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The Teichmüller space of the ideal boundary

Dedicated to Professor Masakazu Shiba for his 60th birthday

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ABSTRACT. In this paper, we consider an analytic kind of structure on the ideal boundary of a Riemann surface, which is finer than the topological one, and show that the set of the natural equivalence classes of mutually quasiconformally related such structures admits a complex Banach manifold structure.

1. The ideal boundary

For an open Riemann surface R, we can consider various kinds of compactifications of R. In this note we consider the Royden's one (cf. [1] and [10]).

To define the Royden compactification, first we take the set $\mathbf{R}(R)$ of bounded continuous (complex) functions f on R which are differentiable in distribution sense and whose Dirichlet integrals

$$D(f) = \int_R df \wedge *\overline{df}$$

are finite. Then

$$\|f\| = \sup_R |f| + \sqrt{D(f)}$$

is a norm on $\mathbf{R}(R)$, and $\mathbf{R}(R)$ is a Banach algebra with respect to this norm. We call this algebra the *Royden algebra* associated with R.

Now there is a compact Hausdorff space R^*, containing R as an open and dense subset, such that every element in $\mathbf{R}(R)$ can be extended to a continuous function on R^* (and hence $\mathbf{R}(R)$ can be considered as a subset of the set $C(R^*)$ of all continuous functions on R^*) and that $\mathbf{R}(R)$ separates points of R^*, i.e. for every pair of points p_1 and p_2 of R^* there is a function f in $\mathbf{R}(R)$ such that $f(p_1) \neq f(p_2)$. Then such an R^* is uniquely determined up to homeomorphisms

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fixing R point-wise, and we call this R^* the Royden compactification of R. Also the compact subset $dR = R^* - R$ is called the *Royden boundary* of R.

Here there are several ways to construct the Royden compactification canonically. One way is to consider the set $X = X(R)$ of all characters on $\mathbf{R}(R)$. Here a multiplicative linear functional χ on $\mathbf{R}(R)$ with $\chi(1) = 1$ is called a *character*. And equipped with the weak* topology, X is a compact Hausdorff space. Moreover, by considering the point evaluations, we can regard R as an open and dense subset of X and X gives a representative of the Royden compactification of R.

In the sequel, *we always consider this compact set $X(R)$ as the Royden compactification R^* of a given R.*

REMARK. $\mathbf{R}(R)$ is dense in $C(R^*)$ with respect to the uniform topology.

Also we recall the following fact.

PROPOSITION 1 ([1], [10]). *Every quasiconformal homeomorphism F of a Riemann surface R_1 onto another R_2 can be extended to a homeomorphism $\tilde{F}$ of R_1^* onto R_2^*.*

Now, we can define another smaller compactification by using, instead of $\mathbf{R}(R)$, the set $\mathbf{KS}(R)$ of continuous functions f, each of which is a constant on every connected component of the complement of some compact set. The Kerékártó-Stoïlow compactification $\hat{R}$ of R is the compact Hausdorff space uniquely determined (up to homeomorphisms fixing R point-wise) by the conditions that R is open and dense in $\hat{R}$, that every element of $\mathbf{KS}(R)$ can be extended to a continuous function on $\hat{R}$, and that $\mathbf{KS}(R)$ separates points of $\hat{R}$.

Clearly, there is the canonical projection π from R^* onto the Kerékártó-Stoïlow compactification $\hat{R}$ of R such that π is the identical map on R. We call the closed set $dR_p = \pi^{-1}(p)$ a *block of dR over p* for every point $p \in \hat{R} - R$. A block dR_p is also open if p is isolated in $\hat{R} - R$.

DEFINITION. When $p \in \hat{R} - R$ corresponds to a puncture of R, we call p a *non-essential point* of $\hat{R} - R$, and the block dR_p a *non-essential* block. Let N be the subset of $\hat{R} - R$ consisting of all non-essential points, and set

$$dR^o = dR - \bigcup_{p \in N} dR_p.$$

Then dR^o is compact, and is called the *essential part* of dR, or the *essential boundary* of R.

