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Proper Lusternik-Schnirelmann π_1-categories

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ABSTRACT. We define new proper homotopy invariants, the proper Lusternik-Schnirelmann π_1-categories p$\tilde{\pi}_1$-cat and p$\tilde{\pi}_1^\infty$-cat. Then, we prove that, if p$\tilde{\pi}_1$-cat (resp. p$\tilde{\pi}_1^\infty$-cat) of a locally path-connected, Hausdorff, locally compact, and paracompact space is equal to or less than n, then there is a proper map to a locally finite polyhedron of dimension $n+1$ that induces an isomorphism of fundamental pro-groups p$\tilde{\pi}_1$ (resp. p$\tilde{\pi}_1^\infty$).

1. Introduction

The L-S category was defined in 1934 in [12] by L. Lusternik and L. Schnirelmann in the course of their studies on calculus of variations, because it gives a lower bound of the number of critical points of a smooth real function on a closed manifold. The L-S category cat X of a space X is the least number of open subsets contractible in X needed to cover X minus one. It is a homotopy invariant, and was early studied by Borsuk [2] and Fox [7]. Also, there is an algebraic counterpart of the L-S category cat_{π_1} defined by using fundamental groups, due to Fox [7]. The L-S π_1-category $\mathrm{cat}_{\pi_1} X$ of X is the least number of open subsets π_1-contractible in X needed to cover X minus one, where a subset of X is π_1-contractible in X if every loop in the subset is contractible to a point in X. It has been studied for example in [6], [8] and [10].

Homotopy invariants, as cat and cat_{π_1}, do not suffice to study open manifolds, and proper homotopy invariants are needed to investigate the behaviour of these spaces at infinity. Ayala, Domínguez, Márquez, and Quintero [1] have defined a proper version of the L-S category. They have introduced two proper invariants, p-cat and p-cat$^\infty$, using subsets that are properly contractible to the image of the half-line $\mathbf{R}_+$.

In this paper we introduce two new proper homotopy invariants, p$\tilde{\pi}_1$-cat and p$\tilde{\pi}_1^\infty$-cat, corresponding to cat_{π_1}. Concretely, p$\tilde{\pi}_1$-cat coincides with cat_{π_1} for compact spaces. In § 2 we define two pro-groups, the fundamental pro-

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group $p\tilde{\pi}_1$ and the fundamental pro-group at infinity $p\tilde{\pi}_1^\infty$. In §3 we define proper L-S π_1-categories corresponding to these two pro-groups, and prove in §4 the proper version (Theorem 4.1) of a result due to Eilenberg-Ganea and Gómez-González ([6], [8], and [10]): if a locally path-connected, Hausdorff, locally compact, and paracompact space has proper L-S π_1-category $p\tilde{\pi}_1$-cat (resp. at infinity $p\tilde{\pi}_1^\infty$-cat) $\leq n$, then there is a proper map into a locally finite polyhedron (i.e. the underlying space of a locally finite simplicial complex) of dimension $n+1$ that induces an isomorphism of fundamental pro-groups (resp. at infinity). Although one might expect to have a locally finite polyhedron of dimension n as in the non-proper case, we will give an example of 4-dimensional manifold whose proper π_1-categories are 3 but there is no 3-dimensional locally finite polyhedron verifying the expected property (Example 4.3).

2. Proper maps and pro-groups

Recall that a map between Hausdorff locally compact topological spaces is proper if it is continuous and the inverse image of every compact set is a compact set. We will denote by $\mathscr{P}$ the category of the Hausdorff locally compact topological spaces and proper maps. Basic facts on proper maps can be found in [3].

We will define the fundamental pro-groups of a space using inverse systems of fundamental groups of subspaces. The base of our fundamental groups will not be a point but any set. Now we will give the precise definitions and facts. Known or straightforward facts will be given as lemmas, remarks, or propositions without proof.

Definition 2.1. Let X be a topological space and M its subset. We define the *fundamental group of X with base points in M* by the family of groups $\tilde{\pi}_1(X, M) = \{\pi_1(X, p) \mid p \in M\}$. Note that we think $\tilde{\pi}_1(X, \varnothing) = \varnothing$ for the empty set $\varnothing$.

Definition 2.2. We define $\mathscr{G}$ by the category with objects of the form $(I, \{G^i \mid i \in I\})$, where I is a set and G^i is a group for each $i \in I$; and morphisms

$$f : (I, \{G^i \mid i \in I\}) \to (J, \{H^j \mid j \in J\})$$

that are pairs $f = (\varphi, \{f^i \mid i \in I\})$, where φ is a map from I to J and $f^i : G^i \to H^{\varphi(i)}$ is a homomorphism of groups for each i. We will call φ the index map and f^i the component maps of f, respectively. Composition is defined component-wise,

$$(\varphi, \{f^i \mid i \in I\}) \circ (\psi, \{g^j \mid j \in J\}) = (\varphi \circ \psi, \{f^{\psi(j)} \circ g^j \mid j \in J\})$$

In the sequel we will denote the families of groups simply as $\{G^i \mid i \in I\}$.

Proposition 2.3. *A continuous map of pairs $f : (X, M) \to (Y, N)$ induces a morphism of families of groups $\tilde{\pi}_1(f) : \tilde{\pi}_1(X, M) \to \tilde{\pi}_1(Y, N)$ given by the index map $f|_M : M \to N$ and the component maps $\pi_1(f) : \pi_1(X, p) \to \pi_1(Y, f(p))$ for any p of M. Indeed, $\tilde{\pi}_1$ is a functor from the category of topological pairs to $\mathscr{G}$.*

Definition 2.4. Let $G = \{G^i \mid i \in I\}$ and $H = \{H^j \mid j \in J\}$ be two families of groups. We say that H is a *subfamily* of G if $J \subset I$ and H^j is a subgroup of G^j for every j of J. If M is a family of subgroups of a group P then $\vee M$ will denote the subgroup of P generated by the union of the subgroups of M. Let $f : G \to H$ be a morphism in $\mathscr{G}$ with $f = (\varphi, \{f^i \mid i \in I\})$. Also, let $A = \{A^i \mid i \in I'\}$ be a subfamily of G and $B = \{B^j \mid j \in J'\}$ a subfamily of H. We define:

$$f(A) = \{\vee\{f^i(A^i) \mid \varphi(i) = j, i \in I'\} \mid j \in \varphi(I')\},$$
$$f^{-1}(B) = \{(f^i)^{-1}(B^{\varphi(i)}) \mid i \in \varphi^{-1}(J')\},$$
$$\operatorname{Ker} f = f^{-1}(\{\Theta \mid j \in J\}) = \{\operatorname{Ker} f^i \mid i \in I\}, \qquad \text{and}$$
$$\operatorname{Im} f = f(G) = \{\vee\{\operatorname{Im} f^i \mid \varphi(i) = j\} \mid j \in \operatorname{Im} \varphi\},$$

where Θ denotes the trivial group.

Definition 2.5. We say that a non-empty family of groups $\{G^i \mid i \in I\}$ is *trivial* if G^i is the trivial group for every i. Also, we say that a morphism of families of groups is *empty* if its domain is the empty family of groups, and we say that it is *null* if its component maps are the null morphisms of groups. Finally, it is *trivial* if it is null or empty.

Lemma 2.6. *A non-empty morphism $f = (\varphi, \{f^i \mid i \in I\})$ in $\mathscr{G}$ is an isomorphism iff φ and f^i are bijective for every i.*

Definition 2.7. A map of pairs $f : (X, M) \to (Y, N)$ is a *homotopy equivalence relative to M* if there is a map $g : (Y, N) \to (X, M)$ such that $g \circ f$ and $f \circ g$ are homotopic to the corresponding identity maps relative to M and N, respectively.

Proposition 2.8. *Let X and Y be two topological spaces, and f and g two homotopic maps from X to Y such that f is injective on a subset M of X. Then, if $\tilde{\pi}_1(f)$ and $\tilde{\pi}_1(g)$ denote the induced morphisms from $\tilde{\pi}_1(X, M)$ to $\tilde{\pi}_1(Y, f(M))$ and $\tilde{\pi}_1(Y, g(M))$ respectively, there is a morphism of families of* *groups* $h : \tilde{\pi}_1(Y, f(M)) \to \tilde{\pi}_1(Y, g(M))$ *such that* $\tilde{\pi}_1(g) = h \circ \tilde{\pi}_1(f)$ *and the component maps of* h *are isomorphisms.*

Corollary 2.9. *The family of groups* $\tilde{\pi}_1(X, M)$ *is determined by the homotopy type of* X *relative to* M.

Now we introduce the inverse systems to define the pro-groups. First we recall the definition of pro-categories (see [5]).

Definition 2.10. Let $\mathscr{C}$ be a category. An *inverse system* in $\mathscr{C}$ is a triple formed by a directed set Λ, a family $\{X_\lambda \mid \lambda \in \Lambda\}$ of objects of $\mathscr{C}$, and a family $\{p_{\lambda\mu} : X_\mu \to X_\lambda \mid \lambda \le \mu\}$ of morphisms in $\mathscr{C}$ that satisfy the following conditions:
(i) $p_{\lambda\lambda} = id_{X_\lambda}$ for every λ of Λ.
(ii) If $\lambda \le \mu \le \nu$ then $p_{\lambda\mu} \circ p_{\mu\nu} = p_{\lambda\nu}$.
The inverse system will be denoted by $(\Lambda, \{X_\lambda\}, \{p_{\lambda\mu}\})$. Λ is called the *index set* and $p_{\lambda\mu}$ the *bonding maps*.

