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Extendibility of negative vector bundles over the complex projective space

Dedicated to the memory of Professor Masahiro Sugawara

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ABSTRACT. By Schwarzenberger's property, a complex vector bundle of dimension t over the complex projective space $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+k}$ for any $k \geq 0$ if and only if it is stably equivalent to a Whitney sum of t complex line bundles. In this paper, we show some conditions for a negative multiple of a complex line bundle over $\mathbf{C}P^n$ to be extendible to $\mathbf{C}P^{n+1}$ or $\mathbf{C}P^{n+2}$, and its application to unextendibility of a normal bundle of $\mathbf{C}P^n$.

1. Introduction and results

An m-dimensional vector bundle V over a space A is called extendible to a space $B \supset A$ when there exists an m-dimensional vector bundle over B whose restriction to A is isomorphic to V. Classically, Schwarzenberger [11], [4, Appendix I] studied extendibility of vector bundles over the real or complex projective spaces. Related results were obtained by Rees [3], [10] and Adams–Mahmud [1]. Extendibility of vector bundles over the real projective spaces and the standard lens spaces are studied extensively by Kobayashi-Maki-Yoshida [8], [9] and so on, and that of vector bundles over the quaternionic projective spaces by [6], [7].

We consider only complex vector bundles, and thus a k-dimensional vector bundle means a $\mathbf{C}^k$-vector bundle. Let ξ be the canonical line bundle over the complex projective space $\mathbf{C}P^n$, and for an integer m

$$\xi^m = \underbrace{\xi \otimes \cdots \otimes \xi}_{m} \quad \text{if } m > 0; \qquad \xi^0 = \underline{\mathbf{C}}^1; \qquad \xi^m = \underbrace{\bar{\xi} \otimes \cdots \otimes \bar{\xi}}_{-m} \quad \text{if } m < 0,$$

where $\bar{\xi}$ is the complex conjugate bundle of ξ and $\underline{\mathbf{C}}^1$ is the trivial line bundle. Then, any line bundle over $\mathbf{C}P^n$ is isomorphic to ξ^m for some m.

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There exists a vector bundle $-\xi^m$ over $\mathbf{C}P^n$ which satisfies $\xi^m \oplus (-\xi^m) \oplus \underline{\mathbf{C}}^j = \underline{\mathbf{C}}^k$ for trivial vector bundles $\underline{\mathbf{C}}^j$ and $\underline{\mathbf{C}}^k$ of some dimensions j and k. Then, $-\xi^m$ is uniquely determined up to stable equivalence, that is, if γ satisfies the relation, then $\gamma \oplus \underline{\mathbf{C}}^{j'} = (-\xi^m) \oplus \underline{\mathbf{C}}^{k'}$ for some j' and k'. A vector bundle $-l\xi^m$ with an integer $l > 0$ is the Whitney sum of l numbers of $-\xi^m$. Then, we can take $-l\xi^m$ as an n-dimensional vector bundle over $\mathbf{C}P^n$ by the following stability property (cf. [5, Chapter 9, Section 1]):

Proposition 1.1 (Stability property). *For any m-dimensional vector bundle α over $\mathbf{C}P^n$ with $m \geq n$, there exists an n-dimensional vector bundle β satisfying $\alpha = \beta \oplus \underline{\mathbf{C}}^{m-n}$. In addition, β is unique for the stably equivalent class of α.*

By Schwarzenberger [4, Appendix I], if a t-dimensional vector bundle α over $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+k}$ for any $k \geq 0$, then α is stably equivalent to a Whitney sum of t line bundles. On the other hand, since the K-group of $\mathbf{C}P^n$ is additively generated by the stably equivalent classes of line bundles ξ^m for $0 \leq m \leq n$ (cf. [2]), any vector bundle over $\mathbf{C}P^n$ is stably equivalent to a Whitney sum of line bundles and vector bundles $-\xi^k$. Our main purpose of this paper is to determine conditions when an n-dimensional vector bundle $-l\xi^m$ over $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+1}$ or $\mathbf{C}P^{n+2}$.

Thomas [14] has characterized the so-called Chern vectors of vector bundles over $\mathbf{C}P^n$, which is applicable to our problem. Using such combinatorial relations of Chern classes, we show the following, where $\binom{a}{b}$ denotes a binomial coefficient.