In this paper, we introduce another structure on the Royden boundary, which is finer than the topological one, and define in § 3 the Teichmüller space of such structures on a given ideal boundary.

Definition. We call a pair (Y, ι_R), of a compact topological space Y and a homeomorphism ι_R of Y onto the essential boundary dR^o of a Rimann surface R, a *primitive pair*. We say that primitive pairs (Y_1, ι_{R_1}) and (Y_2, ι_{R_2}) are *conformally equivalent* if there are a homeomorphism F of a neighborhood U of $dR_1^o = \iota_{R_1}(Y_1)$ in R_1^* into R_2^* and a one ι_{Y_1, Y_2} of Y_1 onto Y_2 such that

$$F \circ \iota_{R_1} = \iota_{R_2} \circ \iota_{Y_1, Y_2}$$

on Y_1 and F is conformal on $U \cap R_1$.

We call the conformal equivalence class of a primitive pair (Y, ι_R) an *ideal boundary*, which we denote by $[Y, \iota_R]$, or simply by a representative Y if R is clear or not important. Also we call such a Riemann surface R a *supporting surface* of Y.

We say that an ideal boundary Y is of *topologically (in)finite type* if a supporting surface R of Y is topologically (in)finite, i.e. the fundamental group of R is (in)finitely generated.

Since an ideal boundary $[Y, \iota_R]$ is determined uniquely by the complex structure of R near Y, we may say that an ideal boundary $[Y, \iota_R]$ represents a "complex structure" on Y.

Proposition 2. *Suppose that primitive pairs (Y_1, ι_{R_1}) and (Y_2, ι_{R_2}) are conformally equivalent. Then we can take the same Riemann surface R, as a supporting surfaces for both of Y_j.*

Hence in the sequel, *if primitive pairs (Y_1, ι_{R_1}) and (Y_2, ι_{R_2}) are conformally equivalent, then we always assume that $R_1 = R_2$, $\iota_{R_1} = \iota_{R_2}$, $Y_1 = Y_2$, and ι_{Y_1, Y_2} is the identical map.*

Proof. First, by replacing Y_2 and ι_{R_2} to Y_1 and $\iota_{R_2} \circ \iota_{Y_1, Y_2}$, we can assume that $Y_1 = Y_2$ and that ι_{Y_1, Y_2} is the identical map. Let $F : U \to R_2^*$ be as in the definition of conformal equivalence between (Y_1, ι_{R_1}) and (Y_2, ι_{R_2}). Here, we may assume that the relative boundary ∂U of $U \cap R_1$ in R_1 consists of a finite number of analytic simple closed curves. Then, there is a Riemann surface R such that $R \supset R_1$ and that $R - R_1$ is compact. We can take this R as a supporting surface of Y_1 instead of R_1. Next, by identifying U and $F(U)$, we can also take R as a supporting surface of Y_2 instead of R_2. □

Next we say that a subsurface S of a Riemann surface R is *almost compact bordered* if the closure $\bar{S}$ of S in the subsurface $\bar{R}^p$ of $\hat{R}$, obtained from R by filling all points of $\hat{R}$ corresponding to punctures of R, is compact and the relative boundary ∂S of S in R consists of a finite number of analytic simple closed curves in R. Furthermore, if every component of ∂S divides $\bar{R}^p$ into two connected components each of which either contains S or is non-compact, then we call the open set

$$U = R^* - S \cup \partial S \cup \left(\bigcup_{p \in N \cap \bar{S}} dR_p \right)$$

a *canonical neighborhood* of the ideal boundary $[Y, \imath_R]$.