Definition 2.11. Let $\underline{X} = (\Lambda, \{X_\lambda\}, \{p_{\lambda\mu}\})$ and $\underline{Y} = (\Gamma, \{Y_\gamma\}, \{q_{\gamma\delta}\})$ be two inverse systems in $\mathscr{C}$. A *system map* from $\underline{X}$ to $\underline{Y}$ is a pair formed by a map $\theta : \Gamma \to \Lambda$ and a family of morphisms in $\mathscr{C}$ $\{f_\gamma : X_{\theta(\gamma)} \to Y_\gamma \mid \gamma \in \Gamma\}$ satisfying that for each $\gamma \le \delta$ of Γ there is a $\lambda \in \Lambda$ such that $\theta(\gamma) \le \lambda$, $\theta(\delta) \le \lambda$, and $f_\gamma \circ p_{\theta(\gamma)\lambda} = q_{\gamma\delta} \circ f_\delta \circ p_{\theta(\delta)\lambda}$. Let $\underline{Z} = (A, \{Z_\alpha\}, \{r_{\alpha\beta}\})$ be another inverse system in $\mathscr{C}$. Given two system maps $f = (\theta, f_\gamma)$ from $\underline{X}$ to $\underline{Y}$ and $g = (\varphi, g_\alpha)$ from $\underline{Y}$ to $\underline{Z}$, their *composition* $g \circ f$ is $(\theta \circ \varphi, \{g_\alpha \circ f_{\varphi(\alpha)} \mid \alpha \in A\})$. The identity map in $\underline{X}$ is $(id_\Lambda, \{id_{X_\lambda}\})$. We say that two system maps $f = (\theta, f_\gamma)$ and $f' = (\theta', f'_\gamma)$ from $\underline{X}$ to $\underline{Y}$ are *equivalent* if for each $\gamma \in \Gamma$ there is a $\lambda \in \Lambda$ such that $\theta(\gamma) \le \lambda$, $\theta'(\gamma) \le \lambda$, and $f_\gamma \circ p_{\theta(\gamma)\lambda} = f'_\gamma \circ p_{\theta'(\gamma)\lambda}$. The above defined relation between system maps is an equivalence relation. The pro-category of $\mathscr{C}$, whose objects are inverse systems in $\mathscr{C}$ and whose morphisms are equivalence classes of system maps, can be defined in the obvious way and denoted by pro-$\mathscr{C}$.

Also in [5] the category $(\mathscr{C}, \text{pro-}\mathscr{C})$ with objects (A, P, f), where A is an object of $\mathscr{C}$, P is an object of pro-$\mathscr{C}$, and f is a morphism in pro-$\mathscr{C}$ from P to A (regarded as a constant inverse system) is defined.

Definition 2.12. When $\underline{M} = (\mathbf{N}, \{M_m\}, \{p_{mn}\})$ is an inverse system of modules we can consider the product $\prod_{n=1}^\infty M_n$ with the module structure given by the component-wise sum and product, and the *shift* homomorphism $s : \prod_{n=1}^\infty M_n \to \prod_{n=1}^\infty M_n$, defined by $s(x_1, x_2, \dots) = (x_1 - p_{12}(x_2), x_2 - p_{23}(x_3), \dots)$. The kernel of s is the *inverse limit* $\varprojlim \underline{M}$ of $\underline{M}$ and the cokernel $(\prod_{n=1}^\infty M_n)/\operatorname{Im} s$ is the *first derived limit* $\varprojlim^1 \underline{M}$ of $\underline{M}$.

Definition 2.13. Let X be a topological space. An *infinity neighbourhood* of X is any subspace of X whose complement is compact. A *system of* *infinity neighbourhoods* of X is any non-empty family $\mathcal{U}$ of infinity neighbourhoods of X such that $\bigcap \mathcal{U} = \varnothing$ and for any two neighbourhoods U and V of $\mathcal{U}$ there is a neighbourhood $W \in \mathcal{U}$ whose closure is contained in $U \cap V$.

Note that for any infinity neighbourhood V of X there is a $U \in \mathcal{U}$ contained in V by the condition of a system of infinity neighbourhoods $\mathcal{U}$. Also, using the local compactness condition it is straightforward that every space of $\mathcal{P}$ has a system of infinity neighbourhoods.

Definition 2.14. Let X be a space of $\mathcal{P}$, M a subset of X, and $\mathcal{U}$ a system of infinity neighbourhoods of X. We define the *fundamental pro-group at infinity* of $(X, M, \mathcal{U})$, denoted by $\mathrm{p}\tilde{\pi}_1^\infty(X, M, \mathcal{U})$, by the inverse system in $\mathcal{G}$ with elements $\tilde{\pi}_1(U, U \cap M)$ for $U \in \mathcal{U}$ bonded by the maps induced by the inclusions. Also, we define the *fundamental pro-group* of $(X, M, \mathcal{U})$ as the pair $(\tilde{\pi}_1(X, M), \mathrm{p}\tilde{\pi}_1^\infty(X, M, \mathcal{U}))$ of $(\mathcal{G}, \text{pro-}\mathcal{G})$, and we denote it by $\mathrm{p}\tilde{\pi}_1(X, M, \mathcal{U})$.

Proposition 2.15. *The above constructions are functorial, where the domain of* $\mathrm{p}\tilde{\pi}_1^\infty$ *and* $\mathrm{p}\tilde{\pi}_1$ *is the category with objects* $(X, M, \mathcal{U})$ *as above and a morphism from* $(X, M, \mathcal{U})$ *to* $(Y, N, \mathcal{V})$ *is simply a proper map of pairs from* (X, M) *to* (Y, N).

Since different choices of systems of infinity neighbourhoods induce naturally equivalent functors, when we need not distinguish between isomorphic objects we will write $\mathrm{p}\tilde{\pi}_1^\infty(X, M)$ and $\mathrm{p}\tilde{\pi}_1(X, M)$. Also, we will denote by $\mathrm{p}\tilde{\pi}_1^\infty(f, M)$ the morphism induced by a proper map $f : (X, M) \to (Y, N)$, or $\mathrm{p}\tilde{\pi}_1^\infty(f)$ when M is clear by the context, and analogously for $\mathrm{p}\tilde{\pi}_1$. Now we will prove an algebraic result needed in the proof of Theorem 4.1.

Definition 2.16. A system map $f : \underline{X} \to \underline{Y}$ is called a *level-preserving map* if $\underline{X}$ and $\underline{Y}$ have the same index set Λ, f is of the form $(id_\Lambda, \{f_\lambda : X_\lambda \to Y_\lambda\})$ and satisfies $f_\lambda \circ p_{\lambda\mu} = q_{\lambda\mu} \circ f_\mu$ for any indices $\lambda \leq \mu$, where $p_{\lambda\mu}$ (resp. $q_{\lambda\mu}$) is the bonding map from X_μ to X_λ (resp. from Y_μ to Y_λ).

Lemma 2.17. *Let* $f : \underline{G} \to \underline{H}$ *be a level-preserving map in* pro-$\mathcal{G}$*, where* $\underline{G} = (\Lambda, \{G_\lambda\}, \{p_{\lambda\mu}\})$ *and* $\underline{H} = (\Lambda, \{H_\lambda\}, \{q_{\lambda\mu}\})$. *Suppose that the index maps of the morphisms of families of groups* $f_\lambda : G_\lambda \to H_\lambda$ *are injective for any* λ. *Then,* f *is an isomorphism iff for any* λ
(i) *there is a* $\mu \geq \lambda$ *such that* $\mathrm{Ker}(f_\mu) \subset \mathrm{Ker}(p_{\lambda\mu})$ *and*
(ii) *there is a* $\nu \geq \lambda$ *such that* $\mathrm{Im}(q_{\lambda\nu}) \subset \mathrm{Im}(f_\lambda)$.

Proof. It is clear that the conditions are necessary. Let us show that they are sufficient. In [14] it is proved that a level-preserving system map is an isomorphism iff for any λ there is a $\nu \geq \lambda$ and a morphism $g_{\lambda\nu} : H_\lambda \to G_\lambda$ such that $g_{\lambda\nu} \circ f_\nu = p_{\lambda\nu}$ and $f_\lambda \circ g_{\lambda\nu} = q_{\lambda\nu}$. Now, let $\lambda \in \Lambda$. Then, there is an index

$\mu \geq \lambda$ such that $\mathrm{Ker}(f_\mu) \subset \mathrm{Ker}(p_{\lambda\mu})$ by (i) and an index $\nu \geq \mu$ such that $\mathrm{Im}(q_{\mu\nu}) \subset \mathrm{Im}(f_\mu)$ by (ii). Let us define $g_{\lambda\nu} : H_\lambda \to G_\lambda$.

Let x be an element of a group H_ν^j of H_ν. Since $\mathrm{Im}(q_{\mu\nu}) \subset \mathrm{Im}(f_\mu)$ and the index map of $f_\mu : G_\mu \to H_\mu$ is injective, there is an unique group G_μ^i of G_μ and there is a $y \in G_\mu^i$ (not unique, in general) satisfying that $q_{\mu\nu}(x) = f_\mu(y)$. We define $g_{\lambda\nu}(x)$ by $p_{\lambda\mu}(y)$. Let us show that the result does not depend on the choice of y. Let y and y' be two elements of G_μ^i such that $f_\mu(y) = f_\mu(y') = q_{\mu\nu}(x)$. Since $f_\mu(y^{-1}y')$ is null, we see that $p_{\lambda\mu}(y^{-1}y')$ is null by (i), and hence $p_{\lambda\mu}(y) = p_{\lambda\mu}(y')$. It is easy to show that this defines a morphism of families of groups that satisfies $g_{\lambda\nu} \circ f_\nu = p_{\lambda\nu}$ and $f_\lambda \circ g_{\lambda\nu} = q_{\lambda\nu}$. □

Definition 2.18. Let $f, g : X \to Y$ be two proper maps. We will say that f and g are *properly homotopic*, $f \simeq_p g$, if there is a homotopy from f to g that is a proper map. On the other hand, we will write $f \simeq g$ if f and g are homotopic in the usual sense.