Theorem 1.2. *Let n, l and m be integers with $n > 0$ and $l > 0$, and $-l\xi^m$ be the n-dimensional vector bundle over $\mathbf{C}P^n$. Then, the following hold:*

(1) *$-l\xi^m$ is extendible to $\mathbf{C}P^{n+1}$ if and only if the following congruence holds:*

$$\binom{n+l}{n+1} m^{n+1} \equiv 0 \pmod{n!}.$$

(2) *If $-l\xi^m$ is extendible to $\mathbf{C}P^{n+2}$, then the congruence in* (1) *and the following congruence hold:*

$$l\left(m - \binom{n+2}{2}\right)\binom{n+l}{n} m^{n+1} \equiv 0 \pmod{(n+2)!}.$$

Conversely, when n is odd, $-l\xi^m$ is extendible to $\mathbf{C}P^{n+2}$ if the above two congruences hold.

Thus, if one of the congruences in Theorem 1.2 does not hold, then $-l\xi^m$ over $\mathbf{C}P^n$ is not stably equivalent to a Whitney sum of less than or equal to

n numbers of line bundles, because the latter is extendible to $\mathbf{C}P^{n+k}$ for any $k \geq 0$.

We also remark that the stable extendibility, introduced in [6], of the n-dimensional vector bundle $-l\xi^m$ over $\mathbf{C}P^n$ is the same as extendibility of it by stability property (Proposition 1.1).

Let $q(n)$ denote the product of all distinct primes less than or equal to n, that is,

$$q(n) = \prod_{\text{prime } p \leq n} p.$$

Then, in special cases, Theorem 1.2 is expressed as follows:

Corollary 1.3. *Assume that* $m \equiv 0 \pmod{q(n)}$. *Then, for the n-dimensional vector bundle $-l\xi^m$ over $\mathbf{C}P^n$ with $n > 0$ and $l > 0$, the following hold:*

(1) *$-l\xi^m$ is extendible to $\mathbf{C}P^{n+1}$.*

(2) *When n is odd, if $n+2$ is not a prime or $m \equiv 0 \pmod{n+2}$, then $-l\xi^m$ is extendible to $\mathbf{C}P^{n+2}$.*

(3) *When $n+2$ is a prime and $m \not\equiv 0 \pmod{n+2}$, $-l\xi^m$ is extendible to $\mathbf{C}P^{n+2}$ if and only if $l \not\equiv 1 \pmod{n+2}$.*

Corollary 1.4. *Let $-\xi^m$ be the n-dimensional vector bundle over $\mathbf{C}P^n$ for $n > 0$. Then, the following hold:*

(1) *$-\xi^m$ is extendible to $\mathbf{C}P^{n+1}$ if and only if $m \equiv 0 \pmod{q(n)}$.*

(2) *If $-\xi^m$ is extendible to $\mathbf{C}P^{n+2}$, then $m \equiv 0 \pmod{q(n+2)}$ or $m \equiv 0 \pmod{q(n)}$ according as $n+2$ is a prime or not. When n is odd, the converse holds.*

Let $\nu(\mathbf{C}P^n)$ be a normal bundle of $\mathbf{C}P^n$ in the sense that $\nu(\mathbf{C}P^n)$ is a complex vector bundle satisfying that $T(\mathbf{C}P^n) \oplus \nu(\mathbf{C}P^n)$ is stably equivalent to a trivial vector bundle, where $T(\mathbf{C}P^n)$ is the complex tangent bundle of $\mathbf{C}P^n$. Then, $\nu(\mathbf{C}P^n)$ exists and is unique up to stable equivalence, and the following holds:

Lemma 1.5. *For $n \geq 2$, $\nu(\mathbf{C}P^n)$ is not stably equivalent to any Whitney sum of line bundles over $\mathbf{C}P^n$.*

Thus, by Schwarzenberger's property, any choice of normal bundle $\nu(\mathbf{C}P^n)$ for $n \geq 2$ is not extendible to $\mathbf{C}P^{n+k}$ for some $k > 0$. Now, by stability property, we can take $\nu(\mathbf{C}P^n)$ as an n-dimensional vector bundle over $\mathbf{C}P^n$. Then, applying Theorem 1.2, we show the following:

THEOREM 1.6. *The n-dimensional normal bundle $\nu(\mathbf{C}P^n)$ is not extendible to $\mathbf{C}P^{n+1}$ for $n \geq 3$. $\nu(\mathbf{C}P^1) = \xi^2$ is extendible to $\mathbf{C}P^k$ for any $k \geq 1$, and $\nu(\mathbf{C}P^2)$ is extendible to $\mathbf{C}P^3$ but not extendible to $\mathbf{C}P^4$.*

The paper is organized as follows: In §2 we prepare some necessary properties about Chern vectors studied in [14], and in §3 we prove Theorem 1.2 and Corollaries 1.3 and 1.4. §4 is devoted to the proof of Lemma 1.5 and Theorem 1.6.

2. Chern vectors of negative line bundles

Let $x \in H^2(\mathbf{C}P^n; \mathbf{Z})$ be the Euler class of the canonical line bundle ξ over $\mathbf{C}P^n$. Then, the cohomology ring $H^*(\mathbf{C}P^n; \mathbf{Z})$ is isomorphic to the truncated polynomial ring $\mathbf{Z}[x]/(x^{n+1})$, and the i-th Chern class $C_i(V)$ of a vector bundle V over $\mathbf{C}P^n$ is represented as an integer $c_i(V)$ multiple of x^i, namely $C_i(V) = c_i(V)x^i$. Then, the Chern vector of V is defined to be an integral vector $(c_1(V), \dots, c_n(V)) \in \mathbf{Z}^n$.

As for the Chern vector of $-l\xi^m$, we have the following:

LEMMA 2.1. *The Chern vector of $-l\xi^m$ with $l > 0$ over $\mathbf{C}P^n$ is equal to*

$$\left(-lm, \binom{l+1}{2}m^2, \dots, (-1)^i\binom{l+i-1}{i}m^i, \dots, (-1)^n\binom{l+n-1}{n}m^n\right).$$

PROOF. Let $C(V) = \sum_{i \geq 0} C_i(V)$ be the total Chern class of a vector bundle V. Then, since $C(V)$ is multiplicative and $C(\xi^m) = 1 + mx$ (cf. [4, §4]),

$$C(-l\xi^m) = (1+mx)^{-l} = \sum_{i=0}^{n}\binom{-l}{i}m^i x^i = \sum_{i=0}^{n}(-1)^i\binom{l+i-1}{i}m^i x^i,$$

and we have the required Chern vector. □

Next, let $s_k : \mathbf{Z}^k \to \mathbf{Z}$ for $k \geq 1$ be a map defined recursively using the Newton's formula as follows: $s_1(m_1) = m_1$; for $k \geq 2$,

$$(2.1) \quad s_k(m_1, \dots, m_k) = \sum_{i=1}^{k-1}(-1)^{i+1}m_i s_{k-i}(m_1, \dots, m_{k-i}) + (-1)^{k+1}km_k.$$

Also, for a vector bundle V over $\mathbf{C}P^n$, we set

$$(2.2) \quad s_k(V) = s_k(c_1(V), \dots, c_k(V)).$$

Then, $s_k(V)$ for $1 \leq k \leq n$ is additive, that is, $s_k(V \oplus W) = s_k(V) + s_k(W)$ holds for vector bundles V and W over $\mathbf{C}P^n$ (cf. [4, §10]), and obviously $s_k(\underline{\mathbf{C}}^j) = 0$ for a trivial vector bundle $\underline{\mathbf{C}}^j$.

For the line bundle ξ^m over $\mathbf{C}P^n$, since $c_1(\xi^m) = m$ and $c_i(\xi^m) = 0$ for $i \geq 2$, we have $s_k(\xi^m) = m^k$ for $k \geq 1$ by definition. Hence, for the vector bundle $-l\xi^m$ over $\mathbf{C}P^n$, we have the following:

Lemma 2.2. *$s_k(-l\xi^m) = -lm^k$ for $1 \leq k \leq n$.*

Let $f_k : \mathbf{Z}^k \to \mathbf{Z}$ for an integer $k \geq 1$ be a map defined recursively by $f_1(m_1) = m_1$ and for $k \geq 2$

$$f_k(m_1, \dots, m_k) = f_{k-1}(m_2, \dots, m_k) - (k-1) f_{k-1}(m_1, \dots, m_{k-1}).$$

The following is straightforward from the definition.

