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Maximal functions for Lebesgue spaces with variable exponent approaching 1

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his seventieth birthday

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Abstract. Our aim in this paper is to deal with maximal functions for Lebesgue spaces with variable exponent approaching 1.

1. Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r. For a locally integrable function f on $\mathbf{R}^n$, we consider the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Following Orlicz [4] and Kováčik and Rákosník [2], we consider a positive continuous function $p(\cdot) : \mathbf{R}^n \to [1,\infty)$ and a measurable function f satisfying

$$\int |f(y)|^{p(y)} dy < \infty.$$

In this paper we are concerned with $p(\cdot)$ satisfying the following log-Hölder condition

$$p(r) = 1 + \frac{a \log(\log(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)}$$

for $0 < r \le r_0 < 1/e$, where $a > 0$ and b is a real number; set $p(0) = 1$ and $p(r) = p(r_0)$ for $r > r_0$. For a bounded open set G in $\mathbf{R}^n$, consider

$$p(x) = p(\delta(x)),$$

where $\delta(x)$ denotes the distance of x from the boundary ∂G of G.

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Cruz-Uribe, Fiorenza and Neugebauer [1] proved the maximal operator M is not bounded on $L^{p(\cdot)}(G)$ if $\inf_G p(x) = 1$. Recently, Hästö [3] has proved that the maximal operator M is bounded from $L^{p(\cdot)}(G)$ to $L^1(G)$ when $a > 1$ and G satisfies a certain regular condition. Our aim in this note is to show that the same conclusion is still valid for $a = 1$.

2. Maximal functions

Throughout this paper, let C denote various constants independent of the variables in question.

Let G be a bounded open set in $\mathbf{R}^n$, and consider a positive continuous function $p(\cdot)$ on G such that

$$p(x) = 1 + \frac{\log(\log(1/\delta(x)))}{\log(1/\delta(x))} + \frac{b}{\log(1/\delta(x))} \tag{1}$$

when $0 < \delta(x) \le r_0 < 1/e$, where b is a real number; assume always that $p(x) > 1$ when $\delta(x) > 0$.

For simplicity, we denote the Lebesgue measure of E by $|E|$.

Let us begin with the following elementary lemmas.

LEMMA 2.1. *Let G be a bounded open set in $\mathbf{R}^n$. For $0 < k \le n$ and $r > 0$, set $G_r = \{x \in G : \delta(x) < r\}$ and assume that*

$$|G_r| \le Cr^k, \tag{2}$$

or the Minkowski $(n-k)$-content of ∂G is finite. Then

$$\int_G \delta(x)^{-k} (\log(1 + \delta(x)^{-1}))^{-\alpha} dx < \infty$$

for every $\alpha > 1$.

LEMMA 2.2. *Set*

$$\varphi(r) = \frac{\log(\log(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)}$$

for a real number b. Then there exists $r_0 > 0$ such that

(i) $\varphi'(r) > 0$ *when* $0 < r < 2r_0$;

(ii) $\varphi''(r) < 0$ *when* $0 < r < 2r_0$;

(iii) $\varphi(s+t) \le \varphi(s) + \varphi(t)$ *when* $0 < s, t < r_0$.

For a locally integrable function f on G, we consider the maximal function Mf defined by

$$Mf(x) = \sup_B \frac{1}{|B|} \int_{G \cap B} |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x, r)$. Define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot), G} = \inf\left\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot)} < \infty$.

Lemma 2.3. *Suppose the Minkowski $(n-1)$-content of ∂G is finite. If f is a measurable function on G with $\|f\|_{p(\cdot)} \le 1$, then*

