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The Herglotz wave function, the Vekua transform and the enclosure method

Masaru Ikehata

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Abstract. This paper gives applications of the enclosure method introduced by the author to typical inverse obstacle and crack scattering problems in two dimensions. Explicit extraction formulae of the convex hull of unknown polygonal sound-hard obstacles and piecewise linear cracks from the far field pattern of the scattered field at a fixed wave number and at most two incident directions are given. The main new points of this paper are: a combination of the enclosure method and the Herglotz wave function; explicit construction of the density in the Herglotz wave function by using the idea of the Vekua transform. By virtue of the construction, one can avoid any restriction on the wave number in the extraction formulae. An attempt for the case when the far field pattern is given on limited angles is also given.

1. Introduction

In this paper we consider typical inverse problems for the Helmholtz equation in two-dimensions. First let us consider an inverse obstacle scattering problem. Let $D \subset \mathbf{R}^2$ be a bounded open set with Lipschitz boundary and satisfy that $\mathbf{R}^2 \backslash \overline{D}$ is connected. Using a variational method (e.g., see [6] and references therein), one knows that: given $k > 0$ and $d \in S^1$ there exists a unique $u \in C^\infty(\mathbf{R}^2 \backslash \overline{D})$ satisfying (1) to (3) described below:

(1) u satisfies the Helmholtz equation

$$\triangle u + k^2 u = 0 \qquad \text{in } \mathbf{R}^2 \backslash \overline{D};$$

(2) there exists a disc B_R with radius R centered at 0 such that $\overline{D} \subset B_R$, $u|_{B_R \backslash \overline{D}} \in H^1(B_R \backslash \overline{D})$ and for all $\phi \in H^1(B_R \backslash \overline{D})$ with $\phi = 0$ on ∂B_R

$$\int_{B_R \backslash \overline{D}} (\nabla u \cdot \nabla \phi - k^2 u \phi) dx = 0; \tag{1.1}$$

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(3) $w = u - e^{ikx\cdot d}$ satisfies the outgoing Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - ikw\right) = 0$$

where $r = |x|$.

Note that the condition $u|_{B_R\setminus\bar{D}} \in H^1(B_R\setminus\bar{D})$ of (2) gives a restriction on the singularity of u in a neighbourhood of ∂D; (1.1) is a weak formulation of the boundary condition $\partial u/\partial \nu = 0$ on ∂D where ν denotes the unit outward normal relative to $\mathbf{R}^2\setminus\bar{D}$.

It is well known that w has the asymptotic expansion as $r \to \infty$ uniformly with respect to $\varphi \in S^1$:

$$w(r\varphi) = \frac{e^{ikr}}{\sqrt{r}} F(\varphi; d, k) + O\left(\frac{1}{r^{3/2}}\right).$$

The coefficient $F(\varphi; d, k)$ is called the far field pattern of the reflected wave w at direction φ.

In this paper we are interested in seeking extraction formulae of information about the location and shape of D from the far field pattern for fixed d and k. Recently the author established an extraction formula of the convex hull of D from the data w on the boundary of any fixed open disc that contains $\bar{D}$ provided D is polygonal ([11]). It is an application of the idea of the enclosure method to the inverse scattering problem (see [9, 12] for a simpler problem). Since the extraction formula in [11] is the starting point, we give a precise description of the formula and point out the problem.

Definition 1.1. We say that D is polygonal if $D = D_1 \cup D_2 \cup \cdots \cup D_m$; each D_j is a simply connected open set and polygon; $\bar{D}_j \cap \bar{D}_{j'} = \emptyset$ for $j \neq j'$.

In this paper we always assume that D is polygonal. Given direction $\omega \in S^1$ define $h_D(\omega) = \sup_{x\in D} x\cdot\omega$. We call the function $\omega \mapsto h_D(\omega)$ the support function of D. From the support function of D one obtains the convex hull of D.

Definition 1.2. We say that the direction ω is regular with respect to D if the set $\{x \mid x\cdot\omega = h_D(\omega)\} \cap \partial D$ consists of only one point.

We have already established the following.

Theorem 1.1 ([11]). *Let ω be regular with respect to D. Then the formula*

$$\lim_{\tau\to\infty} \frac{\log\left|\int_{\partial B_R}\left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u\right) d\sigma\right|}{\tau} = h_D(\omega), \tag{1.2}$$

is valid where $v(x) = e^{x\cdot(\tau\omega + i\sqrt{\tau^2+k^2}\omega^{\perp})}$ *and* $\omega^{\perp} = (\omega_2, -\omega_1)$ *for* $\omega = (\omega_1, \omega_2)$. *Moreover we have:*

if $t \geq h_D(\omega)$, *then*

$$\lim_{\tau\to\infty} e^{-\tau t}\left|\int_{\partial B_R}\left(\frac{\partial u}{\partial \nu}v - \frac{\partial v}{\partial \nu}u\right)d\sigma\right| = 0;$$

if $t < h_D(\omega)$, *then*

$$\lim_{\tau\to\infty} e^{-\tau t}\left|\int_{\partial B_R}\left(\frac{\partial u}{\partial \nu}v - \frac{\partial v}{\partial \nu}u\right)d\sigma\right| = \infty.$$

It is well known that $\partial u/\partial \nu$ on ∂B_R can be computed from u on ∂B_R by using the Dirichlet-to-Neumann map outside B_R. Moreover one can calculate u on ∂B_R from $F(\cdot\,; d, k)$ for fixed d and k by using, e.g., a formula in the point source method (see [16]). Thus one can say that (1.2) essentially gives an extraction formula of the support function from the far field pattern for fixed d and k. However, the computation involves mainly two limiting procedures: the first is the procedure for calculating u and $\partial u/\partial \nu$ on ∂B_R from $F(\cdot\,; d, k)$; the second is the procedure for calculating $h_D(\omega)$ from u and $\partial u/\partial \nu$ on ∂B_R by using (1.2).

In this paper we present a direct formula that extracts the value of the support function of unknown polygonal sound hard obstacles from the far field pattern for fixed d and k just by using only one limiting procedure.

We identify the point $\vartheta = (\vartheta_1, \vartheta_2) \in S^1$ with the complex number $\vartheta_1 + i\vartheta_2$ and denote it by the same symbol ϑ.