Lemma 2.3. (1) *f_k is a linear map, that is, for $x, y \in \mathbf{Z}^k$ and $r, s \in \mathbf{Z}$,*

$$f_k(rx + sy) = r f_k(x) + s f_k(y).$$

(2) $f_k(1, 0, 0, \dots, 0) = (-1)^{k-1}(k-1)!$.
(3) $f_k(0, \dots, 0, 1) = 1$, $f_k(0, \dots, 0, 1, 0) = -\binom{k}{2}$ *for* $k \geq 2$.
(4) ([14, Lemma 3.3(i)]). *For any integer j,*

$$f_k(j, j^2, \dots, j^k) = \prod_{i=0}^{k-1} (j - i).$$

Using the maps f_k, Thomas has shown the following.

Theorem 2.4 ([14, Theorem A, Proposition 3.5]).
(1) *An integral vector $(m_1, \dots, m_n)$ is a Chern vector of a vector bundle over $\mathbf{C}P^n$ if and only if $f_k(s_1, \dots, s_k) \equiv 0 \pmod{k!}$ for $1 \leq k \leq n$, where $s_i = s_i(m_1, \dots, m_i)$.*
(2) *An n-dimensional vector bundle α over $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+1}$ if and only if the following congruence holds:*

$$f_{n+1}(s_1(\alpha), \dots, s_n(\alpha), s_{n+1}(\alpha)) \equiv 0 \pmod{(n+1)!}.$$

Some part of this theorem are slightly generalized as follows:

Proposition 2.5. *If an n-dimensional vector bundle α over $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+k}$ for some $k \geq 1$, then the following congruences hold:*

$$f_{n+i}(s_1(\alpha), \dots, s_n(\alpha), s_{n+1}(\alpha), \dots, s_{n+i}(\alpha)) \equiv 0 \pmod{(n+i)!}$$

for any i with $1 \leq i \leq k$. Furtheremore, when n is odd and $k = 2$, the converse holds.

Proof. If α is extendible to an n-dimensional vector bundle β over $\mathbf{C}P^{n+k}$, then $c_j(\beta) = c_j(\alpha)$ for any $j \geq 1$. Thus, $s_j(\beta) = s_j(\alpha)$ for any $j \geq 1$. Hence, applying Theorem 2.4(1) to β, we have the first required result.

As for the converse, we assume that n is odd and the congruences hold for $k = 2$. Then, by Theorem 2.4(1) and the stability property, there exists an $(n+2)$-dimensional vector bundle γ over $\mathbf{C}P^{n+2}$, which satisfies $c_i(\gamma) = c_i(\alpha)$ for any $i \geq 1$. In particular, we have $c_{n+1}(\gamma) = c_{n+2}(\gamma) = 0$. Then, by Thomas [15, Theorem 3.5], γ has two linearly independent sections, and hence there exists an n-dimensiona vector bundle β over $\mathbf{C}P^{n+2}$ satisfying $\gamma = \beta \oplus \underline{\mathbf{C}}^2$. Then, $c_i(\beta) = c_i(\gamma) = c_i(\alpha)$ for all $i \geq 1$. Since the cohomology group $H^*(\mathbf{C}P^n; \mathbf{Z})$ has no torsion, two vector bundles over $\mathbf{C}P^n$ which have the same Chern classes are stably equivalent. Thus, the restriction of β over $\mathbf{C}P^n$ is stably equivalent to α. Since α and the restriction of β are both n-dimensional vector bundles over $\mathbf{C}P^n$, they are isomorphic by stability property, which completes the proof of the converse. □

3. Proof of Theorem 1.2 and its corollaries

First, we prove Theorem 1.2 using the results in the last section.