DEFINITION. We say that a map f of an ideal boundary $[Y_1, \imath_{R_1}]$ to another $[Y_2, \imath_{R_2}]$ is a *boundary map* (considered as a map of Y_1 to Y_2) if there are a canonical neighborhood U of $dR_1^o = \imath_{R_1}(Y_1)$ in R_1^* and a homeomorphism F of U into R_2^* such that

$$F \circ \imath_{R_1} = \imath_{R_2} \circ f$$

on Y_1. Such a map F as above is called a *supporting map* of f.

If a boundary map f of $[Y, \imath_R]$ to itself or to another $[Y', \imath_{R'}]$ is a surjective homeomorphism (as a map of Y to itself or to Y'), then we call such an f a *boundary self-homeomorphism*, or *boundary homeomorphism*, respectively.

Further, we say that $f : Y \to Y'$ is *conformal*, *quasiconformal*, and *asymptotically conformal* if so is a supporting map F of f on $U \cap R$.

Here, recall that f is *asymptotically conformal* if and only if we can find a $(1+\varepsilon)$-quasiconformal supporting map of f for every $\varepsilon > 0$. (For the basic facts about asymptotically conformal maps, see for instance, [5].)

2. Boundary self-homeomorphims

Let $\mathrm{BH}(Y)$ be the group of all boundary self-homeomorphisms of an ideal boundary $[Y, \imath_R]$. First we recall the following fact.

PROPOSITION 3 ([8], also see [9]). *f is an element of* $\mathrm{BH}(Y)$ *if and only if f is a quasiconformal boundary self-homeomorphism.*

PROOF. Since "if"-part is clear, we assume that $f \in \mathrm{BH}(Y)$. Then there are a Riemann surface R supporting Y and a homeomorphism F of a canonical neighborhood U of dR^o into R^* which supports f. Replacing U to a smaller one if necessary, we can find by Corollary in [8] a quasiconformal homeomorphism of $U \cap R$ into R whose extension to U supports f, which implies the assertion. □

Also note that a boundary self-homeomorphism of Y need not necessarily be the boundary map of a quasiconformal self-homeomorphism of R.

THEOREM 4. *There are an ideal boundary Y and an $f \in$* $\mathrm{BH}(Y)$ *such that, for every supporting surface R of Y, every quasiconformal self-homeomorphism of R supports neither f nor f^{-1}.*

PROOF. Set

$$R_0 = \{z \in \mathbf{C} \mid |\mathrm{Im}\, z| < 1\} - \{n \in \mathbf{Z} \mid n \geq 0\},$$

and $Y = dR_0^o$. Let f be the boundary self-homeomorphism of Y supported by the extension $\tilde{F}_0$ to $R_0^* - \{-1\}$ of the conformal map

$$F_0(z) = z + 1 : R_0 - \{-1\} \to R_0.$$

We show that these Y and f are desired ones.

For this purpose, suppose that there were a Riemann surface R_1 supporting Y and a quasiconformal self-homeomorphism F of R_1 whose extension $\tilde{F}$ to R_1^* supports f.

Take U so small that U can be considered as a canonical neighborhood of Y not only in R_0^* but also in R_1^*. Further, take a smaller $V \subset U$ so that $\tilde{F}_0(V)$ and $\tilde{F}(V)$ are contained in U. Next, F_0 and F restricted to $V \cap R_0$ can be extended to quasiconformal self-homeomorphisms of $\{|\mathrm{Im}\, z| < 1\}$, which in turn can be identified with $\{|z| < 1\}$ by a Riemann map. Moreover, these maps can be extended continuously to $\{|z| \leq 1\}$ and their boundary values coincide, for they support the same f. Hence we conclude that $\Phi = F^{-1} \circ F_0$ can be extended to $\{|z| \leq 1\}$ and has the identical boundary values.