Now we state the results. Given $N = 1, \ldots,$ $\tau > 0$, $\omega \in S^1$ and $k > 0$ define the function $g_N(\cdot\,; \tau, k, \omega)$ on S^1 by the formula

$$g_N(\varphi; \tau, k, \omega) = \frac{1}{2\pi}\sum_{|m|\leq N}\left\{\frac{ik\varphi}{(\tau + \sqrt{\tau^2+k^2})\omega}\right\}^m. \tag{1.3}$$

Theorem 1.2. *Let* ω *be regular with respect to D. Let* β_0 *be the unique positive solution of the equation*

$$\frac{2}{e}s + \log s = 0.$$

Let β *satisfy* $0 < \beta < \beta_0$. *Let* $\{\tau(N)\}_{N=1,\ldots}$ *be an arbitrary sequence of positive numbers satisfying, as* $N \to \infty$

$$\tau(N) = \frac{\beta N}{eR} + O(1).$$

Then the formula

$$\lim_{N\to\infty}\frac{\log\left|\int_{S^1}F(-\varphi;d,k)g_N(\varphi;\tau(N),k,\omega)d\sigma(\varphi)\right|}{\tau(N)}=h_D(\omega),\tag{1.4}$$

is valid. Moreover we have:

if $t\geq h_D(\omega)$, *then*

$$\lim_{N\to\infty}e^{-\tau(N)t}\left|\int_{S^1}F(-\varphi;d,k)g_N(\varphi;\tau(N),k\omega)d\sigma(\varphi)\right|=0;$$

if $t<h_D(\omega)$, *then*

$$\lim_{N\to\infty}e^{-\tau(N)t}\left|\int_{S^1}F(-\varphi;d,k)g_N(\varphi;\tau(N),k,\omega)d\sigma(\varphi)\right|=\infty.$$

Note that there is no restriction on k. (1.4) means that if one knows the Fourier coefficients of the far field pattern

$$\int_{S^1}F(\varphi;d,k)\varphi^m\,d\sigma(\varphi)=(-1)^m\int_{S^1}F(-\varphi;d,k)\varphi^m\,d\sigma(\varphi),\qquad |m|\leq N$$

for sufficiently large N and a disc that contains $\bar{D}$, then one can know an approximate value of $h_D(\omega)$. $g=g_N(\,\cdot\,;\tau(N),k,\omega)$ satisfies, as $N\to\infty$

$$\int_{S^1}e^{iky\cdot\varphi}g(\varphi)d\sigma(\varphi)\approx e^{y\cdot(\tau(N)\omega+i\sqrt{\tau(N)^2+k^2}\omega^\perp)},\qquad y\in\bar{B}_R$$

in an appropriate sense (see Theorem 2.1 in Section 2). The function

$$\int_{S^1}e^{iky\cdot\varphi}g(\varphi)d\sigma(\varphi)$$

is called the Herglotz wave function with density g and satisfies the Helmholtz equation $\triangle v+k^2v=0$ in the whole space.

For a simple explanation of the origin of the desired density we make use of the idea of the Vekua transform which maps harmonic functions into solutions of the Helmholtz equations (see [17, 1] and also [2] for a recent study). Recall the Bessel function of order $m=0,\pm1,\pm2,\ldots$ given by the formula

$$J_m(z)=\left(\frac{z}{2}\right)^m\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+m)}\left(\frac{z}{2}\right)^{2n}.$$

The Vekua transform in two-dimensions takes the form:

$$T_k v(x) = v(x) - \frac{k|x|}{2}\int_0^1 v(tx)J_1(k|x|\sqrt{1-t})\frac{dt}{\sqrt{1-t}}$$

where v is an arbitrary harmonic function in $\mathbf{R}^2$. Using the formulae

$$k|x|\int_0^1 (1-w^2)^m J_1(k|x|w)dw = 1 - \left(\frac{2}{k|x|}\right)^m m! J_m(k|x|), \qquad m = 0, 1, \dots$$

one can easily know that, for $m = 0, 1, 2, \dots$

$$T_k : r^m e^{\pm im\theta} \mapsto \left(\frac{2}{k}\right)^m m! J_m(kr)e^{\pm im\theta}.$$

Using this property (rule), we reduce the construction problem of the density to that of the density in the "harmonic" Herglotz wave function

$$\int_{S^1}\{e^{ik\bar{\varphi}(y_1+iy_2)/2} + e^{ik\varphi(y_1-iy_2)/2} - 1\}g(\varphi)d\sigma(\varphi).$$

See Section 2 for details. Note that one can state and prove the main result of Section 2 (Theorem 2.1) without using the Vekua transform. However, the presence of the "harmonic" Herglotz wave function is something interesting.

(1.4) gives us a direct formula for extracting information about the convex hull from the far field pattern by using only one limiting procedure. An interesting question is to seek such a formula in the case when $F(\cdot\,; d, k)$ is given only on a proper subset of S^1. In Section 4 we present a small first step that gives extraction formulae of the support function from the far field pattern on given nonempty open subset of S^1 provided the center of the coordinates is inside the obstacles.

For another approach to the problem one can cite the no response test introduced in [15]. Therein, for fixed $\varepsilon > 0$ they define the functional of test domains D_t by taking the least upper bound of the quantity

$$\left|\int_{-\Gamma} F(-\varphi; d, k)g(\varphi)d\sigma(\varphi)\right|$$

with respect to all $g \in L^2(-\Gamma)$ satisfying

$$\sup_{y \in D_t}\left|\int_{-\Gamma} e^{iky\cdot\varphi}g(\varphi)d\sigma(\varphi)\right| \le \varepsilon. \tag{1.5}$$

Their idea is to make use of this functional to decide whether D is contained in D_t. In contrast to the enclosure method the form of the density of the Herglotz wave function is not specified except for a restriction on the bound (1.5) of the corresponding Herglotz wave function. Note that Theorem 1.2 gives an explicit way in the case when $\Gamma = S^1$ of deciding whether D is contained in the special test domain $B_R \cap \{x \in \mathbf{R}^2 \mid x \cdot \omega < t\}$ for each t with $|t| < R$.

Note that if one has the complete knowledge of the far field pattern for all d and fixed k, then one has two reconstruction formulae for general D. One is due to Kirsch ([13]) and another is due to the author ([7, 8]).

Next we consider an inverse scattering problem for piecewise linear cracks. Let Σ be the union of finitely many disjoint closed piecewise linear segments $\Sigma_1, \Sigma_2, \dots, \Sigma_m$. Assume that there exists a simply connected open set D such that D is a polygon and each Σ_j is contained in ∂D. We assume that $\bar{D} \subset B_R$. We denote by ν the unit outward normal relative to $B_R \backslash \bar{D}$. Denote by $H^1(B_R \backslash \Sigma)$ the set of all $L^2(B_R)$ functions u such that, $u^+ = u|_{B_R \backslash \bar{D}} \in H^1(B_R \backslash \bar{D})$, $u^- = u|_D \in H^1(D)$ and $u^+ = u^-$ on $\partial D \backslash \Sigma$.

It is well known that: given $k > 0$ and $d \in S^1$ there exists a unique $u \in C^\infty(\mathbf{R}^2 \backslash \Sigma)$ satisfying (4) to (6) described below:

(4) u satisfies the Helmholtz equation

$$\triangle u + k^2 u = 0 \qquad \text{in } \mathbf{R}^2 \backslash \Sigma;$$

(5) $u|_{B_R} \in H^1(B_R \backslash \Sigma)$ and for all $\phi \in H^1(B_R \backslash \Sigma)$ with $\phi = 0$ on ∂B_R

$$\int_{B_R \backslash \Sigma} (\nabla u \cdot \nabla \varphi - k^2 u \varphi) dx = 0; \tag{1.6}$$

(6) $w = u - e^{ikx \cdot d}$ satisfies the outgoing Sommerfeld radiation condition (see (3)).