Proposition 2.19. *Let X and Y be two non-compact spaces of $\mathscr{P}$, and f and g two properly homotopic proper maps from X to Y such that f is injective on a subset M of X that is not contained in any compact subset of X. The maps $\mathrm{p}\tilde{\pi}_1^\infty(f) : \mathrm{p}\tilde{\pi}_1^\infty(X, M) \to \mathrm{p}\tilde{\pi}_1^\infty(Y, f(M))$ and $\mathrm{p}\tilde{\pi}_1^\infty(g) : \mathrm{p}\tilde{\pi}_1^\infty(X, M) \to \mathrm{p}\tilde{\pi}_1^\infty(Y, g(M))$ are defined. Then for any system of infinity neighbourhoods $\mathscr{V}$ of Y there is a morphism h from $\mathrm{p}\tilde{\pi}_1^\infty(Y, f(M), \mathscr{V})$ to $\mathrm{p}\tilde{\pi}_1^\infty(Y, g(M), \mathscr{V})$ such that:*

(i) $\mathrm{p}\tilde{\pi}_1^\infty(g) = h \circ \mathrm{p}\tilde{\pi}_1^\infty(f)$ *as morphisms of pro-groups.*

(ii) *For any $V \in \mathscr{V}$ the corresponding map h_V sends $\tilde{\pi}_1(V, V \cap f(M))$ to $\tilde{\pi}_1(V, V \cap g(M))$, and there is a $W \in \mathscr{V}$ contained in V such that the component map of h_V from $\pi_1(V, f(m))$ to $\pi_1(V, g(m))$ is an isomorphism for any $m \in M \cap f^{-1}(W)$.*

There is a similar result for $\mathrm{p}\tilde{\pi}_1$ (in this case X and Y may be compact).

Proof of Proposition 2.19. Let F be a proper homotopy from f to g. We will define W and h. Then, the rest of the proof is straightforward.

Let $\mathscr{U}$ be a system of infinity neighbourhoods of X and V an element of $\mathscr{V}$. We choose an infinity neighbourhood W of Y contained in V as follows: Take a $U \in \mathscr{U}$ such that $F(U \times [0,1]) \subset V$. Since $F((X - U) \times [0,1])$ is compact there is a W of $\mathscr{V}$ such that $W \subset V \cap (Y - F((X - U) \times [0,1]))$.

Now we define the component map of h_V at $f(m)$. First, when $f(m) \in W$, we see that $m \in U$ and thus $F(m,t) \in V$ for every $t \in [0,1]$. So, we can define $h_V(\beta) = \alpha_m^{-1} \cdot \beta \cdot \alpha_m \in \pi_1(V, g(m))$ for any loop β of $\pi_1(V, f(m))$ by using a path $\alpha_m(t) = F(m,t)$ for every t of $[0,1]$. When $f(m) \notin W$ we can choose a point m' of $M \cap g^{-1}(V)$, because M is not contained in any compact subset of X, and we define $h_V(\beta) = 0 \in \pi_1(V, g(m'))$ for any loop β of $\pi_1(V, f(m))$. $\square$

3. The proper L-S π_1-categories

Remember that the Lusternik-Schnirelmann π_1-category of a space X is the least number of open subsets π_1-contractible in X needed to cover X minus one. The condition that a subset A of X is π_1-contractible in X can be reformulated as stating that the map from $\pi_1(A, a)$ to $\pi_1(X, a)$ induced by the inclusion is trivial for any point a of A. Analogously, we will use the fundamental pro-group $\mathrm{p}\tilde{\pi}_1$ and the fundamental pro-group at infinity $\mathrm{p}\tilde{\pi}_1^\infty$ to define two new proper homotopy invariants. Since the inclusion maps should be proper, a subset is called $\mathrm{p}\tilde{\pi}_1$-categorical (resp. $\mathrm{p}\tilde{\pi}_1^\infty$-categorical) if the inclusion of its closure in X induces a trivial morphism of pro-groups.

Definition 3.1. A morphism in pro-$\mathcal{G}$ is *null* if it has a representative consisting of null morphisms in $\mathcal{G}$ (remember that a morphism in pro-$\mathcal{G}$ is an equivalence class of system maps), and it is *trivial* if it is null or it has a representative consisting of empty morphisms in $\mathcal{G}$. A morphism in $(\mathcal{G}, \text{pro-}\mathcal{G})$ is *trivial* if its two component maps are trivial.

Lemma 3.2. *Let $\underline{G} = (\Lambda, \{G_\lambda\}, \{p_{\lambda\mu}\})$ and $\underline{H} = (A, \{H_\alpha\}, \{q_{\alpha\beta}\})$ be two inverse systems and $f : \underline{G} \to \underline{H}$ a system map in* pro-$\mathcal{G}$. *Then, f is null iff for any α there is a $\lambda \geq \varphi(\alpha)$ such that $f_\alpha \circ p_{\varphi(\alpha)\lambda}$ is null, where $\varphi : A \to \Lambda$ is the map between the index sets. Also, f is trivial iff it is null or there is a λ such that G_λ is empty.*

Note that if X is a space of $\mathcal{P}$ and A is a subset of X, then $A \in \mathcal{P}$ and the inclusion map from A to X is proper iff A is closed in X.

Definition 3.3. Let X be a space of $\mathcal{P}$ and A a subset of X. For every subset M of the closure $\bar{A}$ of A the inclusion map from $\bar{A}$ to X induces morphisms:

$$i_{M,\mathcal{U}} : \mathrm{p}\tilde{\pi}_1^\infty(\bar{A}, M, \mathcal{U}_{\bar{A}}) \to \mathrm{p}\tilde{\pi}_1^\infty(X, M, \mathcal{U})$$

$$j_{M,\mathcal{U}} : \mathrm{p}\tilde{\pi}_1(\bar{A}, M, \mathcal{U}_{\bar{A}}) \to \mathrm{p}\tilde{\pi}_1(X, M, \mathcal{U})$$

for every system of infinity neighbourhoods $\mathcal{U}$ of X, where $\mathcal{U}_{\bar{A}} = \{U \cap \bar{A} \mid U \in \mathcal{U}\}$. We say that A is $\mathrm{p}\tilde{\pi}_1^\infty$-*categorical* (resp. $\mathrm{p}\tilde{\pi}_1$-*categorical*) in X if $i_{M,\mathcal{U}}$ (resp. $j_{M,\mathcal{U}}$) is trivial for some $\mathcal{U}$ and for every $M \subset \bar{A}$.

This definition does not depend on the choice of $\mathcal{U}$, by the remark below Proposition 2.15. Also, for $\mathrm{p}\tilde{\pi}_1^\infty$ we can restrict ourselves to base sets M that are not contained in any compact subset of X, because otherwise $i_{M,\mathcal{U}}$ would be trivial. Finally, let us see that for $\mathrm{p}\tilde{\pi}_1^\infty$ and $\mathrm{p}\tilde{\pi}_1$ we need only one M, if it is chosen appropriately, as explained in the following:

Definition 3.4. Let X be a space of $\mathscr{P}$ and M a subset of X. We say that M *covers the infinity* of X if there is a system of infinity neighbourhoods $\mathcal{U}$ of X such that M intersects every path component of every infinity neighbourhood of $\mathcal{U}$. Also, M is *full* in X if it covers the infinity of X and intersects every path component of X.

It is straightforward that

Proposition 3.5. *Let X be a space of $\mathscr{P}$, A a subset of X and M a subset of $\bar{A}$. If M covers the infinity of $\bar{A}$ and the morphism $i_M : \mathrm{p}\tilde{\pi}_1^\infty(\bar{A}, M) \to \mathrm{p}\tilde{\pi}_1^\infty(X, M)$ is trivial, then A is $\mathrm{p}\tilde{\pi}_1^\infty$-categorical in X. Also, if M is full in $\bar{A}$ and the morphism $j_M : \mathrm{p}\tilde{\pi}_1(\bar{A}, M) \to \mathrm{p}\tilde{\pi}_1(X, M)$ is trivial, then A is $\mathrm{p}\tilde{\pi}_1$-categorical in X.*

Lemma 3.6. *Let X be a space of $\mathscr{P}$. Then, any subset of a $\mathrm{p}\tilde{\pi}_1$-categorical (resp. $\mathrm{p}\tilde{\pi}_1^\infty$-categorical) subset in X is also $\mathrm{p}\tilde{\pi}_1$-categorical (resp. $\mathrm{p}\tilde{\pi}_1^\infty$-categorical) in X.*

Definition 3.7. We define the *proper Lusternik-Schnirelmann π_1-category (at infinity)* of a space X of $\mathscr{P}$ by the least number of open subsets $\mathrm{p}\tilde{\pi}_1$-categorical (resp. $\mathrm{p}\tilde{\pi}_1^\infty$-categorical) in X needed to cover X minus one, and we will denote it by $\mathrm{p}\tilde{\pi}_1\text{-cat}\,X$ (resp. $\mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\,X$). If there is not such a finite cover, we define that $\mathrm{p}\tilde{\pi}_1\text{-cat}\,X = \infty$ (resp. $\mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\,X = \infty$). Also, when X is compact, X itself is a $\mathrm{p}\tilde{\pi}_1^\infty$-categorical subset but it would be better to define $\mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\,X = -1$ exceptionally.

Remark 3.8. The $\mathrm{p}\tilde{\pi}_1^\infty$-category of any non-compact space of $\mathscr{P}$ is equal to the $\mathrm{p}\tilde{\pi}_1^\infty$-category of the closure of any of its infinity neighbourhoods.

Proposition 3.9. $\mathrm{p}\tilde{\pi}_1$-cat *and* $\mathrm{p}\tilde{\pi}_1^\infty$-cat *are proper homotopy invariants.*

Proof. We will prove that $\mathrm{p}\tilde{\pi}_1^\infty$-cat is a proper homotopy invariant, the proof for $\mathrm{p}\tilde{\pi}_1$-cat is analogous. Let X and Y be two spaces of the same proper homotopy type. There are $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq_p \mathrm{id}_Y$ and $g \circ f \simeq_p \mathrm{id}_X$. Note that X is compact iff so is Y, and thus we may assume that X and Y are non-compact.