$$\int_G |f(x)| \log(1 + |f(x)|)dx \le C.$$

Proof. Consider the set

$$G' = \{x \in G : |f(x)| < \delta(x)^{-1}(\log(1/\delta(x)))^{-\alpha}\}$$

for $\alpha > 2$. If $x \in G'$ and $\delta(x) < r_0$ $(< 1/e)$, then

$$|f(x)| \log(1 + |f(x)|) \le C\delta(x)^{-1}(\log(1/\delta(x)))^{-\alpha+1}.$$

Hence we have by Lemma 2.1

$$\int_{G_{r_0} \cap G'} |f(x)| \log(1 + |f(x)|)dx \le C.$$

If $x \notin G'$ and $\delta(x) < r_0$ $(< 1/e)$, then

$$\delta(x) \ge (C|f(x)|(\log|f(x)|)^{\alpha})^{-1},$$

so that Lemma 2.2 yields

$$\begin{aligned}
|f(x)|^{p(x)} &\ge |f(x)| \exp\left(\left(\frac{\log\log(C|f(x)|(\log|f(x)|)^{\alpha})}{\log(C|f(x)|(\log|f(x)|)^{\alpha})}\right.\right. \\
&\qquad \left.\left. + \frac{b}{\log(C|f(x)|(\log|f(x)|)^{\alpha})}\right) \log|f(x)|\right) \\
&\ge C|f(x)| \exp\left(\frac{\log(\log|f(x)| + \log(C(\log|f(x)|)^{\alpha}))}{\log|f(x)| + \log(C(\log|f(x)|)^{\alpha})} \log|f(x)|\right) \\
&\ge C|f(x)| \exp\left(\frac{\log(\log|f(x)|)}{\log|f(x)|} \log|f(x)|\right) \\
&= C|f(x)| \log|f(x)|.
\end{aligned}$$

Here note that

$$\left|\frac{\log(t+s)}{t+s} - \frac{\log t}{t}\right| \le C\left(\frac{\log t}{t}\right)^2 \le \frac{C}{t} \qquad \text{when } 1 < s \le C \log t.$$

Hence it follows that

$$\int_{G_{r_0}\setminus G'} |f(x)| \log(1+|f(x)|)dx \le C\int_G |f(x)|^{p(x)}dx \le C.$$

Finally, since $p(x) \ge p_0 > 1$ when $\delta(x) \ge r_0$, we find

$$\int_{G\setminus G_{r_0}} |f(x)| \log(1+|f(x)|)dx \le C\int_G |f(x)|^{p(x)}dx + C \le C.$$

Consequently, the required assertion is proved. □

Now we are ready to show our main result, which gives an improvement of Hästö [3].

Theorem 2.4. *Suppose the Minkowski* $(n-1)$*-content of* ∂G *is finite. Then*

$$\|Mf\|_1 \le C\|f\|_{p(\cdot)} \qquad \text{for all } f \in L^{p(\cdot)}(G).$$

This is a consequence of Lemma 2.3 and the well-known fact of maximal functions (see Stein [5]).

Remark 2.5. Theorem 2.4 is sharp in the following sense: for instance, if

$$p(x) = 1 + \frac{\log(\log(1/\delta(x)))}{\log(1/\delta(x))} - \frac{\log(\log(\log(1/\delta(x))))}{\log(1/\delta(x))}$$

when $0 < \delta(x) \le r_0 < 1/e$ and $\inf_{\{x:\delta(x)>r_0\}} p(x) > 1$, then we can find $f \in L^{p(\cdot)}(\mathbf{B})$ for which

$$\int_{\mathbf{B}} |f(x)| \log(1+|f(x)|)dx = \infty,$$

where $\mathbf{B} = B(0,1)$ and $\delta(x) = 1-|x|$ for $x \in \mathbf{B}$.

For this purpose, letting $\log_{(1)} t = \log t$ and $\log_{(m+1)} t = \log(\log_{(m)} t)$ for $m = 1, 2, \ldots$, we consider the function

$$f(x) = \delta(x)^{-1}(\log(1/\delta(x)))^{-2}(\log_{(2)}(1/\delta(x)))^{-1}$$

for $x \in \mathbf{B}$ with $\delta(x) < r_0$; set $f(x) = 0$ when $\delta(x) \ge r_0$. Then

$$\int_{\mathbf{B}} f(x) \log(1+f(x))dx \ge C\int_0^{r_0} t^{-1}(\log_{(1)}(1/t))^{-1}(\log_{(2)}(1/t))^{-1}dt = \infty.$$