Now since Φ belongs to $\mathbf{R}(\{|z| < 1\})$, so is $g(z) = \Phi(z) - z$, which identically vanishes on $\{|z| = 1\}$, and hence Φ gives the identical self-map of Y. Here, suppose that there were a sequence of punctures p_n of $V \cap R_0$ (considered as a subsurface of $\{|z| < 1\}$) such that $|p_n| \to 1$ as $n \to +\infty$, and that $g(p_n) \neq 0$ for every n. Since also $|\Phi(p_n)| \to 1$ as $n \to +\infty$, we may further assume, by taking a subsequence if necessary, that

$$\Phi(p_n) \notin \{p_j\}_{j=1}^{\infty}$$

for every n. But then, we could construct a function $P \in \mathbf{R}(R)$ such that $P(p_n) = 1$ but $P(\Phi(p_n)) = 0$ for every n, which would imply that Φ is not the identical map of Y.

Indeed, take a mutually disjoint, simply connected neighborhood U_n of p_n in $\{|z| < 1\}$ so that $\Phi(p_n) \notin U_n$ for every n, and map U_n onto $\{|z| < 1\}$ by a Riemann map g_n so that $g_n(p_n) = 0$. Consider

$$h_n(z) = \frac{-\log(2|z|)}{n^3}$$

on $W_n = \{e^{-n^3}/2 < |z| < 1/2\}$, and set $P_n = h_n \circ g_n$ on $g_n^{-1}(W_n)$. Extend P_n to a continuous function by letting it to be a constant 0 or 1 on each connected component of $R - g_n^{-1}(W_n)$, we have a function P_n in $\mathbf{R}(R)$ such that $D(P_n) = 2\pi/n^3$. And

$$P = \sum_{n=1}^{\infty} P_n$$

is a desired function.

Thus there is a canonical neighborhood V' of Y such that V', $\tilde{F}_0(V')$, $\tilde{F}(V')$ are contained in V, and that $F_0(p) = F(p)$, for every puncture p in V'. But then the number of punctures of R_1 in $V - V'$ is smaller than that of punctures of R_1 $V - \tilde{F}(V')$, which is a contradiction.

Since the case of F_0^{-1} can be treated similarly, we conclude the assertion. □

Next, there are boundary self-homeomorphisms with no fixed points. For instance, rotations give such examples. On the other hand, the following fact seems to be non-trivial.

PROPOSITION 5. *There is an ideal boundary Y such that every element of* $\mathrm{BH}(Y)$ *fixes the same point of Y.*

PROOF. In general, the harmonic boundary d_0R of the Royden boundary is invariant under boundary homeomorphisms ([10] III.7.C Theorem. Also see [10] III.8.C Theorem), and hence by Proposition 3, $d_0R \cap Y$ is invariant under every $f \in BH(Y)$. On the other hand, if a supporting surface R belongs to $O_{HD} - O_G$, a theorem of Royden states that $d_0R \cap Y$ consists of a single point (cf. [10] III.F Theorem), which implies the assertion. □

Finally, conformal equivalence eventually homotopic to the identity is trivial. Here, we say that a conformal boundary self-homeomorphism $f : Y \to Y$ is *eventually homotopic to the identity* if f is supported by a homeomorphism F of a canonical neighborhood U of Y in R^* into R^* such that F on $U \cap R$ is conformal and homotopic to the identical map of $U \cap R$ in R.

PROPOSITION 6. *Suppose that $[Y, \iota_R]$ is an ideal boundary of topologically infinite type. Let $f_1, f_2 \in BH(Y)$. If $f_1^{-1} \circ f_2$ is a conformal boundary self-homeomorphism eventually homotopic to the identity, then $f_1 = f_2$.*

PROOF. By a theorem of Maitani in [6], F as above should be the identical map of U, and hence so is $f_1^{-1} \circ f_2$. □

3. The Teichmüller space

Similarly as before, for ideal boundaries $[Y, \iota_R]$ and $[Y', \iota_{R'}]$, we say that a boundary homeomorphism $f : Y \to Y'$ is *eventually homotopic* to an asymptotically conformal boundary homeomorphism $g : Y \to Y'$ if there are supporting maps $F : U \to (R')^*$ of f and $G : U \to (R')^*$ of g, where U is a canonical neighborhood of Y in R^*, such that F is quasiconformal on $U \cap R$, that G is asymptotically conformal on $U \cap R$, and that F on $U \cap R$ is homotopic to G on $U \cap R$ in R.