Proof of Theorem 1.2. Let α be the $(n+2)$-dimensional vector bundle $-l\xi^m$ over $\mathbf{C}P^{n+2}$. Then, by Lemmas 2.1 and 2.2,

$$c_{n+j}(\alpha) = (-1)^{n+j}\binom{l+n+j-1}{n+j}m^{n+j} \qquad \text{and} \qquad s_{n+j}(\alpha) = -lm^{n+j}$$

for $j = 1, 2$. Thus, for the vector bundle $-l\xi^m$ over $\mathbf{C}P^n$, $s_i(-l\xi^m) = -lm^i$ for $1 \leq i \leq n$, and, by (2.1) and (2.2),

$$\begin{aligned} s_{n+1}(-l\xi^m) &= s_{n+1}(\alpha) - (-1)^n(n+1)c_{n+1}(\alpha) \\ &= -lm^{n+1} + (n+1)\binom{l+n}{n+1}m^{n+1}. \end{aligned}$$

$$\begin{aligned} s_{n+2}(-l\xi^m) &= s_{n+2}(\alpha) - (-1)^n c_{n+1}(\alpha)s_1(\alpha) - (-1)^{n+1}(n+2)c_{n+2}(\alpha) \\ &= -lm^{n+2} - l\binom{l+n}{n+1}m^{n+2} + (n+2)\binom{l+n+1}{n+2}m^{n+2}. \end{aligned}$$

Now, we consider the extendibility of $-l\xi^m$ to $\mathbf{C}P^{n+1}$ in (1). Using Lemma 2.3,

$$\begin{aligned} &f_{n+1}(s_1(-l\xi^m), \ldots, s_{n+1}(-l\xi^m)) \\ &\quad = -lf_{n+1}(m, \ldots, m^{n+1}) + (n+1)\binom{l+n}{n+1}m^{n+1}f_{n+1}(0, \ldots, 0, 1) \\ &\quad = -l\prod_{i=0}^{n}(m-i) + (n+1)\binom{n+l}{n+1}m^{n+1}. \end{aligned}$$

But, concerning the first term of the last equation,

$$\prod_{i=0}^{n}(m-i) = (n+1)!\binom{m}{n+1} \equiv 0 \pmod{(n+1)!}.$$

Hence, by Theorem 2.4(2), $-l\xi^m$ is extendible to $\mathbf{C}P^{n+1}$ if and only if the following congruence holds:

$$\binom{n+l}{n+1}m^{n+1} \equiv 0 \pmod{n!}, \tag{3.1}$$

which is the required result of (1).

As for the extendibility of $-l\xi^m$ to $\mathbf{C}P^{n+2}$ in (2), we can proceed similarly. Using Lemma 2.3,

$$\begin{aligned} f_{n+2}(s_1,\dots,s_{n+2}) &= -l\prod_{i=0}^{n+1}(m-i) - (n+1)\binom{l+n}{n+1}m^{n+1}\binom{n+2}{2} \\ &\quad + \left(-l\binom{l+n}{n+1} + (n+2)\binom{l+n+1}{n+2}\right)m^{n+2} \\ &= -l(n+2)!\binom{m}{n+2} + l\left(m - \binom{n+2}{2}\right)\binom{n+l}{n}m^{n+1}, \end{aligned}$$

where $s_i = s_i(-l\xi^m)$. Hence, by Proposition 2.5, if $-l\xi^m$ is extendible to $\mathbf{C}P^{n+2}$, then the congruence (3.1) and the following congruence hold:

$$l\left(m - \binom{n+2}{2}\right)\binom{n+l}{n}m^{n+1} \equiv 0 \pmod{(n+2)!}.$$

Also, the converse holds by Proposition 2.5 when n is odd. Thus, we have completed the proof. □

In order to prove Corollaries 1.3 and 1.4, we prepare some notations. For a prime p, let $v_p(m) = a$ for an integer m if $m = p^a b$ and b is an integer prime to p, and $\alpha_p(k)$ for an integer $k \geq 1$ be the sum $\sum_{i=0}^{j} a_i$ of the coefficients in the p-adic expansion $k = \sum_{i=0}^{j} a_i p^i$, where $0 \leq a_i \leq p-1$. Then, the following is known, but we give a proof briefly.