Further, we have for $t = \delta(x) < r_0$

$$\begin{aligned} f(x)^{p(x)-1} &\le \exp((\log(1/t))((\log_{(2)}(1/t))/\log(1/t) - (\log_{(3)}(1/t))/\log(1/t))) \\ &= (\log_{(1)}(1/t))(\log_{(2)}(1/t))^{-1}, \end{aligned}$$

so that

$$\int_{\mathbf{B}} f(x)^{p(x)} dx \le C \int_0^{r_0} t^{-1} (\log_{(1)}(1/t))^{-1} (\log_{(2)}(1/t))^{-2} dt < \infty.$$

3. Variable exponent approaching 1 at a point

Suppose $p(\cdot)$ satisfies $\inf_{\{x:|x|>r_0\}} p(x) > 1$ and

$$p(x) = 1 + \frac{1}{n} \frac{\log(\log(1/|x|))}{\log(1/|x|)} + \frac{b}{\log(1/|x|)}$$

for $0 < |x| \le r_0 < 1/e$, where b is a real number. Of course, $p(0) = 1$ as before.

Theorem 3.1. *If* $\|f\|_{p(\cdot)} \le 1$, *then*

$$\int_{\mathbf{B}} |f(x)| \log(1 + |f(x)|) dx \le C;$$

and hence

$$\|Mf\|_1 \le C\|f\|_{p(\cdot)} \qquad \text{for all } f \in L^{p(\cdot)}(\mathbf{B}).$$

Proof. As in the proof of Lemma 2.3, we consider the set

$$B' = \{x \in \mathbf{B} : |f(x)| < |x|^{-n} (\log(1 + |x|^{-1}))^{-\alpha}\}$$

for $\alpha > 2$. Then we have

$$\begin{aligned}\int_{B'} |f(x)| \log(1 + |f(x)|) dx &\le C \int_{\mathbf{B}} |x|^{-n} (\log(1 + |x|^{-1}))^{-\alpha+1} dx \\ &\le C \int_0^1 t^{-1} (\log(1 + t^{-1}))^{-\alpha+1} dt < \infty\end{aligned}$$

with the aid of Lemma 2.1. If $x \in \mathbf{B} \backslash B'$, then we see that

$$|x| \ge (C|f(x)|^{1/n} (\log|f(x)|)^{\alpha/n})^{-1},$$

which yields

$$|f(x)|^{p(x)} \ge C|f(x)| \log|f(x)|.$$

Hence we obtain

$$\int_{\mathbf{B} \backslash B'} |f(x)| \log(1 + |f(x)|) dx \le C \int_{\mathbf{B} \backslash B'} |f(x)|^{p(x)} dx \le C,$$

as required. □

Remark 3.2. As stated in Hästö [3] we have a general result:
Consider a compact subset F of a bounded open set G, and denote by

$\delta(x) = \mathrm{dist}(x, F)$ the distance of x from F. For $0 \le m < n$ and $0 < r < r_0$ with r_0 small enough let $G_r = \{x \in G : \delta(x) < r\}$ and assume

$$|G_r| \le Cr^{n-m}. \tag{3}$$

Further $p(\cdot)$ is a continuous function on G such that

$$p(x) = 1 + \frac{1}{n-m} \frac{\log(\log(1/\delta(x)))}{\log(1/\delta(x))} + \frac{b}{\log(1/\delta(x))} \tag{4}$$

when $0 < \delta(x) \le r_0 < 1/e$ for some real number b and $\inf_{\{x:\delta(x)>r_0\}} p(x) > 1$. Then we claim that if f is a locally integrable function on G satisfying $\|f\|_{p(\cdot)} \le 1$, then

$$\int_G |f(x)| \log(1 + |f(x)|)dx \le C,$$

so that $M : L^{p(\cdot)}(G) \to L^1(G)$ is bounded.

To prove this, as in proofs of Theorems 2.4 and 3.1, it suffices to consider the set

$$G' = \{x \in G : |f(x)| < \delta(x)^{m-n}(\log(1/\delta(x)))^{-\alpha}\}$$

for $\alpha > 2$.

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