If $\mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\,Y \le n$, then Y can be covered by $n+1$ open subsets $A_0, \dots, A_n$ that are $\mathrm{p}\tilde{\pi}_1^\infty$-categorical in Y. For every integer k between 0 and n there is a commutative diagram:

$$\begin{tikzcd}
f^{-1}(\overline{A_k}) \arrow[r, "f_k"] \arrow[d, hook, "i"] & \overline{A_k} \arrow[d, hook, "j"] & \\
X \arrow[r, "f"] \arrow[rr, bend right=60, "\mathrm{id}_X"'] & Y \arrow[r, "g"] \arrow[d, phantom, "\simeq_p"] & X \\
 & {} &
\end{tikzcd}$$

where $\overline{A_k}$ denotes the closure of A_k, i and j are inclusion maps, f_k denotes $f|_{f^{-1}(\overline{A_k})}$ and the symbol $\simeq_p$ means that the corresponding part of the diagram is commutative up to proper homotopy.

Taking the fundamental pro-groups at infinity we obtain:

$$\begin{tikzcd}
\mathrm{p}\tilde{\pi}_1^\infty(f^{-1}(\overline{A_k}), M) \arrow[r, "(f_k)_*"] \arrow[d, "i_*"] & \mathrm{p}\tilde{\pi}_1^\infty(\overline{A_k}, f(M)) \arrow[d, "j_*"] & \\
\mathrm{p}\tilde{\pi}_1^\infty(X, M) \arrow[r, "f_*"] \arrow[drr, "\mathrm{id}"'] & \mathrm{p}\tilde{\pi}_1^\infty(Y, f(M)) \arrow[r, "g_*"] & \mathrm{p}\tilde{\pi}_1^\infty(X, g(f(M))) \\
 & & \mathrm{p}\tilde{\pi}_1^\infty(X, M) \arrow[u, "h"']
\end{tikzcd}$$

where $\psi_* = \mathrm{p}\tilde{\pi}_1^\infty(\psi)$ for any map ψ, M is a subset of $f^{-1}(\overline{A_k})$ not contained in any compact subset and h is the map defined in Proposition 2.19 for a system of infinity neighbourhoods $\mathscr{V}$ of X. Since j_* is null, $j_* \circ (f_k)_*$ is also null. Then, $g_* \circ f_* \circ i_*$ must be null and so is $h \circ i_*$. Since $h \circ i_*$ is null, for any $U \in \mathscr{V}$ there is a $V \subset U$ such that $h_U \circ (i_*)_U \circ p_{UV}$ is null by Lemma 3.2, where p_{UV} is the bonding map of $\mathrm{p}\tilde{\pi}_1^\infty(f^{-1}(\overline{A_k}), M)$ determined by $U \cap f^{-1}(\overline{A_k})$ and $V \cap f^{-1}(\overline{A_k})$. Moreover, by Proposition 2.19 there is an infinity neighbourhood W of $\mathscr{V}$ contained in V such that the component maps of h_V corresponding to base points in $M \cap W$ are isomorphisms. And since $M \cap W$ is not empty, $(i_*)_U \circ p_{UW}$ is null, which means that i_* is null. Thus, $f^{-1}(A_k)$ is $\mathrm{p}\tilde{\pi}_1^\infty$-categorical in X and we have proved that Π-cat $X \leq n$. □

Proposition 3.10. *If $(K, |K|)$ is a finite-dimensional locally finite simplicial complex, then* $\mathrm{p}\tilde{\pi}_1\text{-cat}|K| \leq \dim K$.

Proof. Suppose that the dimension of K is n. We will construct an open cover $\{A, \dots, A_n\}$ of $|K|$ by $n+1$ subsets that are $\mathrm{p}\tilde{\pi}_1$-categorical in $|K|$. Let i be an integer between 0 and n. First, let B_i be the 0-dimensional subcomplex formed by the barycenters of the i-simplexes of K. Next, let C_i be the regular neighbourhood of B_i in the second barycentric subdivision $\mathrm{sd}^2(K)$ of K.

Finally, let A_i be the interior of the regular neighbourhood of C_i in the third barycentric subdivision $\mathrm{sd}^3(K)$ of K. Since the closure of A_i is a disjoint union of sets that are π_1-contractible in $|K|$, A_i is $\mathrm{p}\tilde{\pi}_1$-categorical in $|K|$. Since $\{A_0, \dots, A_n\}$ is an open cover of $|K|$, the result holds. $\square$

It is straightforward that $\mathrm{cat}_{\pi_1} X \leq \mathrm{p}\tilde{\pi}_1\text{-cat}\, X \geq \mathrm{p}\tilde{\pi}_1^{\infty}\text{-cat}\, X$ for any X of $\mathscr{P}$. An example for which $\mathrm{p}\tilde{\pi}_1$-cat is greater than $\mathrm{p}\tilde{\pi}_1^{\infty}$-cat is the wedge of the circumference and the half-line $\mathbf{R}_+ = [0, \infty)$. And $\mathrm{p}\tilde{\pi}_1$-cat is greater than cat_{π_1} for the plane. Let us recall a little about the proper categories defined by Quintero and others in [1] and [4].

DEFINITION 3.11. Let X be a non-compact space of $\mathscr{P}$. A closed subset A of X is called *properly inessential* in X if the inclusion map $i : A \to X$ factorizes up to proper homotopy through the half-line. That is, there are proper maps $f : A \to \mathbf{R}_+$ and $g : \mathbf{R}_+ \to X$ such that $g \circ f \simeq_p i$. Note that properly inessential subsets are closed. A subset of X is called *properly categorical* in X if it is contained in a closed subset of X which is properly inessential in X. Any proper map from the half-line to a space is called a *ray*. A *properly based space* is a pair (X, α) where α is a ray in X.

DEFINITION 3.12. Let X be a non-compact space of $\mathscr{P}$. The *proper L-S category* $\text{p-cat}\, X$ of X is the least number of open subsets properly categorical in X needed to cover X minus one. The *proper L-S category at infinity* $\text{p-cat}^{\infty} X$ of X is the least number of open subsets properly categorical in X whose union is an infinity neighbourhood of X minus one.

Note that the definitions in [1] and [4] are the number of elements of the covers defined above, so they are equal to the categories defined above plus one.

It is easy to prove that $\mathrm{p}\tilde{\pi}_1\text{-cat}\, X \leq \text{p-cat}\, X$ and $\mathrm{p}\tilde{\pi}_1^{\infty}\text{-cat}\, X \leq \text{p-cat}^{\infty} X$ for any non-compact space X of $\mathscr{P}$ (for the last inequality, see Remark 3.8). A space for which the inequalities are not equalities is obtained by pasting to the half-line a copy of the 2-sphere at each natural number.

4. Main theorem

Now we will prove the main theorem of this paper, a proper version of a result due to Eilenberg-Ganea and Gómez-González (see [6], [8], and [10]).

THEOREM 4.1. *Let X be a Hausdorff, locally compact, paracompact and locally pathwise-connected space and Π one of the functors $\mathrm{p}\tilde{\pi}_1^{\infty}$ or $\mathrm{p}\tilde{\pi}_1$. If $\Pi\text{-cat}\, X \leq n$ then there is a locally finite simplicial complex $(L, |L|)$ of dimension $\leq n+1$ and a proper map $f : X \to |L|$ such that $\Pi(f) : \Pi(X, M) \to$*

$\Pi(|L|, f(M))$ is an isomorphism for every non-empty subset M of X for which f is injective on M. Also, there is a full subset of X on which f is injective.

Before giving the proof, we remark the following:

REMARK 4.2. Let X be a Hausdorff, locally compact, paracompact and locally pathwise-connected space, $(L, |L|)$ a locally finite simplicial complex of dimension $\leq n$, $f : X \to |L|$ a proper map and M a subset of X such that $\Pi(f) : \Pi(X, M) \to \Pi(|L|, f(M))$ is an isomorphism for $\Pi = \mathrm{p}\tilde{\pi}_1^\infty$ or $\mathrm{p}\tilde{\pi}_1$. If M covers the infinity of X and $\Pi = \mathrm{p}\tilde{\pi}_1^\infty$, then $\mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\, X \leq n$; and if M is full in X and $\Pi = \mathrm{p}\tilde{\pi}_1$, then $\mathrm{p}\tilde{\pi}_1\text{-cat}\, X \leq n$.

Our main theorem is not the converse of this remark but it is the best possible result due to Example 4.3.

PROOF OF REMARK 4.2. We will prove this remark for $\Pi = \mathrm{p}\tilde{\pi}_1^\infty$, the proof for $\mathrm{p}\tilde{\pi}_1$ is analogous. Since the result is trivial for compact spaces, we may assume that X is not compact. Let us show that $\mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\, X \leq \mathrm{p}\tilde{\pi}_1^\infty\text{-cat}|L|$. Let $\{A_i \mid 0 \leq i \leq \mathrm{p}\tilde{\pi}_1^\infty\text{-cat}|L|\}$ be an open cover of $|L|$ by $\mathrm{p}\tilde{\pi}_1^\infty$-categorical sets. It suffices to prove that the sets $f^{-1}(A_i)$ form an open cover of X by $\mathrm{p}\tilde{\pi}_1^\infty$-categorical sets. Let $g_i : f^{-1}(\overline{A_i}) \hookrightarrow X$ be the inclusion map and N_i any subset of $f^{-1}(\overline{A_i})$ not contained in a compact subset. Since A_i is $\mathrm{p}\tilde{\pi}_1^\infty$-categorical in $|L|$, the morphism from $\mathrm{p}\tilde{\pi}_1^\infty(\overline{A_i}, f(N_i))$ to $\mathrm{p}\tilde{\pi}_1^\infty(|L|, f(N_i))$ induced by the inclusion map is null, and thus the composition $\mathrm{p}\tilde{\pi}_1^\infty(f, N_i) \circ \mathrm{p}\tilde{\pi}_1^\infty(g_i, N_i)$ is null.