(1.6) is a weak formulation of the boundary condition $\partial u / \partial \nu = 0$ on Σ where ν stands for the unit outward normal relative to $B_R \backslash \bar{D}$.

Define

$$h_\Sigma(\omega) = \sup_{x \in \Sigma} x \cdot \omega, \qquad \omega \in S^1.$$

We call h_Σ the support function of Σ. We say that the direction $\omega \in S^1$ is regular with respect to Σ if the set $\{x \mid x \cdot \omega = h_\Sigma(\omega)\} \cap \Sigma$ consists of only one point.

Given $\omega = (\omega_1, \omega_2) \in S^1$ set $\omega^\perp = (\omega_2, -\omega_1)$. Define the indicator function $I_\omega^d(\tau, t)$ by the formula

$$I_\omega^d(\tau, t) = e^{-\tau t} \left| \int_{|x|=R} \left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) d\sigma \right|$$

where $-\infty < t < \infty$, $\tau > 0$ and $v = e^{x \cdot (\tau\omega + i\sqrt{\tau^2 + k^2}\omega^\perp)}$.

In [11] we gave an extraction formula of the convex hull of Σ from the indicator function for a single incident direction d.

Theorem 1.3 ([11]). *Let ω be regular with respect to Σ. If every end point of $\Sigma_1, \Sigma_2, \dots, \Sigma_m$ satisfies $x \cdot \omega < h_\Sigma(\omega)$, then the formula*

$$\lim_{\tau \to \infty} \frac{\log I_\omega^d(\tau, 0)}{\tau} = h_\Sigma(\omega),$$

is valid. Moreover, we have:

if $t \geq h_\Sigma(\omega)$, then $\lim_{\tau \to \infty} I_\omega^d(\tau, t) = 0$;

if $t < h_\Sigma(\omega)$, then $\lim_{\tau \to \infty} I_\omega^d(\tau, t) = \infty$.

If there is an end point y of some Σ_j such that $y \cdot \omega = h_\Sigma(\omega)$, then, for d that is not perpendicular to ν on Σ_j near the point, the same conclusions as above are valid.

Note that ν on Σ_j in a neighbourhood of y becomes a constant vector if y is an end point of Σ_j.

From the definition of regularity of ω one knows that the set $\{x \,|\, x \cdot \omega = h_\Sigma(\omega)\} \cap \Sigma$ consists of a single point (say x_0). Then every endpoint of $\Sigma_1, \dots, \Sigma_m$ satisfies $x \cdot \omega < h_\Sigma(\omega)$ if and only if there exists j such that $x_0 \in \Sigma_j$ and ν on Σ_j has discontinuity at x_0.

One of two implications of Theorem 1.3 is: one can delete the sentences "If every end point of $\Sigma_1, \Sigma_2, \dots, \Sigma_m$ satisfies $x \cdot \omega < h_\Sigma(\omega)$" and "If there is an end point … are valid" in Theorem 1.3 by introducing the new indicator function

$$I_\omega(\tau, t) = I_\omega^{d_1}(\tau, t) + I_\omega^{d_2}(\tau, t)$$

where d_1, d_2 are two arbitrary linearly independent incident directions and fixed. This indicator function makes use of two reflected waves for two incident plane waves at a fixed wave number. This idea is coming from that of the multi probe method (see [10]). We obtain the following.

Theorem 1.4. *Let ω be regular with respect to Σ. The formula*

$$\lim_{\tau \to \infty} \frac{\log I_\omega(\tau, 0)}{\tau} = h_\Sigma(\omega),$$

is valid. Moreover, we have:

if $t \geq h_\Sigma(\omega)$, then $\lim_{\tau \to \infty} I_\omega(\tau, t) = 0$;

if $t < h_\Sigma(\omega)$, then $\lim_{\tau \to \infty} I_\omega(\tau, t) = \infty$.

The next theorem corresponds to Theorem 1.2.

Theorem 1.5. *Fix two arbitrary linearly independent directions d_1 and d_2. Let ω be regular with respect to Σ. Let $\{\tau(N)\}_{N=1,\dots}$ be the same as that of Theorem 1.2. Then the formula*

$$\lim_{N\to\infty} \frac{\log\left\{\sum_{j=1}^{2}\left|\int_{S^1} F(-\varphi; d_j, k) g_N(\varphi; \tau(N), k, \omega) d\sigma(\varphi)\right|\right\}}{\tau(N)} = h_\Sigma(\omega),$$

is valid. Moreover we have:
if $t \geq h_\Sigma(\omega)$, *then*

$$\lim_{N\to\infty} e^{-\tau(N)t} \sum_{j=1}^{2}\left|\int_{S^1} F(-\varphi; d_j, k) g_N(\varphi; \tau(N), k, \omega) d\sigma(\varphi)\right| = 0;$$

if $t < h_\Sigma(\omega)$, *then*

$$\lim_{N\to\infty} e^{-\tau(N)t} \sum_{j=1}^{2}\left|\int_{S^1} F(-\varphi; d_j, k) g_N(\varphi; \tau(N), k, \omega) d\sigma(\varphi)\right| = \infty.$$

The outline of this paper is as follows. In Section 2 we describe the derivation of the density given by (1.3) by using the idea of the Vekua transform and establish the desired property. The proof of Theorem 1.2, 1.4 and comment on the proof of Theorem 1.5 are found in Section 3. In Section 4 a case when the far field pattern is given on limited angles is discussed.

2. Construction of the density in the Herglotz wave function

The aim of this section is to construct a density $g \in L^2(S^1)$ explicitly such that

$$\int_{S^1} e^{iky\cdot\varphi} g(\varphi) d\sigma(\varphi) \approx e^{y\cdot(\tau\omega + i\sqrt{\tau^2+k^2}\omega^\perp)}, \qquad y \in \overline{B}_R.$$

We start with an elementary fact.

Proposition 2.1. *Let m be an arbitrary integer. Let ξ be an arbitrary complex vector. We have*

$$\frac{1}{2\pi}\int_{S^1} e^{\vartheta\cdot\xi}(\vartheta_1 + i\vartheta_2)^m d\sigma(\vartheta) = \begin{cases} \left(\dfrac{z}{2}\right)^m \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!(m+n)!}\left(\frac{1}{2}\right)^{2n}(z^*z)^n, & \text{if } m \geq 0, \\ \left(\dfrac{z^*}{2}\right)^{-m} \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!(-m+n)!}\left(\frac{1}{2}\right)^{2n}(z^*z)^n, & \text{if } m < 0 \end{cases} \tag{2.1}$$

where $\xi = (\xi_1, \xi_2)$, $z = \xi_1 + i\xi_2$ *and* $z^* = \xi_1 - i\xi_2$.

