_{\mathbf{J}}"'] \arrow[ll, "\chi_{\mathbf{J}}"] \end{tikzcd}$$

(3) *We have the following orbit decomposition:*

$$W_{\mathbf{J}}^{\infty} = \coprod_{\mathbf{K} \subsetneq \mathbf{J}} W_{\mathbf{J}}.z_{\mathbf{J}}^{\mathbf{K}}.$$

Proof. (1) Put $y = (\bar{x}^{\mathbf{K}})^{-1}x$. Then $y \in \mathring{W}_{\mathbf{K}} T_{\mathbf{J}}$. By Proposition 2.15, we have

$$\Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) = \{\Phi_{\mathbf{J}}(y) \backslash (-\Lambda)\} \amalg \{y\Delta_{\mathbf{J}}^{\mathbf{K}}(1,-) \backslash \Lambda\}, \tag{7.7}$$

where $\Lambda = y\Delta_{\mathbf{J}}^{\mathbf{K}}(1,-) \cap \Delta_{\mathbf{J}-}^{re}$. Since $\#\Lambda < \infty$ we have $\Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) \doteq \Delta_{\mathbf{J}}^{\mathbf{K}}(1,-)$ by (7.1). Thus, by Lemma 6.1(2) we get

$$\Delta_{\mathbf{J}}^{\mathbf{K}}(1,-) \subset \Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}), \tag{7.8}$$

$$\Delta_{\mathbf{J}}^{\mathbf{K}}(1,+) \cap \Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) = \emptyset. \tag{7.9}$$

By (7.2), (7.3), (7.6), and (7.7), we see that

$$\Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) \cap \Delta_{\mathbf{K}+}^{re} = \Phi_{\mathbf{J}}(y) \cap \Delta_{\mathbf{K}+}^{re} = \Phi_{\mathbf{K}}(z_x). \tag{7.10}$$

By (7.8)–(7.10) with (5.7), we have

$$\begin{aligned} \Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) &= \{\Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) \cap \Delta^{\mathbf{J}}(1,-)\} \amalg \{\Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) \cap \Delta_{\mathbf{K}+}^{re}\} \\ &= \Delta_{\mathbf{J}}^{\mathbf{K}}(1,-) \amalg \Phi_{\mathbf{K}}(z_x) = \nabla_{\mathbf{J}}(\mathbf{K}, 1, z_x). \end{aligned}$$

Hence, by Proposition 2.15 and (6.2), we get

$$\Phi_{\mathbf{J}}^{\infty}(x.z_{\mathbf{J}}^{\mathbf{K}}) = \Phi_{\mathbf{J}}^{\infty}(\bar{x}^{\mathbf{K}}.y.z_{\mathbf{J}}^{\mathbf{K}}) = \Phi(\bar{x}^{\mathbf{K}}) \amalg \bar{x}^{\mathbf{K}}\Phi_{\mathbf{J}}^{\infty}(y.z_{\mathbf{J}}^{\mathbf{K}}) = \nabla_{\mathbf{J}}(\mathbf{K}, \bar{x}^{\mathbf{K}}, z_x).$$

(2) By (1), we have

$$\Phi_{\mathbf{J}}^{\infty}(uy.z_{\mathbf{J}}^{\mathbf{K}}) = \nabla_{\mathbf{J}}(\mathbf{K}, u, y) \tag{7.11}$$

for each $(\mathbf{K}, u, y) \in \mathscr{P}_{\mathbf{J}}$. Hence $\Phi_{\mathbf{J}}^{\infty} \circ \chi_{\mathbf{J}} = \nabla_{\mathbf{J}}$, which implies the surjectivity of $\Phi_{\mathbf{J}}^{\infty}$ since $\nabla_{\mathbf{J}}$ is bijective (see Theorem 6.7). Moreover, since $\Phi_{\mathbf{J}}^{\infty}$ is injective, $\Phi_{\mathbf{J}}^{\infty}$ is bijective, so is $\chi_{\mathbf{J}}$.