In particular, if $[Y, \iota_R] = [Y', \iota_{R'}]$ and G is the identical map, then again we say that f is *eventually homotopic to the identity*.

Theorem 7. *For every ideal boundary Y, there is a non-identical asymptotically conformal boundary self-homeomorphism of Y eventually homotopic to the identity.*

Proof. Let U be a canonical neighborhood of Y in R^*, where R is a supporting surface of Y. Take a sequence of points p_n on $U \cap R$ escaping from any compact set of R, and a mutually disjoint, simply connected open neighborhood U_n of p_n for every n. Map each U_n onto $\{|z| < 1\}$ by a Riemann map g_n so that $g_n(p_n) = 0$.

Set

$$\varphi_n(z) = \frac{z + (1/n)}{1 + (1/n)\bar{z}}$$

on $\{|z| < 1\}$, and φ_n is a $(1/n)$-quasiconformal self-homeomorphism of $\{|z| < 1\}$ and $\varphi_n(z) = z$ on $\{|z| = 1\}$. Hence we can define a $(1/n)$-quasiconformal homeomorphism Φ of U into R^* by setting $g_n^{-1} \circ \varphi_n \circ g_n$ on U_n for every n, and to be the identical map outside $\bigcup_{n=1}^{\infty} U_n$. Then Φ gives an asymptotically conformal boundary self-homeomorphism f of Y eventually homotopic to the identity.

Next similarly as before, set

$$h_n(z) = \frac{-\log(n|z|)}{n^3}$$

on $W_n = \{(1/n)e^{-n^3} < |z| < (1/n)\}$. Then we have an element P_n of $\mathbf{R}(R)$ by setting $P_n = h_n \circ g_n$ on $g_n^{-1}(W_n)$ and by letting it to be a constant 0 or 1 on each component of $R - g_n^{-1}(W_n)$. Since $D(P_n) = 2\pi/n^3$, $P = \sum_{n=1}^{\infty} P_n$ also belongs to $\mathbf{R}(R)$, and $P(p_n) = 1$ and $P(\Phi(p_n)) = 0$ for every n. Thus f is not the identical map. □

We say that two ideal boundaries $Y_1 = [Y_1, \iota_{R_1}]$ and $Y_2 = [Y_2, \iota_{R_2}]$ are *quasiconformally related* if there is a (quasiconformal) boundary homeomorphism of Y_1 onto Y_2. Then we can define the Teichmüller space of quasiconformally related ideal boundaries.

Definition. For a given ideal boundary $Y_0 = [Y_0, \iota_{R_0}]$, consider a pair

$(Y, f) = ([Y, \iota_R], f)$ of an ideal boundary $Y = [Y, \iota_R]$ quasiconformally related to Y_0 and a boundary homeomorphism $f : Y_0 \to Y$, which is called a *marking* of Y.

We say that two pairs (Y_1, f_1) and (Y_2, f_2) are *Teichmüller equivalent* if there is an asymptotically conformal boundary homeomorphism of Y_1 to Y_2 eventually homotopic to $f_2 \circ f_1^{-1}$.

We call the set of all Teichmüller equivalence classes $[Y, f] = [[Y, \iota_R], f]$ of such pairs (Y, f) the *Teichmüller space* of Y_0, which is denoted by $T(Y_0)$. A point of $T(Y_0)$ is called a *marked ideal boundary*.

Here, note that if Y_0 is an ideal boundary of analytically finite type, i.e. obtained from a closed surface by deleting a finite number of points, then Y_0 is empty, and hence $T(Y_0)$ consists of a single point (which can be compared with results in [2], [4]). It is remarkable that the Teichmüller space of every ideal boundary admits a natural complex structure.