Proof. Let $w = e^{i\theta}$. Then

$$\xi_1 \cos\theta + \xi_2 \sin\theta = \frac{\xi_1}{2}(w + w^{-1}) + \frac{\xi_2}{2i}(w - w^{-1})$$
$$= \frac{1}{2}(z^* w + z w^{-1}).$$

This gives the expression

$$\int_{S^1} e^{\vartheta\cdot\xi}(\vartheta_1 + i\vartheta_2)^m d\sigma(\vartheta) = \int_0^{2\pi} e^{\xi_1\cos\theta + \xi_2\sin\theta} e^{im\theta}\, d\theta$$
$$= \frac{1}{i}\int_{|w|=1} w^{m-1} e^{(1/2)(z^* w + z w^{-1})}\, dw$$
$$= 2\pi\, \mathrm{Res}_{w=0}(w^{m-1} e^{(1/2)(z^* w + z w^{-1})}).$$

Write

$$w^{m-1} e^{(1/2)(z^* w + z w^{-1})} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n (z^* w + z w^{-1})^n w^{m-1}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \binom{n}{r} (z^*)^{n-r} z^r w^{n-2r+m-1}. \tag{2.2}$$

Consider the case when $m \geq 0$. If $n < m$, then $n + m > 2n \geq 2r$ for $0 \leq r \leq n$. Thus $n - 2r + m - 1 \neq -1$. This gives

$$\mathrm{Res}_{w=0}(w^{m-1} e^{(1/2)(z^* w + z w^{-1})})$$
$$= \mathrm{Res}_{w=0}\left(\sum_{n=m}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \binom{n}{r} (z^*)^{n-r} z^r w^{n+m-2r-1}\right)$$
$$= \mathrm{Res}_{w=0}\left(\sum_{l=0}^{\infty} \frac{1}{(m+l)!}\left(\frac{1}{2}\right)^{m+l} \sum_{r=0}^{m+l} \binom{m+l}{r} (z^*)^{m+l-r} z^r w^{2(m-r)+l-1}\right).$$

If $l = 2l'$, $l' = 0, \dots,$ then $2(m-r) + l - 1 = 2(m + l' - r) - 1$. Since $0 \leq r \leq m + 2l'$, $2(m-r) + l - 1$ becomes -1 only for $r = m + l'$. If $l = 2l' + 1$, $l' = 0, \dots,$ then $2(m-r) + l - 1 = 2(m + l' - r)$ and thus never become -1.

Therefore

$$\mathrm{Res}_{w=0}(w^{m-1} e^{(1/2)(z^* w + z w^{-1})})$$
$$= \sum_{l'=0}^{\infty} \frac{1}{(m+2l')!}\left(\frac{1}{2}\right)^{m+2l'} \binom{m+2l'}{m+l'} (z^*)^{m+2l'-(m+l')} z^{m+l'}$$

$$= \sum_{l'=0}^{\infty} \frac{1}{l'!(m+l')!} \left(\frac{1}{2}\right)^{m+2l'} (z^*)^{l'} z^{m+l'}$$

$$= \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{1}{n!(m+n!)} \left(\frac{1}{2}\right)^{2n} (z^* z)^n. \tag{2.3}$$

This gives (2.1) in the case when $m \geq 0$. Consider the case when $m < 0$. If $0 \leq n < -m$, then $n + m < 0$ and thus $n + m - 1 - 2r < -2r - 1 \leq -1$ for $0 \leq r \leq n$. Then from (2.2) we have

$$\mathrm{Res}_{w=0}(w^{m-1} e^{(1/2)(z^* w + z w^{-1})})$$

$$= \mathrm{Res}_{w=0}\left(\sum_{n=-m}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \binom{n}{r} (z^*)^{n-r} z^r w^{n+m-2r-1}\right)$$

$$= \mathrm{Res}_{w=0}\left(\sum_{l=0}^{\infty} \frac{1}{(-m+l)!}\left(\frac{1}{2}\right)^{-m+l} \sum_{r=0}^{-m+l} \binom{-m+l}{r} (z^*)^{-m+l-r} z^r w^{l-2r-1}\right)$$

$$= \mathrm{Res}_{w=0}\left(\sum_{l'=0}^{\infty} \frac{1}{(-m+2l')!}\left(\frac{1}{2}\right)^{-m+2l'} \sum_{r=0}^{-m+2l'} \binom{-m+2l'}{r} (z^*)^{-m+2l'-r} z^r w^{2(l'-r)-1}\right)$$

$$= \sum_{l'=0}^{\infty} \frac{1}{(-m+2l')!}\left(\frac{1}{2}\right)^{-m+2l'} \binom{-m+2l'}{l'} (z^*)^{-m+l'} z^{l'}$$

$$= \sum_{l'=0}^{\infty} \frac{1}{l'!(-m+l')!}\left(\frac{1}{2}\right)^{-m+2l'} z^{l'} (z^*)^{-m+l'}$$

$$= \left(\frac{z^*}{2}\right)^{-m} \sum_{l'=0}^{\infty} \frac{1}{l'!(-m+l'!)}\left(\frac{1}{2}\right)^{2l'} (z^* z)^{l'}. \qquad \square$$

Given $\omega = (\omega_1, \omega_2) \in S^1$ set $\omega^\perp = (\omega_2, -\omega_1)$. In this paper we denote the complex number $\omega_1 + i\omega_2$ by ω again and thus $\bar{\omega} = \omega_1 - i\omega_2$.

We give another expression of (2.1) for $\xi = r(\tau\omega + i\sqrt{\tau^2 + k^2}\omega^\perp)$ with $r > 0$. Since $\xi_1 = r(\tau\omega_1 + i\sqrt{\tau^2 + k^2}\omega_2)$ and $\xi_2 = r(\tau\omega_2 - i\sqrt{\tau^2 + k^2}\omega_1)$, we have

$$z = r(\tau + \sqrt{\tau^2 + k^2})\omega, \qquad z^* = r(\tau - \sqrt{\tau^2 + k^2})\bar{\omega}$$

and $z^* z = -(rk)^2$. Then (2.1) gives

$$\frac{1}{2\pi}\int_{S^1} e^{\vartheta\cdot\xi}(\vartheta_1 + i\vartheta_2)^m d\sigma(\vartheta) = \begin{cases} \left\{\dfrac{(\tau + \sqrt{\tau^2 + k^2})\omega}{k}\right\}^m J_m(kr), & \text{if } m \geq 0, \\ \left\{\dfrac{(\tau - \sqrt{\tau^2 + k^2})\bar{\omega}}{k}\right\}^{-m} J_{-m}(kr), & \text{if } m < 0. \end{cases} \tag{2.4}$$

Note that using the equation $J_{-m}(kr) = (-1)^m J_m(kr)$ one can rewrite (2.4) as for all integer m

$$\frac{1}{2\pi}\int_{S^1} e^{r\vartheta\cdot(\tau\omega+i\sqrt{\tau^2+k^2}\omega^\perp)}(\vartheta_1+i\vartheta_2)^m d\sigma(\vartheta) = \left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{k}\right\}^m J_m(kr),$$

however, for our purpose (2.4) is more convenient. From (2.4) we obtain the expansion formula

$$e^{y\cdot(\tau\omega+i\sqrt{\tau^2+k^2}\omega^\perp)} = \sum_{m=0}^{\infty}\left\{\frac{(\tau-\sqrt{\tau^2+k^2})\bar{\omega}}{k}\right\}^m J_m(kr)e^{im\theta} + \sum_{m=0}^{\infty}\left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{k}\right\}^m J_m(kr)e^{-im\theta} - J_0(kr) \quad (2.5)$$

where $y = (r\cos\theta, r\sin\theta)$.