Lemma 3.1. *For a prime p and a positive integer k,*

$$v_p(k!) = \frac{k - \alpha_p(k)}{p-1}.$$

Proof. When $k = 1$, it is clear. Thus, inductively, assume that the result is true for an integer $k \geq 1$. We put $k+1 = bp^t$ with $t \geq 0$ and

$b \not\equiv 0 \pmod{p}$. Then, $v_p(k+1) = t$ and $\alpha_p(k+1) = \alpha_p(b)$. Since $k = b - 1$ if $t = 0$ and since

$$k = bp^t - 1 = (b-1)p^t + (p-1)p^{t-1} + \cdots + (p-1)p + (p-1)$$

if $t > 0$, we have $\alpha_p(k) = \alpha_p(b) - 1 + t(p-1)$. Thus, we have

$$v_p((k+1)!) = v_p(k!) + v_p(k+1) = \frac{k - \alpha_p(k)}{p-1} + t$$

$$= \frac{k - \alpha_p(b) + 1}{p-1} = \frac{(k+1) - \alpha_p(k+1)}{p-1},$$

which completes the induction. □

Let $q(n)$ be the product of all distinct primes less than or equal to a positive integer n, as is introduced in §1. Then, we have the following:

LEMMA 3.2. *For integers $k \geq 1$ and m, if $m^i \equiv 0 \pmod{k!}$ for some $i \geq 1$, then $m \equiv 0 \pmod{q(k)}$. Conversely, if $m \equiv 0 \pmod{q(k)}$, then $m^{k-1} \equiv 0 \pmod{k!}$.*

PROOF. First, assume that $m^i \equiv 0 \pmod{k!}$ for some $i \geq 1$. Then, $m \equiv 0 \pmod{p}$ for any prime $p \leq k$, and thus $m \equiv 0 \pmod{q(k)}$. Conversely, assume that $m \equiv 0 \pmod{q(k)}$, and let p be any prime with $p \leq k$. Then, $m \equiv 0 \pmod{p}$, and by Lemma 3.1 we have $v_p(k!) \leq k - 1 \leq (k-1)v_p(m) = v_p(m^{k-1})$. Hence, we have $m^{k-1} \equiv 0 \pmod{k!}$, as is required. □

Now, we prove the corollaries.

PROOF OF COROLLARY 1.3. Assume that $m \equiv 0 \pmod{q(n)}$. As for (1), the congruence in Theorem 1.2(1) holds by Lemma 3.2, and we have the required result.

Concernig the proof of (2), we first assume that n is odd and $n+2$ is not a prime. Let p be any prime with $p \leq n$. We shall show

$$v_p((n+2)!) \leq v_p(m^{n+1}). \tag{3.2}$$

Then, since

$$l\binom{n+l}{n}m^{n+1} = \binom{n+l}{n+1}(n+1)m^{n+1}$$

and $(n+1)m^{n+1} \equiv 0 \pmod{(n+2)!}$ by (3.2), we obtain the required result in this case by Theorem 1.2(2). Now, we prove (3.2). We notice that $v_p(m^{n+1}) \geq n+1$ by the first assumption. We put $n+1 = ap^k + \sum_{i=0}^{k-1}(p-1)p^i$ for some $k \geq 0$ and $a \geq 0$ with $a \not\equiv p-1 \pmod{p}$, where we consider the second term of the right hand side of the equality is 0 when $k = 0$. Then, $\alpha_p(n+1) =$

$\alpha_p(a) + k(p-1)$ and $\nu_p(n+2) = k$, and thus we obtain (3.2) using lemma 3.1 as follows:

$$\nu_p((n+2)!) = \nu_p((n+1)!) + k = \frac{(n+1) - \alpha_p(a)}{p-1} \le n+1 \le \nu_p(m^{n+1}).$$

Next, assume that $m \equiv 0 \pmod{n+2}$ and $n+2$ is a prime. Then, since $n+1$ is not a prime, we have $m \equiv 0 \pmod{q(n+2)}$ by the assumptions $m \equiv 0 \pmod{q(n)}$ and $m \equiv 0 \pmod{n+2}$. Hence, $m^{n+1} \equiv 0 \pmod{(n+2)!}$ by Lemma 3.2, which establishes the congruence in Theorem 1.2(2), and thus we have (2).