THEOREM 8. *Let Y_0 be an ideal boundary. Then the Teichmüller space $T(Y_0)$ of Y_0 has a complex Banach manifold structure.*

PROOF. A theorem of Miyaji in [7] implies that the asymptotic Teichmüller spaces $AT(R_0)$ of R_0 are mutually biholomorphic for all supporting surfaces R_0 of Y_0. Indeed, if R_1 and R_2 are such surfaces, then there is another supporting surface R_3 of Y_0 and analytically finite Riemann surfaces S_1 and S_2 such that R_3 and S_j are obtained from R_j by applying a conformal 2-surgery along a dividing simple closed curve for each j. And Reducing Theorem in [7] states that the asymptotic Teichmüller space $AT(R_j)$ is biholomorphic to the product $AT(S_j) \times AT(R_3)$ for each j. Here, since $AT(S_j)$ are trivial, we have a canonical biholomorphic map between $AT(R_j)$. (For the details of the asymptotic Teichmüller theory, see [5], [2], and [3].)

Next, fix a supporting surface R_0 of Y_0. Then we can construct a natural bijection from $T(Y_0)$ onto $AT(R_0)$ as follows. Take any element $[Y, f]$ of $T(Y_0)$. Then there is a quasiconformal homeomorphism F of $U \cap R_0$ into R whose extension to U supports f. Here, U is a canonical neighborhood of Y_0 in R_0 and R is a supporting surface of Y. Such an F can be extended to a quasiconformal map of R_0 onto another supporting surface R' of Y (possibly different from R), which gives a point in $AT(R_0)$. By the definitions, we see that this map ι induces a bijection of $T(Y_0)$ to $AT(R_0)$.

Indeed, if pairs (Y_1, f_1) and (Y_2, f_2) belong to the same point of $T(Y_0)$, then there is an asymptotically conformal boundary homeomorphism $g : Y_1 \to Y_2$ eventually homotopic to $f_2 \circ f_1^{-1}$. Hence we can find a canonical neighborhood U of Y_0, asymptotically conformal maps F_j of $U \cap R_0$ into R_j for each j, where R_j is a supporting surface of Y_j, and an asymptotically conformal map G of

$F_1(U \cap R_0)$ into R_2 supporting g and homotopic to $F_2 \circ F_1^{-1}$. Here taking a smaller U and changing supporting surfaces if necessary, we may also assume that F_j can be extended to a quasiconformal map $\hat{F}_j$ of R_0 onto R_j for each j. Then $\hat{F}_2^{-1} \circ \hat{F}_1$ is homotopic to an asymptotically conformal homeomorphism. Hence ι is well-defined.

Conversely, if there are quasiconformal maps $\hat{F}_j$ of R_0 onto R_j for each j such that $\hat{F}_2^{-1} \circ \hat{F}_1$ is homotopic to an asymptotically conformal homeomorphism. Then by definition, the boundary maps supported by these $\hat{F}_j$ are Teichmüller equivalent. Hence ι is injective. Finally, since every element of $AT(R_0)$ determines an ideal boundary Y quasiconformally related to Y_0 and a boundary homeomorphism of Y_0 onto Y, ι is also surjective. Thus we have proved the assertion. □

Remark. We say that two boundary self-homeomorpihsms f_1 and f_2 in $\mathrm{BH}(Y_0)$ are *AC-equivalent* if $f_2 \circ f_1^{-1}$ is homotopic to an asymptotically conformal self-homeomorphism of Y. The equivalence class of f is called an *AC-mapping class*, and denoted by $[f]$.

Now every element f of $\mathrm{BH}(Y_0)$ naturally induces an automoprhism f^* of $T(Y_0)$, by setting

$$f^*([Y,g]) = [(Y, g \circ f^{-1})].$$

Then it is clear from the definition that $f_1^* = f_2^*$ if and only if $[f_1] = [f_2]$.

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