(3) Since $\chi_{\mathbf{J}}$ is surjective, we have $W_{\mathbf{J}}^{\infty} = \bigcup_{\mathbf{K} \subsetneq \mathbf{J}} W_{\mathbf{J}}.Z_{\mathbf{J}}^{\mathbf{K}}$. Hence, it suffices to show that this union is disjoint. By (7.5), (7.11), and the injectivity of $\Phi_{\mathbf{J}}^{\infty}$, we have the following equality:

$$x.Z_{\mathbf{J}}^{\mathbf{K}} = \bar{x}^{\mathbf{K}} z_x.Z_{\mathbf{J}}^{\mathbf{K}} \tag{7.12}$$

for each $x \in W_{\mathbf{J}}$. Suppose that $x.Z_{\mathbf{J}}^{\mathbf{K}} = y.Z_{\mathbf{J}}^{\mathbf{L}}$ for some $\mathbf{L} \subsetneq \mathbf{J}$ and $y \in W_{\mathbf{J}}$. By (7.12), we have $\bar{x}^{\mathbf{K}} z_x.Z_{\mathbf{J}}^{\mathbf{K}} = \bar{y}^{\mathbf{L}} z_y.Z_{\mathbf{J}}^{\mathbf{L}}$ with a unique $z_y \in W_{\mathbf{L}}$. Thus we get $\mathbf{K} = \mathbf{L}$ since $\chi_{\mathbf{J}}$ is injective. □

Remark. The existence and uniqueness of the element $z_x \in W_{\mathbf{K}}$ satisfying (7.6) are guaranteed by Lemma 2.5(2) and Corollary 5.5(1).

Lemma 7.5. *If B is a biconvex set in $\Delta_{\mathbf{J}+}$, then we have either $B \subset \Delta_{\mathbf{J}+}^{re}$ or $\Delta_{+}^{im} \subset B$.*

Proof. We claim that if $B \cap \Delta_{+}^{im} \neq \varnothing$ then $\Delta_{+}^{im} \subset B$. Indeed, if $m\delta \in B$ for some $m \in \mathbf{N}$, then $\delta \in B$ by the convexity of $\Delta_{\mathbf{J}+} \backslash B$, and hence $m\delta \in B$ for all $m \in \mathbf{N}$ by the convexity of B, i.e., $\Delta_{+}^{im} \subset B$. Thus we have either $B \subset \Delta_{\mathbf{J}+}^{re}$ or $\Delta_{+}^{im} \subset B$. □

Corollary 7.6. *Let B be a subset of $\Delta_{\mathbf{J}+}$. Then B is a biconvex set in $\Delta_{\mathbf{J}+}$ if and only if one of the following* (a)–(d) *holds:*

(a) $B = \Phi_{\mathbf{J}}(z)$; (b) $B = \Delta_{\mathbf{J}+} \backslash \Phi_{\mathbf{J}}(z)$; (c) $B = \Phi_{\mathbf{J}}^{\infty}(Z)$; (d) $B = \Delta_{\mathbf{J}+} \backslash \Phi_{\mathbf{J}}^{\infty}(Z)$,

where z is an element of $W_{\mathbf{J}}$ and Z is an element of $W_{\mathbf{J}}^{\infty}$.

Proof. The "if part" is obvious. Let us prove the "only if part". By Lemma 7.5, we have either $B \subset \Delta_{\mathbf{J}+}^{re}$ or $\Delta_{+}^{im} \subset B$. If $B \subset \Delta_{\mathbf{J}+}^{re}$ and $\#B < \infty$, then $B = \Phi_{\mathbf{J}}(z)$ with $z \in W_{\mathbf{J}}$ by Theorem 2.6. If $B \subset \Delta_{\mathbf{J}+}^{re}$ and $\#B = \infty$, then $B = \Phi_{\mathbf{J}}^{\infty}(Z)$ with $Z \in W_{\mathbf{J}}^{\infty}$ by Theorem 7.4(2). If $B \subset \Delta_{+}^{im}$, then $\Delta_{\mathbf{J}+} \backslash B$ is a real biconvex set in $\Delta_{\mathbf{J}+}$. Hence we have either $\Delta_{\mathbf{J}+} \backslash B = \Phi_{\mathbf{J}}(z)$ or $\Delta_{\mathbf{J}+} \backslash B = \Phi_{\mathbf{J}}^{\infty}(Z)$, i.e., $B = \Delta_{\mathbf{J}+} \backslash \Phi_{\mathbf{J}}(z)$ or $B = \Delta_{\mathbf{J}+} \backslash \Phi_{\mathbf{J}}^{\infty}(Z)$, where $z \in W_{\mathbf{J}}$ and $Z \in W_{\mathbf{J}}^{\infty}$. □

Remark. By the corollary, we see that a subset $B \subset \Delta_{\mathbf{J}+}$ is a biconvex set in $\Delta_{\mathbf{J}+}$ if and only if B satisfies the conditions ${}_{\mathbf{Q}}\mathrm{C(i)}'$ and ${}_{\mathbf{Q}}\mathrm{C(ii)}'$ with replacing Δ_{+} by $\Delta_{\mathbf{J}+}$ (see the remarks below Theorem 2.6).