We take any system $\mathscr{U}^L$ of infinity neighbourhoods of $|L|$ and put $\mathscr{U} = \{f^{-1}(U) \mid U \in \mathscr{U}^L\}$. Then, $\mathscr{U}$ is a system of infinity neighbourhoods of X and $\mathrm{p}\tilde{\pi}_1^\infty(f)$ is level-preserving. Since $\mathrm{p}\tilde{\pi}_1^\infty(f, M) : \mathrm{p}\tilde{\pi}_1^\infty(X, M) \to \mathrm{p}\tilde{\pi}_1^\infty(|L|, f(M))$ is an isomorphism, it satisfies the condition (i) of Lemma 2.17 for $\mathscr{U}$ and $\mathscr{U}^L$. So, for $U_1 \in \mathscr{U}$ there is a $U_2^L \in \mathscr{U}^L$ such that for any loop α in $U_2 = f^{-1}(U_2^L)$ with base point in M, if $f \circ \alpha$ is null-homotopic in U_2^L then α is null-homotopic in U_1. Let us prove that $\mathrm{p}\tilde{\pi}_1^\infty(f, N_i)$ also satisfies the condition (i) of Lemma 2.17 for $\mathscr{U}$ and $\mathscr{U}^L$. Since M covers the infinity of X, there is a system of infinity neighbourhoods $\mathscr{U}'$ of X such that M intersects any path-component of every element of $\mathscr{U}'$. Moreover, there are $U_2' \in \mathscr{U}'$ and $U_3^L \in \mathscr{U}^L$ such that $U_3 \subset U_2' \subset U_2$, where $U_3 = f^{-1}(U_3^L)$. Let β be a loop in U_3 with base point in N_i such that $f \circ \beta$ is null-homotopic in U_3^L. There is a path γ in U_2' from the base point of β to some point of M. Since $f \circ (\gamma^{-1} \cdot \beta \cdot \gamma)$ is null-homotopic in U_2^L, $\gamma^{-1} \cdot \beta \cdot \gamma$ is null-homotopic in U_1, which implies that β is null-homotopic in U_1 and thus the condition (i) holds for $\mathrm{p}\tilde{\pi}_1^\infty(f, N_i)$.

Now, let us show that $\mathrm{p}\tilde{\pi}_1^\infty(g_i, N_i)$ is null. Let $U_1 \in \mathscr{U}$. Since $\mathrm{p}\tilde{\pi}_1^\infty(f, N_i)$ satisfies the condition (i) of Lemma 2.17, there is a $U_2^L \in \mathscr{U}^L$ such that $U_2 = f^{-1}(U_2^L)$ is contained in U_1 and for any loop α in U_2 with base point in N_i, if $f \circ \alpha$ is null-homotopic in U_2^L then α is null-homotopic in U_1. Since

$\mathrm{p}\tilde{\pi}_1^\infty(f, N_i) \circ \mathrm{p}\tilde{\pi}_1^\infty(g_i, N_i)$ is null, there is a $U_3 \in \mathcal{U}$ contained in U_2 such that $f \circ \beta$ is null-homotopic in U_2^L for any loop β in $f^{-1}(\overline{A_i}) \cap U_3$ with base point in N_i. Applying the result just proved in the above paragraph, β must be null-homotopic in U_1, which implies that $\mathrm{p}\tilde{\pi}_1^\infty(g_i, N_i)$ is null and thus the set $f^{-1}(A_i)$ is $\mathrm{p}\tilde{\pi}_1^\infty$-categorical in X.

Hence, $\mathrm{p}\tilde{\pi}_1^\infty$-cat $X \le \mathrm{p}\tilde{\pi}_1^\infty$-cat$|L| \le \dim L \le n$ by Proposition 3.10. □

Proof of Theorem 4.1. We may assume that X is not compact. In fact, if X is compact, $\mathrm{p}\tilde{\pi}_1^\infty$-cat $X = -1$ and the map to the one point space induces an isomorphism of pro-groups $\mathrm{p}\tilde{\pi}_1^\infty$, and we see easily that $\mathrm{p}\tilde{\pi}_1$-cat $X = \mathrm{cat}_{\pi_1} X$ and thus the theorem for the non-proper case can be applied. First, we will prove the theorem for the σ-compact spaces in steps one and two, and then for paracompact spaces in step three. In the σ-compact case; in step one, we will define $(L, |L|)$ and f, and in step two we will prove that f induces an isomorphism of pro-groups.

Now, let us suppose that X is σ-compact and Π-cat $X \le n$.

Step one: Definition of $(L, |L|)$ and f.

Let $\mathcal{U} = \{U_m \mid m \in \mathbf{N}\}$ be a countable system of infinity neighbourhoods of X such that $\overline{U_{m+1}} \subset U_m$ for any m and $U_1 = X$. In fact, such a system exists when X is σ-compact.

We define the sets:

$$G_1 = X - \overline{U_2}$$

$$G_m = U_{m-1} - \overline{U_{m+1}} \qquad (\forall m \ge 2).$$

Note that G_m does not intersect $G_{m'}$ if m and m' are not consecutive. Since Π-cat $X \le n$, there is an open cover $\{A_i \mid 0 \le i \le n\}$ of X by subsets that are Π-categorical in X. Let $\{\Psi_m^G \mid m \in \mathbf{N}\}$ and $\{\Psi_i^A \mid 0 \le i \le n\}$ be partitions of unity of X subordinate to the covers $\{G_m \mid m \in \mathbf{N}\}$ and $\{A_i \mid 0 \le i \le n\}$, that is, the supports of Ψ_m^G and Ψ_i^A are contained in G_m and A_i, respectively, for any i and m. The sets $G_m \cap A_i$ form an open cover of X and the products $\Psi_m^G \cdot \Psi_i^A$ form a partition of unity subordinate to it.

At first we will reconstruct the cover and the partition of unity so that every map of the partition is positive in every point of its corresponding set of the cover, as in [10].

For any non-empty subsets Γ of $\mathbf{N}$ and S of $\{0, \dots, n\}$ we put:

$$\tilde{G}_\Gamma = \{x \in X \mid \Psi_p^G(x) > 0, \Psi_p^G(x) > \Psi_q^G(x) \ \ (\forall p \in \Gamma, \forall q \notin \Gamma)\} \qquad \text{and}$$

$$\tilde{A}_S = \{x \in X \mid \Psi_i^A(x) > 0, \Psi_i^A(x) > \Psi_j^A(x) \ \ (\forall i \in S, \forall j \notin S)\}.$$

Note that Γ is of the form $\{k\}$ or $\{k, k+1\}$; otherwise $\tilde{G}_\Gamma = \emptyset$. We define $D_\Gamma^S = \tilde{G}_\Gamma \cap \tilde{A}_S$ for any Γ and S. The sets D_Γ^S form a locally finite open cover $\mathscr{D}$ of X. For any point x of X and any Γ and S as above we define:

$$\varphi_\Gamma^G(x) = \max\{\min\{\Psi_m^G(x) \mid m \in \Gamma\} - \max\{\Psi_m^G(x) \mid m \notin \Gamma\}, 0\}$$

$$\varphi_S^A(x) = \max\{\min\{\Psi_i^A(x) \mid i \in S\} - \max\{\Psi_i^A(x) \mid i \notin S\}, 0\}$$

$$\varphi_\Gamma^S(x) = \varphi_S^A(x) \cdot \varphi_\Gamma^G(x) \qquad \Phi_\Gamma^S(x) = \frac{\varphi_\Gamma^S(x)}{\sum_{\Delta, T} \varphi_\Delta^T(x)}$$

where the sum of the last formula is taken over all the sets D_Δ^T of the cover $\mathscr{D}$. The maps Φ_Γ^S form a partition of unity subordinate to $\mathscr{D}$ that satisfies the property: Φ_Γ^S is positive in any point of D_Γ^S and null outside D_Γ^S.

Next we need the sets of the cover to be path-connected. For any $D_\Gamma^S \in \mathscr{D}$ and any point $x \in D_\Gamma^S$ we define $V_\Gamma^S(x)$ by the path component of D_Γ^S that contains x; we define $V_\Gamma^S(x) = \emptyset$ when $x \notin D_\Gamma^S$. Let $\mathscr{V} = \{V_\Gamma^S(x) \mid D_\Gamma^S \in \mathscr{D}, x \in D_\Gamma^S\}$, where we distinguish $V_\Gamma^S(x)$ and $V_{\Gamma'}^{S'}(x)$ when $(\Gamma, S) \neq (\Gamma', S')$ even if $V_\Gamma^S(x) = V_{\Gamma'}^{S'}(x)$ as a subset of X. Since X is locally path-connected, $\mathscr{V}$ is an open cover of X.

Let V be a path component of D_Γ^S. Since V is open in X, Φ_V defined by

$$\Phi_V(x) = \begin{cases} \Phi_\Gamma^S(x) & \text{if } x \in V \\ 0 & \text{otherwise} \end{cases}$$

is continuous. These functions form a partition of unity subordinate to $\mathscr{V}$.

Moreover, we need a cover that has a locally finite nerve. For any index Γ there is a finite subfamily $\mathscr{V}_\Gamma$ of $\mathscr{V}$ that covers the closure of $\tilde{G}_\Gamma$, because this set is compact. Let $\mathscr{W}$ be the union of the families $\mathscr{V}_\Gamma$ for all Γ. It is clear that $\mathscr{W}$ is an open cover of X and that any element of $\mathscr{W}$ only intersects a finite number of other elements of $\mathscr{W}$, and thus its nerve is locally finite. Next, we will define a subcover $\mathscr{V}'$ of $\mathscr{W}$ such that any $V \in \mathscr{V}'$ contains a point that does not belong to other elements of $\mathscr{V}'$. We will use this property to construct a full subset of X on which f is injective.