Lastly, we prove (3). Thus, we assume that $n+2$ is a prime and $m \not\equiv 0 \pmod{n+2}$. Then, since $n+1$ is even and $m^{n-1} \equiv 0 \pmod{n!}$ by the first assumption and Lemma 3.2, the following term in the second congruence in Theorem 1.2 satisfies

$$l\binom{n+2}{2}\binom{n+l}{n}m^{n+1} = \frac{n+1}{2}\binom{n+l}{n+1}(n+1)(n+2)m^{n+1} \equiv 0 \pmod{(n+2)!}.$$

Thus, by Theorem 1.2(2) and (1), $-l\xi^m$ is extendible to $\mathbf{C}P^{n+2}$ if and only if the congruence

$$l\binom{n+l}{n}m^{n+2} = \binom{n+l}{n+1}(n+1)m^{n+2} \equiv 0 \pmod{(n+2)!}$$

holds. Since $(n+1)m^{n+2} \equiv 0 \pmod{(n+1)!}$ and $(n+1)m^{n+2} \not\equiv 0 \pmod{(n+2)!}$, the congruence is equivalent to

$$\binom{n+l}{n+1} \equiv 0 \pmod{n+2}. \tag{3.3}$$

Then, putting $n+l = c(n+2)+d$ for some integers $c \ge 0$ and $0 \le d \le n+1$ and using a well known property of binomial coefficients modulo a prime (cf. [12, Lemma 2.6]), we have

$$\binom{n+l}{n+1} \equiv \binom{d}{n+1} \pmod{n+2}.$$

Hence, (3.3) holds if and only if $0 \le d \le n$, that is, if and only if $l \not\equiv 1 \pmod{n+2}$, and thus we have completed the proof. □

Proof of Corollary 1.4. As for (1), by Theorem 1.2(1), $-\xi^m$ over $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+1}$ if and only if the congruence $m^{n+1} \equiv 0 \pmod{n!}$ holds since $l = 1$ in this case. Then, the congruence is equivalent to the required congruence $m \equiv 0 \pmod{q(n)}$ by Lemma 3.2.

Concerning (2), assume first that $n+2$ is a prime. Then, if $m \equiv 0 \pmod{q(n+2)}$, then $m^{n+1} \equiv 0 \pmod{(n+2)!}$ by Lemma 3.2. Thus, $-\xi^m$ is extendible to $\mathbf{C}P^{n+2}$ by Theorem 1.2(2). Conversely, if $-\xi^m$ is extendible to $\mathbf{C}P^{n+2}$, then $m^{n+1} \equiv 0 \pmod{n!}$ by the congruence in Theorem 1.2(1), and thus $m \equiv 0 \pmod{q(n)}$ by Lemma 3.2. Then, by Corollary 1.3(2) and (3), we have $m \equiv 0 \pmod{n+2}$ since $l = 1$, and thus $m \equiv 0 \pmod{q(n+2)}$ as is required. Similarly, when n is odd and $n+2$ is not a prime, $-\xi^m$ is extendible to $\mathbf{C}P^{n+2}$ if $m \equiv 0 \pmod{q(n)}$ by Corollary 1.3(2), and the converse follows from the congruence in Theorem 1.2(1) and Lemma 3.2. Thus, we have completed the proof. □

4. Unxtendibility of normal bundle

First, we prove Lemma 1.5 using the K-ring structure of $\mathbf{C}P^n$.

PROOF OF LEMMA 1.5. Let $X = [\xi - \underline{\mathbf{C}}^1]$ be the stably equivalent class of ξ over $\mathbf{C}P^n$. Then, the K-ring $K(\mathbf{C}P^n)$ of $\mathbf{C}P^n$ is a truncated polynomial ring $\mathbf{Z}[X]/(X^{n+1})$ (cf. [2]). The tangent bundle $T(\mathbf{C}P^n)$ of $\mathbf{C}P^n$ satisfies $T(\mathbf{C}P^n) \oplus \underline{\mathbf{C}}^1 = (n+1)\bar{\xi} = (n+1)\xi^{-1}$ (cf. [13, Chapter V]). Thus, a normal bundle $\nu(\mathbf{C}P^n)$ is stably equivalent to $-(n+1)\xi^{-1}$. Since $\xi \otimes \xi^{-1} = \underline{\mathbf{C}}^1$, we have $(X+1)([\xi^{-1} - \underline{\mathbf{C}}^1] + 1) = 1$ in $K(\mathbf{C}P^n)$. Hence,