Example. Suppose that Δ is of the type $A_2^{(1)}$ and $\mathbf{J} = \mathring{\mathbf{I}} = \{1, 2\}$. Then $\{\varnothing, \{1\}, \{2\}\}$ is the set of all proper subsets of $\mathring{\mathbf{I}}$ and the following equalities hold:

$$\mathring{W}^{\varnothing} = \mathring{W} = \langle s_1, s_2 \rangle, \qquad \mathring{W}^{\{1\}} = \{1, s_2, s_1 s_2\}, \qquad \mathring{W}^{\{2\}} = \{1, s_1, s_2 s_1\}.$$

Thus the following set

$$\mathscr{P} = \{(\emptyset, u, 1) \mid u \in \langle s_1, s_2 \rangle\} \amalg \{(\{1\}, u, y) \mid u \in \{1, s_2, s_1 s_2\}, y \in \langle s_{\delta-\alpha_1}, s_1 \rangle\}$$
$$\amalg \{(\{2\}, u, y) \mid u \in \{1, s_1, s_2 s_1\}, y \in \langle s_{\delta-\alpha_2}, s_2 \rangle\}$$

parametrizes the set $\mathfrak{B}^\infty$ of all infinite real biconvex set by the following mapping:

$$\nabla(\mathbf{K}, u, y) := \Delta^{\mathbf{K}}(u, -) \amalg u\Phi_{\mathbf{K}}(y).$$

In particular, the following three sets are fundamental infinite real biconvex sets:

$$\nabla(\emptyset, 1, 1) = \Delta^{\emptyset}(1, -) = \{m\delta - \alpha_1, m\delta - \alpha_1 - \alpha_2, m\delta - \alpha_2 \mid m \in \mathbf{N}\},$$
$$\nabla(\{1\}, 1, 1) = \Delta^{\{1\}}(1, -) = \{m\delta - \alpha_1 - \alpha_2, m\delta - \alpha_2 \mid m \in \mathbf{N}\},$$
$$\nabla(\{2\}, 1, 1) = \Delta^{\{2\}}(1, -) = \{m\delta - \alpha_1, m\delta - \alpha_1 - \alpha_2 \mid m \in \mathbf{N}\}.$$

[1] Set $\lambda_\emptyset = \alpha_1 + \alpha_2$. Then $\lambda_\emptyset$ is an element of $\mathring{Q}^\vee = \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_2$ such that $(\alpha_1|\lambda_\emptyset) = (\alpha_2|\lambda_\emptyset) = 1 > 0$, and hence $\Phi(t_{\lambda_\emptyset}) = \{\delta - \alpha_1 - \alpha_2, \delta - \alpha_1, 2\delta - \alpha_1 - \alpha_2, \delta - \alpha_2\}$. Since $t_{\lambda_\emptyset} = s_0 s_2 s_1 s_2$ is a reduced expression, the infinite sequence $\boldsymbol{s}_\emptyset := (s_0, s_2, s_1, s_2)^\infty$ is an infinite reduced word such that $\Phi^\infty([\boldsymbol{s}_\emptyset]) = \nabla(\emptyset, 1, 1)$. For example, set $u := s_1 s_2$ and $\boldsymbol{s}'_\emptyset := (s_1, s_2)\boldsymbol{s}_\emptyset$. Then $\boldsymbol{s}'_\emptyset$ is an infinite reduced word satisfying $u.[\boldsymbol{s}_\emptyset] = [\boldsymbol{s}'_\emptyset]$ and

$$\Phi^\infty([\boldsymbol{s}'_\emptyset]) = \nabla(\emptyset, u, 1) = \{m\delta + \alpha_1 + \alpha_2, m\delta + \alpha_1, n\delta - \alpha_2 \mid m \in \mathbf{Z}_{\geq 0}, n \in \mathbf{N}\}.$$

[2] Set $\lambda_{\{1\}} := \alpha_1 + 2\alpha_2$. Then $\lambda_\emptyset$ is an element of $\mathring{Q}^\vee$ such that $(\alpha_1|\lambda_{\{1\}}) = 0$ and $(\alpha_2|\lambda_{\{1\}}) = 3 > 0$, and hence $\Phi(t_{\lambda_{\{1\}}}) = \{m\delta - \alpha_1 - \alpha_2, m\delta - \alpha_2 \mid m = 1, 2, 3\}$. Since $t_{\lambda_{\{1\}}} = s_0 s_1 s_2 s_0 s_1 s_2$ is a reduced expression, the infinite sequence $\boldsymbol{s}_{\{1\}} := (s_0, s_1, s_2)^\infty$ is an infinite reduced word such that $\Phi^\infty([\boldsymbol{s}_{\{1\}}]) = \nabla(\{1\}, 1, 1)$. The group $W_{\{1\}}$ is isomorphic to the infinite dihedral group and satisfies the equality:

$$W_{\{1\}} = \{(s_{\delta-\alpha_1} s_1)^n, (s_{\delta-\alpha_1} s_1)^n s_{\delta-\alpha_1}, (s_1 s_{\delta-\alpha_1})^n, (s_1 s_{\delta-\alpha_1})^n s_1 \mid n \in \mathbf{Z}_{\geq 0}\}.$$

For example, set $u := s_2$ and $y := (s_{\delta-\alpha_1} s_1)^n$, then $uy(s_0 s_1 s_2)^{2n} = s_2(s_0 s_2 s_1 s_2)^{2n}$, and hence the infinite sequence $\boldsymbol{s}'_{\{1\}} := (s_2)(s_0, s_2, s_1, s_2)^{2n}\boldsymbol{s}_{\{1\}}$ is an infinite reduced word satisfying $uy.[\boldsymbol{s}_{\{1\}}] = [\boldsymbol{s}'_{\{1\}}]$ and

$$\Phi^\infty([\boldsymbol{s}'_{\{1\}}]) = \nabla(\{1\}, u, y) = \{m\delta - \alpha_1 - \alpha_2 \mid 1 \leq m \leq 2n\} \amalg \Delta^{\{1\}}(u, -),$$

where $\Delta^{\{1\}}(u, -) = \{m\delta - \alpha_1, n\delta + \alpha_2 \mid m \in \mathbf{N}, n \in \mathbf{Z}_{\geq 0}\}$. For one more example, set $u' := s_1 s_2$ and $y' = (s_1 s_{\delta-\alpha_1})^n$, then $u'y'(s_0 s_1 s_2)^{2n-1} = s_1 s_2 s_1 (s_0 s_2 s_1 s_2)^{2n-1}$, and hence the infinite sequence $\boldsymbol{s}''_{\{1\}} := (s_1, s_2, s_1)(s_0, s_2, s_1, s_2)^{2n-1}\boldsymbol{s}_{\{1\}}$ is an infinite reduced word satisfying $u'y'.[\boldsymbol{s}_{\{1\}}] = [\boldsymbol{s}''_{\{1\}}]$ and

$$\Phi^{\infty}([s''_{\{1\}}]) = \nabla(\{1\}, u', y') = \{m\delta + \alpha_2 \mid 0 \le m \le 2n-1\} \amalg \Delta^{\{1\}}(u, -),$$

where $\Delta^{\{1\}}(u', -) = \{m\delta + \alpha_1 + \alpha_2, m\delta + \alpha_1 \mid m \in \mathbf{Z}_{\ge 0}\}$.

References

[1] J. Beck, Convex bases of PBW-type for quantum affine algebras, Comm. Math. Phys. **165** (1994), 193–199.

[2] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[3] V. V. Deodhar, On the root system of a Coxeter group, Comm. Algebra. **10** (1982), 611–630.

[4] V. V. Deodhar, Some characterizations of Coxeter groups, Enseign. Math. **32** (1986), 111–120.

[5] K. Ito, The classification of convex orders on affine root systems, Comm. Algebra. **29** (2001), 5605–5630.

[6] V. G. Kac, Infinite dimensional Lie algebras, third edition, Cambridge Univ. Press, Cambridge, UK, 1990.

[7] A. N. Kirillov and N. Reshetikhin, q-Weyl group and a multiplicative formula for universal R-matrices, Comm. Math. Phys. **134** (1990), 421–431.

[8] S. M. Khoroshkin and V. N. Tolstoy, The Cartan-Weyl basis and the universal R-matrix for quantum Kac-Moody algebras and superalgebras, In: Quantum Symmetries, Proc. of the Int. Workshop on Math. Physics (Clausthal, 1991), World Sci. Publishing, River Edge, NJ, 1993, pp. 336–351.

[9] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, Boston, 1993.

[10] P. Papi, A characterization of a special ordering in a root system, Proc. Amer. Math. Soc. **120** (1994), 661–665.

[11] P. Cellini and P. Papi, The structure of total reflection orders in affine root systems, J. Algebra. **205** (1998), 207–226.

Ken Ito
Science Division
Center for General Education
Aichi Institute of Technology
Yakusa-cho, Toyota 470-0392*, Japan*
e-mail: ken-ito@aitech.ac.jp