Define the harmonic function $e_\omega(y;\tau,k)$ in the whole space by the formula

$$e_\omega(y;\tau,k) = e^{(\tau-\sqrt{\tau^2+k^2})\bar{\omega}(y_1+iy_2)/2} + e^{(\tau+\sqrt{\tau^2+k^2})\omega(y_1-iy_2)/2} - 1.$$

Then from (2.5) one immediately obtains the following.

Proposition 2.2. *The Vekua transform of the harmonic function $e_\omega(y;\tau,k)$ coincides with $e^{y\cdot(\tau\omega+i\sqrt{\tau^2+k^2}\omega^\perp)}$.*

Next consider the case when ξ in (2.1) is given by the complex vector $\xi = ikr\varphi$, $\varphi\in S^1$, $r>0$. Then $z = ikr\varphi$, $z^* = ikr\bar{\varphi}$ and $z^*z = -r^2k^2$. Thus from (2.1) we have

$$\frac{1}{2\pi}\int_{S^1} e^{ikr\vartheta\cdot\varphi}(\vartheta_1+i\vartheta_2)^m d\sigma(\vartheta) = \begin{cases}(i\varphi)^m J_m(kr), & \text{if } m\ge 0,\\ (i\bar{\varphi})^{-m}J_{-m}(kr), & \text{if } m<0.\end{cases}$$

This gives the Jacobi-Anger expansion

$$e^{iky\cdot\varphi} = \sum_{m=0}^{\infty}(i\bar{\varphi})^m J_m(kr)e^{im\theta} + \sum_{m=0}^{\infty}(i\varphi)^m J_m(kr)e^{-im\theta} - J_0(kr) \quad (2.6)$$

where $y = (r\cos\theta, r\sin\theta)$. Then the following statement becomes trivial.

Proposition 2.3. *The Vekua transform of the harmonic function*

$$e^{ik\bar{\varphi}(y_1+iy_2)/2} + e^{ik\varphi(y_1-iy_2)/2} - 1$$

coincides with $e^{iky\cdot\varphi}$.

Let Γ be a non empty open subset of S^1. Given $g\in L^2(S^1)$ the function

$$\int_{\Gamma}\{e^{ik\bar{\varphi}(y_1+iy_2)/2}+e^{ik\varphi(y_1-iy_2)/2}-1\}g(\varphi)d\sigma(\varphi)$$

is harmonic in the whole space. As a corollary of Proposition 2.3 one knows that the Vekua transform of this harmonic function coincides with the Herglotz wave function with density g

$$\int_{\Gamma} e^{iky\cdot\varphi}g(\varphi)d\sigma(\varphi).$$

Taking account of Proposition 2.2 and the fact mentioned above, it suffices to construct g in such a way that

$$\int_{\Gamma}\{e^{ik\bar{\varphi}(y_1+iy_2)/2}+e^{ik\varphi(y_1-iy_2)/2}-1\}g(\varphi)d\sigma(\varphi)\approx e_{\omega}(y;\tau,k). \tag{2.7}$$

Using the power series expansion of the exponential function, one knows that if g satisfies the system of equations

$$\left(\frac{ik}{2}\right)^m\int_{\Gamma}(\bar{\varphi})^m g(\varphi)d\sigma(\varphi)=\left\{\frac{(\tau-\sqrt{\tau^2+k^2})\bar{\omega}}{2}\right\}^m, \qquad m=0,1,\dots \tag{2.8}$$

and

$$\left(\frac{ik}{2}\right)^m\int_{\Gamma}\varphi^m g(\varphi)d\sigma(\varphi)=\left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{2}\right\}^m, \qquad m=1,\dots, \tag{2.9}$$

then g satisfies (2.7) exactly. We construct g in the form

$$g(\varphi)=\sum_{m=0}^{\infty}\beta_m\varphi^m+\sum_{m=1}^{\infty}\beta_{-m}\bar{\varphi}^m.$$

Now consider the case when $\Gamma=S^1$. Since

$$\frac{1}{2\pi}\int_{S^1}\bar{\varphi}^m g(\varphi)d\sigma(\varphi)=\beta_m$$

and

$$\frac{1}{2\pi}\int_{S^1}\varphi^m g(\varphi)d\sigma(\varphi)=\beta_{-m},$$

from (2.8) and (2.9) we get

$$\beta_m=\frac{1}{2\pi}\left\{\frac{ik}{(\tau+\sqrt{\tau^2+k^2})\omega}\right\}^m, \qquad m=0,\dots$$

and

$$\beta_{-m} = \frac{1}{2\pi}\left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{ik}\right\}^m, \qquad m = 1, \ldots.$$

Then g becomes

$$g(\varphi) = \sum_{m=0}^{\infty}\frac{1}{2\pi}\left\{\frac{ik\varphi}{(\tau+\sqrt{\tau^2+k^2})\omega}\right\}^m + \sum_{m=1}^{\infty}\frac{1}{2\pi}\left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{ik\varphi}\right\}^m \tag{2.10}$$

The first term is convergent and has the form

$$\frac{1}{2\pi}\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{(\tau+\sqrt{\tau^2+k^2})\omega - ik\varphi}.$$

However the second term is always divergent since $\tau+\sqrt{\tau^2+k^2} > k$. So we consider a truncation of (2.10):

$$g_N(\varphi;\tau,k,\omega) = \frac{1}{2\pi}\sum_{m=0}^{N}\left\{\frac{ik\varphi}{(\tau+\sqrt{\tau^2+k^2})\omega}\right\}^m + \frac{1}{2\pi}\sum_{m=1}^{N}\left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{ik\varphi}\right\}^m$$

where $N = 1, \ldots.$ This coincides with the expression given by (1.3). Then one obtains

$$\begin{aligned}
&\int_{S^1}\{e^{ik\bar{\varphi}(y_1+iy_2)/2} + e^{ik\varphi(y_1-iy_2)/2} - 1\}g_N(\varphi;\tau,k,\omega)d\sigma(\varphi) - e_\omega(y;\tau,k)\\
&\quad = -\sum_{m>N}\frac{1}{m!}\left\{\frac{(\tau-\sqrt{\tau^2+k^2})\bar{\omega}}{2}\right\}^m (y_1+iy_2)^m\\
&\qquad - \sum_{m>N}\frac{1}{m!}\left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{2}\right\}^m (y_1-iy_2)^m.
\end{aligned} \tag{2.11}$$

This shows $g_N(\,\cdot\,;\tau,k,\omega)$ satisfies (2.7) in this sense. Taking the Vekua transform of the both sides of (2.11) we obtain the equation

$$\begin{aligned}
&\int_{S^1} e^{iky\cdot\varphi}g_N(\varphi;\tau,k,\omega)d\sigma(\varphi) - e^{y\cdot(\tau\omega+i\sqrt{\tau^2+k^2}\omega^\perp)}\\
&\quad = -\sum_{m>N}\left\{\frac{(\tau-\sqrt{\tau^2+k^2})\bar{\omega}}{k}\right\}^m J_m(kr)e^{im\theta}\\
&\qquad - \sum_{m>N}\left\{\frac{(\tau+\sqrt{\tau^2+k^2})\omega}{k}\right\}^m J_m(kr)e^{-im\theta}
\end{aligned} \tag{2.12}$$

where $y = (r\cos\theta, r\sin\theta)$. Note that this can be checked also directly. For our purpose we have to consider how to choose τ depending on N. One answer to this question is the following and it is the main result of this section.