Let us define $\mathscr{V}'$. Since $\mathscr{W}$ is countable, we can give an order to their elements and write $\mathscr{W} = \{V_k \mid k \in \mathbf{N}\}$. We define $\mathscr{V}'$ as a subfamily $\{V_{k_i} \mid i \in \mathbf{N}\}$ of $\mathscr{W}$ recursively. Let V_{k_1} be the first element of $\mathscr{W}$ that contains a point that does not belong to V_k for any $k > k_1$. This V_{k_1} must exist, because only a finite number of elements of $\mathscr{W}$ intersect each set $\tilde{G}_\Gamma$. It is clear that $\{V_k \mid k \geq k_1\}$ is a cover of X. Next, suppose that we have defined V_{k_j} for any $j \leq i$, that the family $\{V_{k_j} \mid 1 \leq j \leq i\} \cup \{V_k \mid k > k_i\}$ is a cover of X and that for any $j \leq i$ the subset V_{k_j} contains a point that does not belong to any other element of this cover. We define $V_{k_{i+1}}$ by the first element of this cover such that $k_{i+1} > k_i$ and $V_{k_{i+1}}$ contains a point that does not belong to any element of $\{V_{k_j} \mid 1 \leq j \leq i\} \cup \{V_k \mid k > k_{i+1}\}$. This set must exist, again because only a finite number of elements of $\mathscr{W}$ intersect each set $\tilde{G}_\Gamma$. Thus we have defined $\mathscr{V}'$.

The partition of unity subordinate to $\mathscr{V}'$ is as follows. For any x of X and $V \in \mathscr{V}'$ we define:

$$\Phi'_V(x) = \frac{\Phi_V(x)}{\sum_W \Phi_W(x)}$$

where the sum is taken over each element W of $\mathscr{V}'$ that contains x.

Let $(N_{\mathscr{V}'}, |N_{\mathscr{V}'}|)$ be the nerve of $\mathscr{V}'$. We define a map $\tilde{f} : X \to |N_{\mathscr{V}'}|$ as follows. For any point x of X let $\{V_1, \dots, V_k\}$ be the elements of $\mathscr{V}'$ that contain x. Then, $[V_1], \dots, [V_k]$ are the vertices of a simplex of $N_{\mathscr{V}'}$, where we use square brackets to distinguish vertices from path components. We define $\tilde{f}(x)$ by the point of this simplex given by the barycentric coordinates $\Phi'_{V_1}(x)[V_1] + \cdots + \Phi'_{V_k}(x)[V_k]$. This map is well-defined because $\mathscr{V}'$ is locally finite, and continuous by the definition of topology on $|N_{\mathscr{V}'}|$.

The dimension of $(N_{\mathscr{V}'}, |N_{\mathscr{V}'}|)$ may be greater than $n+1$. Indeed, in the following paragraphs we will show that $2n+1$ is an upper bound. But we need to define a simplicial complex $(L, |L|)$ of dimension $\leq n+1$ and a proper map from $|N_{\mathscr{V}'}|$ to $|L|$. Since $\mathscr{V}'$ is a subcover of $\mathscr{V}$, let us study the dimension of the nerve $(N_{\mathscr{V}}, |N_{\mathscr{V}}|)$ of $\mathscr{V}$, that is, estimate the maximum number of elements of $\mathscr{V}$ that have non-empty intersection.

First, note that if $\tilde{A}_S$ intersects $\tilde{A}_{S'}$ then $S \subset S'$ or $S' \subset S$, for if there are i of $S - S'$ and j of $S' - S$, then for any $x \in \tilde{A}_S \cap \tilde{A}_{S'}$ the inequalities $\Psi_i^A(x) > \Psi_j^A(x)$ and $\Psi_j^A(x) > \Psi_i^A(x)$ must hold, which is impossible. Analogously it can be proved that if $\tilde{G}_\Gamma$ intersects $\tilde{G}_{\Gamma'}$ then $\Gamma \subset \Gamma'$ or $\Gamma' \subset \Gamma$. By induction, if the intersection $\tilde{A}_{S_1} \cap \cdots \cap \tilde{A}_{S_k}$ is not empty, then there is a permutation $\{i_1, \dots, i_k\}$ of $\{1, \dots, k\}$ such that $S_{i_1} \subset \cdots \subset S_{i_k}$. Note that in the case of the sets of the form $\tilde{G}_\Gamma$, the intersection of three distinct sets is always empty.

Now, let $V_\Gamma^S(x)$ and $V_{\Gamma'}^{S'}(x)$ be two elements of $\mathscr{V}$ with non-empty intersection. Since $D_\Gamma^S \cap D_{\Gamma'}^{S'} = D_\Gamma^S \cap D_{\Gamma'}^S \cap D_\Gamma^{S'} \cap D_{\Gamma'}^{S'}$ by the definition of the elements of $\mathscr{D}$, the intersection $V_\Gamma^S(x) \cap V_{\Gamma'}^S(x) \cap V_\Gamma^{S'}(x) \cap V_{\Gamma'}^{S'}(x)$ is not empty. Applying induction it can be seen that for any point x of X the intersection of the elements of $\mathscr{V}$ that contain x is a set of the form:

(i) $V_\Gamma^{S_1}(x) \cap \cdots \cap V_\Gamma^{S_p}(x)$ or

(ii) $V_\Gamma^{S_1}(x) \cap \cdots \cap V_\Gamma^{S_p}(x) \cap V_{\Gamma'}^{S_1}(x) \cap \cdots \cap V_{\Gamma'}^{S_p}(x)$

where $S_1 \subsetneq \cdots \subsetneq S_p$ and $\Gamma \subsetneq \Gamma'$. Since the sets S_i are contained in $\{0, \dots, n\}$ for any i we see that $p \leq n+1$. Also, the above result implies that every simplex of $N_{\mathscr{V}}$ is a face of a simplex corresponding to the form (i) or (ii).

Thus, any simplex of $N_{\mathcal{V}}$ is a face of a simplex whose dimension is at most $2n+1$, and hence $\dim N_{\mathcal{V}} \leq 2n+1$ and $\dim N_{\mathcal{V}'} \leq 2n+1$.

To define $(L, |L|)$ we will define the space $|L|$ at first and afterwards triangulate it. We will define $|L|$ as a quotient space of $|N_{\mathcal{V}'}|$ by an equivalence relation using a map h from $|N_{\mathcal{V}'}|$ to a topological space $|C|$ that we construct as follows:

Consider the family of the sets of the form $\{k\}$ or $\{k, k+1\}$ with k any natural number. We define the abstract simplicial complex C_Γ by the 1-dimensional complex with vertices the elements of this family and with 1-simplices $\langle \Gamma, \Gamma' \rangle$ where $\Gamma \subset \Gamma'$. Also, we define the abstract simplicial complex C_S with the non-empty subsets of $\{0, \dots, n\}$ as vertices and with the sets $\{S_1, \dots, S_i\}$ such that $S_1 \subset \cdots \subset S_i$ as simplices. This complex is isomorphic to the barycentric subdivision of the canonical n-simplex, and thus can be embedded in $\mathbf{R}^n$. Indeed, if $\langle a_0, \dots, a_n \rangle$ is the canonical n-simplex, the map that sends a vertex $S = \{j_1, \dots, j_k\}$ of C_S to the barycenter of $\langle a_{j_1}, \dots, a_{j_k} \rangle$ in $\mathbf{R}^n$ is an isomorphism. On the other hand, C_Γ can be linearly embedded in $\mathbf{R}^+$ by mapping the vertex $\Gamma = \{k\}$ to $2k-1$ and the vertex $\Gamma = \{k, k+1\}$ to $2k$ for any k. We define the topological space $|C|$ by the union of the cylinders $\varepsilon \times \rho$ of $\mathbf{R}^{n+1}$ such that $\varepsilon \in C_S$ and $\rho \in C_\Gamma$.

We define a piecewise linear map $h : |N_{\mathcal{V}'}| \to |C|$ by $h([V_\Gamma^S(x)]) = (S, \Gamma) \in |C| \subset \mathbf{R}^{n+1}$ for any vertex $[V_\Gamma^S(x)]$. In fact, the images of the vertices of any simplex $\sigma = \langle V_{\Gamma 1}^{S_1}(x), \dots, V_{\Gamma k}^{S_k}(x) \rangle$ of $N_{\mathcal{V}'}$ by h are vertices of a cylinder of $|C|$, because $\tilde{A}_{S_1} \cap \cdots \cap \tilde{A}_{S_k} \neq \varnothing$ and $\tilde{G}_{\Gamma_1} \cap \cdots \cap \tilde{G}_{\Gamma_k} \neq \varnothing$, and thus we can extend h linearly to any point of σ.

Now, we identify two points x, y of $|N_{\mathcal{V}'}|$ and write $x \approx y$ if they belong to the same simplex of $N_{\mathcal{V}'}$ and $h(x) = h(y)$. But $\approx$ is not an equivalence relation, and thus we take the equivalence relation $\sim$ induced by $\approx$. This equivalence relation defines a quotient topological space $|L|$. We define g by the canonical projection from $|N_{\mathcal{V}'}|$ to $|L|$ and $f = g \circ \tilde{f}$. Then, there is a unique continuous map $\tilde{h}$ such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & {|N_{\mathcal{V}'}|} \arrow[r, "h"] \arrow[d, "g"] & {|C| \subset \mathbf{R}^{n+1}} \\
& {|L|} \arrow[ur, "\tilde{h}"'] &
\end{tikzcd}$$

We will triangulate $|C|$ appropriately and then use $\tilde{h}$ to triangulate $|L|$. For any simplex $\sigma \in N_{\mathcal{V}'}$ the restriction of $\tilde{h}$ to $g(\sigma)$ is injective. Also $h(\sigma)$ is known to be a convex hull inside a cylinder. We will define a triangulation C of $|C|$ that satisfies the following property: for any cylinder $\langle S_1, \dots, S_p \rangle \times \langle \Gamma, \Gamma' \rangle$, the convex hull of any subset of $\{S_1, \dots, S_p\} \times \{\Gamma, \Gamma'\}$ is the underlying space of a subcomplex of C. Thus, each $h(\sigma)$ will be the underlying subspace of a subcomplex of C.