$$[\xi^{-1} - \underline{\mathbf{C}}^1] = (X+1)^{-1} - 1 = \sum_{i=1}^{n} (-1)^i X^i,$$

and thus

$$[\nu(\mathbf{C}P^n) - \underline{\mathbf{C}}^N] = -(n+1)[\xi^{-1} - \underline{\mathbf{C}}^1] \equiv (n+1)X - (n+1)X^2 \pmod{X^3},$$

where $n \geq 2$ and $N = \dim \nu(\mathbf{C}P^n)$.

Now, we suppose that $\nu(\mathbf{C}P^n)$ is stably equivalent to a Whitney sum $\xi^{k_1} \oplus \cdots \oplus \xi^{k_j}$ of line bundles, and induce a contradiction. Under the hypothesis, we have

$$[\nu(\mathbf{C}P^n) - \underline{\mathbf{C}}^N] = \sum_{i=1}^{j} (1+X)^{k_i} - j \equiv \sum_{i=1}^{j} k_i X + \sum_{i=1}^{j} \binom{k_i}{2} X^2 \pmod{X^3}.$$

Thus, comparing the coefficients of X and X^2 in the above two congruences,

$$\sum_{i=1}^{j} k_i = n+1 \qquad \text{and} \qquad \sum_{i=1}^{j} \binom{k_i}{2} = -(n+1).$$

But, these two equalities are not compatible since $\sum_{i=1}^{j} k_i^2 \neq -(n+1)$, and thus we have completed the proof. $\square$

Lastly, we prove Theorem 1.6.

Proof of Theorem 1.6. Since the n-dimensional vector bundles $\nu(\mathbf{C}P^n)$ and $-(n+1)\xi^{-1}$ over $\mathbf{C}P^n$ are stably equivalent each other as is mentioned in the above, they are actually isomorphic by stability property.

The line bundle $\nu(\mathbf{C}P^1)$ is isomorphic to ξ^2 over $\mathbf{C}P^1$, because they have the same Chern classes. Thus, $\nu(\mathbf{C}P^1)$ is extendible to $\mathbf{C}P^k$ for any $k \geq 1$. As for the 2-dimensional vector bundle $\nu(\mathbf{C}P^2) = -3\xi^{-1}$, since the congruence in Theorem 1.2(1) is satisfied and the second congruence in Theorem 1.2(2) is not in the case of $n = 2$, $m = -1$ and $l = 3$, we have the required result.

Thus, we assume $n \geq 3$, and show that the n-dimensional vector bundle $-(n+1)\xi^{-1}$ is not extendible to $\mathbf{C}P^{n+1}$. By Theorem 1.2(1), it is sufficient to show

$$\binom{2n+1}{n+1} \not\equiv 0 \pmod{n!}.$$

But, the incongruence follows if we prove the inequality

$$\nu_2\binom{2n+1}{n+1} < \nu_2(n!). \tag{4.1}$$

As for the right hand side of (4.1), we have $\nu_2(n!) = n - \alpha_2(n)$ by Lemma 3.1. Since

$$\binom{2n+1}{n+1} = \frac{(2n+1)!}{(n+1)!n!} = \frac{2^n n!(2h+1)}{(n+1)!n!} = \frac{2^n(2h+1)}{(n+1)!}$$

for some integer $h > 0$, we have

$$\begin{aligned}\nu_2\binom{2n+1}{n+1} &= \nu_2(2^n) - \nu_2((n+1)!) \\ &= n - ((n+1) - \alpha_2(n+1)) = \alpha_2(n+1) - 1.\end{aligned}$$

Then, the following inequality is easily shown by the induction on $n \geq 3$:

$$\nu_2(n!) - \nu_2\binom{2n+1}{n+1} = n + 1 - \alpha_2(n) - \alpha_2(n+1) > 0.$$

Hence (4.1) holds, and thus we have completed the proof. $\square$

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