Theorem 2.1. *Let β_0 be the unique positive solution of the equation*

$$\frac{2}{e}s + \log s = 0.$$

Let β satisfy $0 < \beta < \beta_0$. Let $\{\tau(N)\}_{N=1,\dots}$ be an arbitrary sequence of positive numbers satisfying, as $N \to \infty$

$$\tau(N) = \frac{\beta N}{eR} + O(1).$$

Then we have, as $N \to \infty$

$$\begin{aligned} &e^{R\tau(N)} \sup_{|y|\le R} \left| \int_{S^1} e^{iky\cdot\varphi} g_N(\varphi; \tau(N), k, \omega)\, d\sigma(\varphi) - e^{y\cdot(\tau(N)\omega + i\sqrt{\tau(N)^2+k^2}\,\omega^{\perp})} \right| \\ &\quad + e^{R\tau(N)} \sup_{|y|\le R} \left| \nabla \left\{ \int_{S^1} e^{iky\cdot\varphi} g_N(\varphi; \tau(N), k, \omega)\, d\sigma(\varphi) - e^{y\cdot(\tau(N)\omega + i\sqrt{\tau(N)^2+k^2}\,\omega^{\perp})} \right\} \right| \\ &\quad = O(N^{-\infty}). \end{aligned} \tag{2.13}$$

Proof. We give a direct proof without referring the property of the Vekua transform. Set

$$R_N(y;\tau) = \sum_{m>N} \left\{ \frac{(\tau - \sqrt{\tau^2+k^2})\bar{\omega}}{k} \right\}^m J_m(kr) e^{im\theta}$$

$$S_N(y;\tau) = \sum_{m>N} \left\{ \frac{(\tau + \sqrt{\tau^2+k^2})\omega}{k} \right\}^m J_m(kr) e^{-im\theta}$$

$$E(\tau;N) = \frac{1}{N!} \left\{ \frac{R(\tau + \sqrt{\tau^2+k^2})}{2} \right\}^N e^{R(\tau+\sqrt{\tau^2+k^2})/2}.$$

The estimate

$$|J_m(kr)| \le \left(\frac{kr}{2}\right)^m \frac{1}{m!},$$

is well known. Then we have, for all y with $|y| \le R$

$$|S_N(y;\tau)| \le \frac{1}{(N+1)!}\left\{\frac{R(\tau+\sqrt{\tau^2+k^2})}{2}\right\}^{N+1} e^{R(\tau+\sqrt{\tau^2+k^2})/2} = E(\tau;N+1). \tag{2.14}$$

Now let $\tau = \tau(N)$. Since

$$E(\tau;N) = \frac{1}{N}\frac{R(\tau+\sqrt{\tau^2+k^2})}{2}E(\tau;N-1),$$

from (2.14) we have, as $N \to \infty$

$$|S_{N-1}(y;\tau(N))| + |S_N(y;\tau(N))| + |S_{N+1}(y;\tau(N))| = O(E(\tau(N);N-1)). \tag{2.15}$$

On the other hand, we have

$$\begin{aligned}
&|R_N(y;\tau)| \\
&\le \frac{1}{(N+1)!}\left\{\frac{Rk^2}{2(\tau+\sqrt{\tau^2+k^2})}\right\}^{N+1} \exp\frac{k^2R}{2(\tau+\sqrt{\tau^2+k^2})} \\
&= \frac{1}{(N+1)!}\left\{\frac{R(\tau+\sqrt{\tau^2+k^2})}{2}\right\}^{N+1} \exp\frac{R(\tau+\sqrt{\tau^2+k^2})}{2}\left\{\frac{2}{R(\tau+\sqrt{\tau^2+k^2})}\right\}^{N+1} \\
&\quad \times \left\{\frac{Rk^2}{2(\tau+\sqrt{\tau^2+k^2})}\right\}^{N+1} \exp\left\{\frac{k^2R}{2(\tau+\sqrt{\tau^2+k^2})} - \frac{R(\tau+\sqrt{\tau^2+k^2})}{2}\right\} \\
&= O(E(\tau;N+1))
\end{aligned}$$

and thus this yields

$$|R_{N-1}(y;\tau(N))| + |R_N(y;\tau(N))| + |R_{N+1}(y;\tau(N))| = O(E(\tau(N);N-1)). \tag{2.16}$$

Here we claim

$$e^{R\tau(N)}E(\tau(N);N-1) = O(N^{-\infty}). \tag{2.17}$$

This is proved as follows. Using the Stirling formula

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\left\{1+O\left(\frac{1}{x}\right)\right\}$$

as $x \to \infty$ and

$$\tau(N)R + \frac{R(\tau(N)+\sqrt{\tau(N)^2+k^2})}{2} = 2\beta N/e + O(1),$$

one gets

$$
\begin{aligned}
&e^{R\tau(N)}E(\tau(N); N-1)\\
&\quad = e^{R\tau(N)}\sqrt{\frac{N}{2\pi}}\left(\frac{e}{N}\right)^N\left\{\frac{R(\tau(N)+\sqrt{\tau(N)^2+k^2})}{2}\right\}^{N-1}\\
&\qquad\times e^{R(\tau(N)+\sqrt{\tau(N)^2+k^2})/2}\left\{1+O\left(\frac{1}{N}\right)\right\}\\
&\quad = O\left(e^{2\beta N/e}e^{N\log(e/N)}\sqrt{\frac{N}{2\pi}}\{\beta N/e+O(1)\}^{N-1}\left\{1+O\left(\frac{1}{N}\right)\right\}\right)\\
&\quad = O\left(\exp\left(\frac{2\beta N}{e}+N\log\frac{e}{N}+N\log\{\beta N/e+O(1)\}\right)\right)\\
&\quad = O\left(\exp N\left\{\frac{2\beta}{e}+\log\frac{e(\beta N/e+O(1))}{N}\right\}\right) \qquad (2.18)
\end{aligned}
$$

Since

$$\frac{e(\beta N/e+O(1))}{N} = \beta + O\left(\frac{1}{N}\right),$$

we get

$$\log\frac{\{e(\beta N/e+O(1))\}}{N} = \log\beta + O\left(\frac{1}{N}\right).$$

Then from (2.18) one obtains

$$e^{R\tau(N)}E(\tau(N); N-1) = O\left(\exp N\left(\frac{2\beta}{e}+\log\beta\right)\right).$$

Since

$$\frac{2\beta}{e}+\log\beta < 0,$$

we obtain (2.17).