We triangulate each cylinder of $|C|$ by an induction on dimension. Any cylinder of dimension 1 is a 1-simplex, and thus it is already triangulated. Now suppose that all the cylinders of dimension $\leq m$ have been triangulated and the triangulations agree in the intersection of every pair of cylinders. Let $\varepsilon \times \langle \Gamma, \Gamma' \rangle$ be an $(m+1)$-dimensional cylinder of $|C|$. For each m-dimensional cylinder $\rho \times \langle \Gamma, \Gamma' \rangle$ of $\partial\varepsilon \times \langle \Gamma, \Gamma' \rangle$ we form two cone complexes with vertices (S, Γ) and (S, Γ') respectively, where S is the vertex of ε that does not belong to ρ. We obtain $2m+2$ cone complexes contained in $\varepsilon \times \langle \Gamma, \Gamma' \rangle$. The intersection of two simplices of different complexes is the convex hull of a finite set of points of $\mathbf{R}^{n+1}$ (see [16], 2.6). If we form all the possible intersections of pairs of simplices of the cone complexes, we obtain a family of convex hulls. We can triangulate all these convex hulls, without introducing new vertices, to form a triangulation of $\varepsilon \times \langle \Gamma, \Gamma' \rangle$ that contains the cone complexes above defined and the simplices $\varepsilon \times \{\Gamma\}$ and $\varepsilon \times \{\Gamma'\}$ as subcomplexes (see [16], 2.8 (5) and 2.9). The union of the simplicial complexes of each $m+1$-dimensional cylinder form a simplicial complex. By induction this defines a triangulation of $|C|$ that satisfies the desired property.

Now we define L. Let σ be a simplex of $N_{\mathscr{V}'}$ and α a simplex of C such that $\alpha \subset h(\sigma)$. The restriction of $\tilde{h}$ to $g(\sigma)$, $\tilde{h}|_{g(\sigma)} : g(\sigma) \to h(\sigma)$, is bijective and continuous. Since $g(\sigma)$ is compact and $|C|$ is Hausdorff, it is a homeomorphism. We define α_σ by the inverse image of α by $\tilde{h}|_{g(\sigma)}$. We denote the inverse map of $\tilde{h}|_{\alpha_\sigma} : \alpha_\sigma \to \alpha$ by $\hat{h}_\alpha^\sigma$. We will show that the family of maps $\{\hat{h}_\alpha^\sigma ; \alpha \to \alpha_\sigma \mid \alpha \in C, \sigma \in N_{\mathscr{V}'}, \alpha \subset h(\sigma)\}$ define a Δ-complex structure on $|L|$, that is:

(i) The restriction of $\hat{h}_\alpha^\sigma$ to $\mathring{\alpha}$ is injective for any map $\hat{h}_\alpha^\sigma$ of the family, and for each point $p \in |L|$ there is a unique map $\hat{h}_p$ $(= \hat{h}_\alpha^\sigma$ for some α and $\sigma)$ of the family such that $\hat{h}_p(\mathring{\alpha})$ contains p.

(ii) For any map $\hat{h}_\alpha^\sigma$ and any face β of α the restriction of $\hat{h}_\alpha^\sigma$ to β belongs to the family.

(iii) $|L|$ has the weak topology with respect to the family of subsets $\{\tilde{a}_\sigma \mid \alpha \in C, \sigma \in N_{\mathscr{V}'}, \alpha \subset h(\sigma)\}$.

Here $\mathring{\alpha}$ is the interior of the simplex α. Note that a map of this family may correspond to several pairs of simplices (α, σ): for example, $\hat{h}_\alpha^\sigma = \hat{h}_\alpha^\tau$ if τ is a face of σ. The condition (i) states that the map $\hat{h}_\alpha^\sigma$ such that $p \in \hat{h}_\alpha^\sigma(\mathring{\alpha})$ must be unique, not the simplex σ. In fact, we will see that two maps $\hat{h}_\alpha^\sigma$ and $\hat{h}_\beta^\tau$ are equal iff $\hat{h}_\alpha^\sigma(\mathring{\alpha}) = \hat{h}_\beta^\tau(\mathring{\beta})$. The second barycentric subdivision of a Δ-complex is a simplicial complex (see [11] for details), by which we will define L. We will check the conditions (i), (ii) and (iii) above.

Since each map $\hat{h}_\alpha^\sigma$ is injective, its restriction is also injective, thus the first statement of (i) holds. The condition (ii) is also trivial, because the restriction of

$\hat{h}^\sigma_\alpha$ to β is $\hat{h}^\sigma_\beta$. Before concluding the proof of (i) we will show that the sets $g(\sigma)$ form a locally finite closed cover of $|L|$ in order to prove the condition (iii).

To prove that $g(\sigma)$ is closed in $|L|$ for any simplex $\sigma \in N_{\mathscr{V}'}$, it suffices to show that $g^{-1}(g(\sigma))$ is compact, because g is a quotient map and $|N_{\mathscr{V}'}|$ is Hausdorff. To do this we will decompose $g^{-1}(g(\sigma))$ as a finite union of compact sets. For any subset B of $|N_{\mathscr{V}'}|$ let $\xi(B)$ be the set of the points of $|N_{\mathscr{V}'}|$ related to the points of B by the relation $\approx$ used to define $|L|$. Since ξ commutes with the union operator, $\xi(B) = \bigcup_\tau \xi(B \cap \tau)$, where τ ranges over the simplices of $N_{\mathscr{V}'}$. By the definition of $\sim$, the equivalence relation generated by $\approx$, $g^{-1}(g(B)) = \bigcup_{k \geq 0} \xi^k(B)$, where ξ^k denotes the composition of ξ with itself k times. Let us show that if $B = \sigma$ this union is finite and the sets $\xi^k(\sigma)$ are compact, which implies that $g^{-1}(g(\sigma))$ is compact. Let K be a compact subset of $|N_{\mathscr{V}'}|$. If τ is a simplex of $N_{\mathscr{V}'}$, $\xi(K \cap \tau)$ is compact, because it is the intersection of $h^{-1}(h(K \cap \tau))$ and the union of the simplices that contain τ. And since K only intersects a finite number of simplices of $N_{\mathscr{V}'}$, $\xi(K)$ is compact. Thus, applying induction, $\xi^k(\sigma)$ is compact for any k. Now let us check that there is an l such that $\xi^k(\sigma) = \xi^l(\sigma)$ for any $k \geq l$, and then the above union is finite.

Note that $\xi(B) \supset B$ and hence $\xi^k(B) \supset \xi^l(B)$ for $k \geq l$. Since ξ commutes with the union operator, $\xi^k(\sigma)$ is the union of the sets $\xi(\xi(\cdots\xi(\xi(\sigma \cap \tau_1) \cap \tau_2) \cap \cdots) \cap \tau_k)$ where $\tau_1, \ldots, \tau_k$ are simplices of $N_{\mathscr{V}'}$. If $\xi(\cdots\xi(\sigma \cap \tau_1) \cap \cdots \cap \tau_k) \neq \varnothing$ then each point of this set is equivalent by $\sim$ to points of σ and points of τ_i for any i by the definition of ξ, and thus $h(\tau_i) \cap h(\sigma) \neq \varnothing$ for any i. There is only a finite number l of simplices $\tau \in N_{\mathscr{V}'}$ such that $h(\tau) \cap h(\sigma) \neq \varnothing$, because $h(\tau)$ and $h(\sigma)$ must be contained in two cylinders of $|C|$ with non-empty intersection, and then the elements of $\mathscr{V}'$ corresponding to the vertices of τ must be contained in some compact subset of X. Now let us prove that $\xi(\xi(\cdots\xi(B \cap \tau_1) \cap \cdots \cap \tau_i) \cap \tau_1) \subset \xi(B \cap \tau_1)$ for any $B \subset |N_{\mathscr{V}'}|$ and $i \geq 1$. Indeed, $\xi(\cdots\xi(B \cap \tau_1) \cap \cdots \cap \tau_i) \subset h^{-1}(h(B \cap \tau_1))$, because the points of $\xi(\cdots\xi(B \cap \tau_1) \cap \cdots \cap \tau_i)$ are equivalent to points of $B \cap \tau_1$. Thus

$$\begin{aligned} h^{-1}(h(\xi(\cdots\xi(B \cap \tau_1) \cap \cdots \cap \tau_i) \cap \tau_1)) &\subset h^{-1}(h(h^{-1}(h(B \cap \tau_1)) \cap \tau_1)) \\ &\subset h^{-1}(h(B \cap \tau_1)) \end{aligned}$$

and by taking the intersection with the union of the simplices that contain τ_1, the result holds. Let $k > l$ and let $\tau_1, \ldots, \tau_k$ be simplices of $N_{\mathscr{V}'}$ such that $\xi(\cdots\xi(\sigma \cap \tau_1) \cap \cdots \cap \tau_k) \neq \varnothing$. Since $k > l$, there are i and j such that $\tau_i = \tau_j$, with $j > i$, and thus we see that $\xi(\xi(\cdots\xi(\xi(\cdots\xi(\sigma \cap \tau_1) \cap \cdots) \cap \tau_i) \cap \cdots) \cap \tau_j)$ is contained in $\xi(\xi(\cdots\xi(\sigma \cap \tau_1) \cap \cdots) \cap \tau_i)$ by taking

$$B = \xi(\xi(\cdots\xi(\sigma \cap \tau_1) \cap \cdots) \cap \tau_{i-1}).$$

It follows that $\xi(\cdots\xi(\sigma\cap\tau_1)\cap\cdots\cap\tau_k)$ is contained in

$$\xi(\cdots\xi(\xi(\cdots\xi(\sigma\cap\tau_1)\cap\cdots\cap\tau_i)\cap\tau_{j+1})\cap\cdots\cap\tau_k).$$

Hence, $\xi^k(\sigma)\subset\xi^{k-j+i}(\sigma)$. If $k-j+i>l$, repeat the argument. Then, we see that $\xi^k(\sigma)\subset\xi^l(\sigma)$. Thus, we have proved that $g(\sigma)$ is closed in $|L|$.