Using the recurrence relation

$$J_{m+1}(kr) = \frac{m}{kr}J_m(kr) - J_m'(kr)$$

$$J_{m-1}(kr) = \frac{m}{kr}J_m(kr) + J_m'(kr)$$

and the formulae

$$\frac{\partial}{\partial y_1} = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r} + i\frac{1}{r}\frac{\partial}{\partial \theta}\right) + \frac{e^{-i\theta}}{2}\left(\frac{\partial}{\partial r} - i\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial y_2} = \frac{-ie^{i\theta}}{2}\left(\frac{\partial}{\partial r} + i\frac{1}{r}\frac{\partial}{\partial \theta}\right) + \frac{ie^{-i\theta}}{2}\left(\frac{\partial}{\partial r} - i\frac{1}{r}\frac{\partial}{\partial \theta}\right),$$

one knows that: $\partial R_N(y;\tau)/\partial y_j$ can be written as a linear combination of $R_{N+1}(y;\tau)$ and $R_{N-1}(y;\tau)$ whose coefficients are at most algebraic growing as $\tau \to \infty$; $\partial S_N(y;\tau)/\partial y_j$ can be written as a linear combination of $S_{N+1}(y;\tau)$ and $S_{N-1}(y;\tau)$ whose coefficients are at most algebraic growing as $\tau \to \infty$. Using those facts, and (2.12), (2.15), (2.16) and (2.17), one obtains the desired conclusion. □

3. Proof of Theorems 1.2, 1.4 and comment on the proof of Theorem 1.5

The starting point is the representation formula given below. The proof is taken from that of (2.9) in [3]. Therein they made use of the formula to establish an interesting equation that connects an eigenvalue of the operator with the integral kernel $K(\varphi, d) = F(\varphi; d, k)$ acting on the functions on S^1 with an absorbing medium. Here for reader's convenience we give a brief description of the proof.

Lemma 3.1. *Let $\Gamma \subset S^1$ be measurable with respect to the standard measure on S^1. Let $u \in C^\infty(\mathbf{R}^2\backslash B_R)$ satisfy $\triangle u + k^2 u = 0$ in $\mathbf{R}^2\backslash \overline{B}_R$; the outgoing Sommerfeld radiation condition $\lim_{r\to\infty}\sqrt{r}(\partial w/\partial r - ikw) = 0$ where $r = |x|$ and $w = u - e^{ikx\cdot d}$. Then the formula*

$$\int_\Gamma F(-\varphi; d, k) g(\varphi) d\sigma(\varphi) = -\frac{e^{i\pi/4}}{\sqrt{8\pi k}}\int_{\partial B_R}\left(\frac{\partial u}{\partial \nu} v_g - \frac{\partial v_g}{\partial \nu} u\right) d\sigma,$$

is valid where v_g is the Herglotz wave function with density $g \in L^2(\Gamma)$

$$v_g(y) = \int_\Gamma e^{iky\cdot\varphi} g(\varphi) d\sigma(\varphi)$$

and ν is the unit outward normal relative to B_R.

Proof. From the representation formula of w outside B_R ([4]) one obtains the formula

$$F(\varphi; d, k) = -\frac{e^{i\pi/4}}{\sqrt{8\pi k}}\int_{\partial B_R}\left(\frac{\partial u}{\partial \nu} e^{-ik\varphi\cdot y} - \frac{\partial e^{-ik\varphi\cdot y}}{\partial \nu} u\right) d\sigma(y).$$

Thus replacing φ with $-\varphi$, we have

$$F(-\varphi; d, k) = -\frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial B_R} \left(\frac{\partial u}{\partial \nu} e^{ik\varphi\cdot y} - \frac{\partial e^{ik\varphi\cdot y}}{\partial \nu} u \right) d\sigma(y). \tag{3.1}$$

Multiplying the both sides by $g(\varphi)$ and integrating the resultant on Γ, we obtain the desired formula. $\square$

Now we give a proof of Theorem 1.2. Using Lemma 3.1 for $\Gamma = S^1$, we write

$$\begin{aligned}
&-e^{-\tau(N)h_D(\omega)} \frac{\sqrt{8\pi k}}{e^{i\pi/4}} \int_{S^1} F(-\varphi; d, k) g_N(\varphi; \tau(N), k, \omega) d\varphi \\
&\quad = e^{-\tau(N)h_D(\omega)} \int_{\partial B_R} \left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) d\sigma \\
&\qquad + e^{-\tau(N)h_D(\omega)} \int_{\partial B_R} \left\{ \frac{\partial u}{\partial \nu} (v_{g_N} - v) - \frac{\partial}{\partial \nu} (v_{g_N} - v) u \right\} d\sigma \\
&\quad \equiv I_1 + I_2
\end{aligned} \tag{3.2}$$

where $v = e^{y\cdot(\tau\omega + i\sqrt{\tau^2+k^2}\omega^\perp)}$ and v_{g_N} is the Herglotz wave function with density $g_N(\cdot\,; \tau(N), k, \omega)$. In [11] we have already proven that there exist $\mu > 0$ and $A > 0$ such that

$$\lim_{N\to\infty} \tau(N)^\mu |I_1| = A. \tag{3.3}$$

Theorem 2.1 gives the estimate

$$\begin{aligned}
\tau(N)^\mu |I_2| &\le \tau(N)^\mu e^{R\tau(N)} C_R(u) \left\{ \sup_{|y|\le R} |v_{g_N}(y) - v(y)| + \sup_{|y|\le R} |\nabla\{v_{g_N}(y) - v(y)\}| \right\} \\
&= O(N^{-\infty})
\end{aligned} \tag{3.4}$$

where $C_R(u)$ is a positive constant depending on u, R and independent of N.

From (3.2), (3.3) and (3.4) one gets

$$\tau(N)^\mu e^{-\tau(N)h_D(\omega)} \left| \int_{S^1} F(-\varphi; d, k) g_N(\varphi; \tau(N), k, \omega) d\sigma(\varphi) \right| \to \frac{A}{\sqrt{8\pi k}} \tag{3.5}$$

as $N \to \infty$. Then (1.4) and other all conclusions come from (3.5). $\square$

In the following we say that: a function $f(\tau)$ decays algebraically as $\tau \to \infty$ in strict sense if $\tau^\lambda |f(\tau)|$ converges to a positive number as $\tau \to \infty$ for a positive constant λ; $f(\tau)$ is decaying at most algebraic if $f(\tau) = O(\tau^{-\mu})$ as $\tau \to \infty$ for a suitable positive constant μ.

Now we give a proof of Theorem 1.4. Let y denote the only one point of the set $\{x \mid x \cdot \omega = h_{\Sigma}(\omega)\} \cap \Sigma$. First consider the case when every end points of $\Sigma_1, \Sigma_2, \dots, \Sigma_m$ satisfies $x \cdot \omega < h_{\Sigma}(\omega)$. Then y should be a point where two segments in some Σ_j meet. Then from Theorem 4.2 in [11] and a fact similar to Lemma 5.1 in [11] one obtains that both $I_{\omega}^{d_1}(\tau, h_{\Sigma}(\omega))$ and $I_{\omega}^{d_2}(\tau, h_{\Sigma}(\omega))$ decays algebraically as $\tau \to \infty$ in strict sense. This yields the algebraic decaying of $I_{\omega}(\tau, h_{\Sigma}(\omega))$ as $\tau \to \infty$ in strict sense. Then we automatically obtain the desired results.

The problem is the case when y is an end point of some Σ_j. Since ν on Σ_j becomes a constant vector in a neighbourhood of y, we denote the constant vector by ν_j. Using Theorem 4.3 in [11] for u with $d = d_1, d_2$ one concludes that $I_{\omega}(\tau, h_{\Sigma}(\omega))$ decays at most algebraically as $\tau \to \infty$. Moreover, an argument similar to that of the proof of Lemma 5.1 in [11] yields that if both of $I_{\omega}^{d_1}(\tau, h_{\Sigma}(\omega))$ and $I_{\omega}^{d_2}(\tau, h_{\Sigma}(\omega))$ decay rapidly then $d_1 \cdot \nu_j = d_2 \cdot \nu_j = 0$. However, this is impossible. Thus one concludes that one of them has to be algebraically decaying as $\tau \to \infty$ in strict sense. Then we obtain the algebraic decaying of $I_{\omega}(\tau, h_{\Sigma}(\omega))$ as $\tau \to \infty$ in strict sense. This completes the proof. $\square$

It is easy to see that completely the same formula for $d = d_1, d_2$ as Lemma 3.1 is valid. Then the proof of Theorem 1.5 can be done along the same line as that of Theorem 1.2.

4. Limited aperture

In this section we consider the case when the far field pattern with limited aperture is given. Here we point out an effect of a priori information on the formulae in Theorem 1.2. Let Γ be a non empty open subset of S^1.

We assume that:

(1) the far field pattern on Γ is known for fixed d and k;

(2) $0 \in D$.

(1) means that Γ is an aperture. (2) means that the center of the coordinates is inside D and we know it in advance.

The starting point is a theorem established in [5]. Define

$$W(B_R) = \{v \in C^2(B_R) \cap C^1(\overline{B}_R) \mid \triangle v + k^2 v = 0 \text{ in } B_R\}.$$

We denote by $\overline{W(B_R)}$ the $H^1(B_R)$ closure of $W(B_R)$. Given $g \in L^2(-\Gamma)$ define

$$Hg(y) = \int_{-\Gamma} e^{iky\cdot\varphi} g(\varphi) d\sigma(\varphi) \qquad y \in B_R.$$

Then $Hg \in \overline{W(B_R)}$ and the map $H : L^2(-\Gamma) \to \overline{W(B_R)}$ is bounded linear. They proved the following.

Theorem 4.1 (Theorem 2.6 of [5]). *The range of H is dense in $\overline{W(B_R)}$.*

Note that: therein they considered only the case when $-\Gamma = S^1$, however, by using the real analyticity of the far field pattern, one knows that the proof is still valid.

Given $v \in \overline{W(B_R)}$ and $\delta > 0$ an element $g_0 \in L^2(-\Gamma)$ is called minimum norm solution of $Hg = v$ with discrepancy δ, if g_0 satisfies $\|Hg - v\|_{H^1(B_R)} \le \delta$ and

$$\|g_0\|_{L^2(-\Gamma)} = \inf\{\|g\|_{L^2(-\Gamma)} : \|Hg - v\|_{H^1(B_R)} \le \delta\}.$$

Now given $\tau > 0$ and $\omega \in S^1$ set $v = e^{x\cdot(\tau\omega + i\sqrt{\tau^2+k^2}\omega^{\perp})} (\in \overline{W(B_R)})$. Theorem 4.1 ensures the existence of the minimum norm solution of $Hg = v$ with discrepancy δ (see Theorem 16.11 in [14]). It is given by the formula

$$g = (\alpha I + H^*H)^{-1} H^* v$$

where $\alpha > 0$ is any zero of the function

$$\|H(\alpha I + H^*H)^{-1} H^* v - v\|_{H^1(B_R)}^2 - \delta^2.$$

We denote the minimum norm solution by $g = g_{\tau,\delta}(\cdot\,; k, \omega)$. This satisfies

$$\|Hg_{\tau,\delta}(\cdot\,; k, \omega) - v\|_{H^1(B_R)} \le \delta. \tag{4.1}$$

Then we obtain the following theorem.

Theorem 4.2. *Assume that $0 \in D$. Let ω be regular with respect to D. Then the formula*

$$\lim_{\tau\to\infty} \frac{\log\left|\int_{\Gamma} F(\varphi; d, k) g_{\tau,\delta}(-\varphi; k, \omega) d\sigma(\varphi)\right|}{\tau} = h_D(\omega),$$

is valid. Moreover we have:

if $t \ge h_D(\omega)$, then

$$\lim_{\tau\to\infty} e^{-\tau t}\left|\int_{\Gamma} F(\varphi; d, k) g_{\tau,\delta}(-\varphi; k, \omega) d\sigma(\varphi)\right| = 0;$$

if $t < h_D(\omega)$, then

$$\lim_{\tau\to\infty} e^{-\tau t}\left|\int_{\Gamma} F(\varphi; d, k) g_{\tau,\delta}(-\varphi; k, \omega) d\sigma(\varphi)\right| = \infty.$$

PROOF. Using Lemma 3.1, we write

$$
\begin{aligned}
&- e^{-\tau h_D(\omega)} \frac{\sqrt{8\pi k}}{e^{i\pi/4}} \int_\Gamma F(\varphi; d, k) g_{\tau,\delta}(-\varphi; k, \omega) d\varphi \\
&\quad = e^{-\tau h_D(\omega)} \int_{\partial B_R} \left(\frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) d\sigma \\
&\quad\quad + e^{-\tau h_D(\omega)} \int_{\partial B_R} \left\{ \frac{\partial u}{\partial \nu} (Hg_{\tau,\delta} - v) - \frac{\partial}{\partial \nu} (Hg_{\tau,\delta} - v) u \right\} d\sigma \qquad (4.2)
\end{aligned}
$$

where $v = e^{y\cdot(\tau\omega + i\sqrt{\tau^2+k^2}\omega^\perp)}$ and $Hg_{\tau,\delta}$ is the Herglotz wave function with density $g_{\tau,\delta}(\cdot\,; k, \omega)$. Using the trace theorem, from (4.1) we know that

$$
\|Hg_{\tau,\delta} - v\|_{H^{1/2}(\partial B_R)} + \left\| \frac{\partial}{\partial \nu} \{Hg_{\tau,\delta} - v\} \right\|_{H^{-1/2}(\partial B_R)} \le C\delta \qquad (4.3)
$$

where $C > 0$ is independent of τ. The assumption gives $h_D(\omega) > 0$. Thus from (4.3) one knows that the second term of the right hand side of (4.2) is exponentially decaying as $\tau \to \infty$ and of course, the first term is decaying algebraically as $\tau \to \infty$ in strict sense by the same reason as described in the proof of Theorem 1.2. $\square$

In a forthcoming paper we consider algorithms that are based on Theorems 1.2, 1.5 and 4.2.

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Masaru Ikehata
Department of Mathematics
Faculty of Engineering
Gunma University
Kiryu 376-8515, *Japan*
E-mail: ikehata@math.sci.gunma-u.ac.jp