Next we will prove that the cover $\{g(\sigma)\,|\,\sigma\in N_{\mathscr{V}'}\}$ is locally finite. Any point of $|L|$ is contained in the preimage of the simplicial star of a vertex of C by $\tilde{h}$. Remember that the simplicial star $\mathrm{St}(p,C)$ of a point p of $|C|$ is an open subset defined by the finite union of the interiors of the simplices of C that contain p. Let α be a simplex that contains p. If σ is a simplex of $N_{\mathscr{V}'}$ such that $g(\sigma)\cap\tilde{h}^{-1}(\mathring{\alpha})\neq\varnothing$ then $h(\sigma)\cap\mathring{\alpha}\neq\varnothing$, which implies that $\alpha\subset h(\sigma)$, because $h(\sigma)$ is the underlying space of a subcomplex of C. But if $h(\sigma)$ and $h(\tau)$ contain α then $h(\sigma)\cap h(\tau)\neq\varnothing$, and thus there is only a finite number of simplices σ such that $g(\sigma)$ intersects $\tilde{h}^{-1}(\alpha)$. Hence, only a finite number of sets of the cover $\{g(\sigma)\,|\,\sigma\in N_{\mathscr{V}'}\}$ intersect the preimage of the star, and thus the cover is locally finite.

Now, we can prove the condition (iii), that a set F is closed in $|L|$ iff $F\cap\alpha_\sigma$ is closed in α_σ for any α_σ. The necessity is obvious. For the sufficiency, suppose that $F\cap\alpha_\sigma$ is closed in α_σ for any α_σ. Then $F\cap\alpha_\sigma$ is also closed in $g(\sigma)$, because α_σ is closed in $g(\sigma)$. And since the sets $g(\sigma)$ form a locally finite closed cover of $|L|$, F is closed.

Finally, let us prove the second part of (i). First, let us show that such a map exists. Let p be a point of $|L|$. There is a simplex σ of $N_{\mathscr{V}'}$ such that $p\in g(\sigma)$, because $\{g(\sigma)\,|\,\sigma\in N_{\mathscr{V}'}\}$ is a cover of $|L|$. On the other hand, there is a simplex α of C such that $\tilde{h}(p)\in\mathring{\alpha}$. Since $\tilde{h}(p)\in h(\sigma)\cap\mathring{\alpha}$, we see that $\alpha\subset h(\sigma)$ and hence $p\in\hat{h}^\sigma_\alpha(\mathring{\alpha})$.

If α is a simplex of C and σ a simplex of $N_{\mathscr{V}'}$ such that $\mathring{\alpha}\cap h(\sigma)\neq\varnothing$ then there is a face ρ of σ such that $\mathring{\alpha}\subset h(\mathring{\rho})$. In fact, if $\mathring{\alpha}$ intersects $h(\sigma)$ then $\alpha\subset h(\sigma)$. If $\mathring{\alpha}\not\subset h(\mathring{\sigma})$ then $\mathring{\alpha}$ intersects $h(\partial\sigma)$, thus there is a proper face ρ' of σ such that $\mathring{\alpha}$ intersects $h(\rho')$, and hence $\alpha\subset h(\rho')$. If $\mathring{\alpha}\not\subset h(\mathring{\rho}')$ we apply the same argument to ρ', and after some repetitions of the argument we obtain the desired face ρ.

Now, let us check the uniqueness of $\hat{h}^\sigma_\alpha$. We prove that if two images $\hat{h}^\sigma_\alpha(\mathring{\alpha})$ and $\hat{h}^\tau_\beta(\mathring{\beta})$ have non-empty intersection then $\hat{h}^\sigma_\alpha=\hat{h}^\tau_\beta$. If $\hat{h}^\sigma_\alpha(\mathring{\alpha})$ intersects $\hat{h}^\tau_\beta(\mathring{\beta})$, then taking the images by $\tilde{h}$, $\mathring{\alpha}$ must intersect $\mathring{\beta}$, and thus $\alpha=\beta$. We may assume that $\mathring{\alpha}\subset h(\mathring{\sigma})$, for if $\mathring{\alpha}\not\subset h(\mathring{\sigma})$ then there is a face σ' of σ such that $\mathring{\alpha}\subset h(\mathring{\sigma}')$, and since $g(\sigma')\subset g(\sigma)$, we see that $\alpha_\sigma=\alpha_{\sigma'}$, which implies that $\hat{h}^\sigma_\alpha=\hat{h}^{\sigma'}_\alpha$, and thus we can redefine σ by σ'. Similarly, we may assume that $\mathring{\alpha}\subset h(\mathring{\tau})$.

Then, we can prove that $\hat{h}^\sigma_\alpha(\mathring{\alpha})=\hat{h}^\tau_\alpha(\mathring{\alpha})$. It suffices to show that $\hat{h}^\sigma_\alpha(\mathring{\alpha})$ is contained in $\hat{h}^\tau_\alpha(\mathring{\alpha})$; the other inclusion follows in the same way. Since

$\hat{h}^\sigma_\alpha(\mathring{\alpha}) \cap \hat{h}^\tau_\alpha(\mathring{\alpha}) \neq \emptyset$, there are two equivalent points $x_\sigma \in \sigma$ and $x_\tau \in \tau$ such that $h(x_\sigma) = h(x_\tau) \in \mathring{\alpha}$. Now we prove that for any $y_\sigma \in \sigma$ such that $h(y_\sigma) \in \mathring{\alpha}$ there is a $y_\tau \in \tau$ equivalent to y_σ. In fact, since $x_\sigma \sim x_\tau$, there are points $z_1, \dots, z_m$ such that $x_\sigma \approx z_1 \approx \cdots \approx z_m \approx x_\tau$. Thus, there are simplices $\varepsilon_1, \dots, \varepsilon_{m-1}$ such that $z_1 \in \sigma \cap \varepsilon_1$, $z_{i+1} \in \varepsilon_i \cap \varepsilon_{i+1}$ for any i, and $z_m \in \varepsilon_m \cap \tau$. Since $\mathring{\alpha}$ intersects $h(\varepsilon_i \cap \varepsilon_{i+1})$ for any i, $\mathring{\alpha} \subset h(\varepsilon_i \cap \varepsilon_{i+1})$. Analogously, $\mathring{\alpha}$ is contained in $h(\sigma \cap \varepsilon_1)$ and $h(\varepsilon_{m-1} \cap \tau)$. Thus, there are points $w_1 \in \sigma \cap \varepsilon_1$, $w_m \in \varepsilon_{m-1} \cap \tau$ and $w_{i+1} \in \varepsilon_i \cap \varepsilon_{i+1}$ for any i such that $h(w_1) = h(w_{i+1}) = h(w_m) = h(y_\sigma)$. Thus, $y_\sigma \approx w_1 \approx \cdots \approx w_n$, and if we define $y_\tau = w_m$, the result holds.

Since $g(\sigma)$ is closed in $|L|$, the closure of $\hat{h}^\sigma_\alpha(\mathring{\alpha})$ in $|L|$ is equal to its closure in $g(\sigma)$. And since $\tilde{h}|_{g(\sigma)}$ is an homeomorphism, this closure is equal to the inverse image by this map of the closure of $\mathring{\alpha}$ in $h(\sigma)$, that is α. Thus, the closure of $\hat{h}^\sigma_\alpha(\mathring{\alpha})$ in $|L|$ is α_σ. Analogously, the closure of $\hat{h}^\tau_\alpha(\mathring{\alpha})$ in $|L|$ is α_τ. Thus, $\alpha_\sigma = \alpha_\tau$, which implies that $\hat{h}^\sigma_\alpha = \hat{h}^\tau_\alpha$, by the definition of these maps.

Hence we have defined a simplicial complex L. Also, L is locally finite, because $\{g(\sigma) \mid \sigma \in N_{\mathscr{V}'}\}$ is a locally finite cover of $|L|$ and each $g(\sigma)$ only contains a finite number of simplices of L. We define $f = g \circ \tilde{f}$. It is continuous. We will prove that g and $\tilde{f}$ are proper and thus f is proper. The map from $g^{-1}(g(\sigma))$ to $g(\sigma)$ that coincides with g on any point of $g^{-1}(g(\sigma))$ is proper, because $g^{-1}(g(\sigma))$ is compact. Since $\{g(\sigma) \mid \sigma \in N_{\mathscr{V}'}\}$ is a locally finite closed cover of $|L|$, g is proper, by Proposition 3 of I.72 of [3]. Similarly, since the cover of $|N_{\mathscr{V}'}|$ formed by its closed stars is locally finite and the inverse image by $\tilde{f}$ of any closed star is compact (because it is closed and contained in some $X - U_j$), $\tilde{f}$ is proper. Thus, f is proper. Also, $\tilde{h}^{-1}(h(\sigma))$ is compact for any simplex σ of $N_{\mathscr{V}'}$, because it is contained in a finite union of sets of the form $g(\tau)$, where τ is a simplex of $N_{\mathscr{V}'}$. And since $\{h(\sigma) \mid \sigma \in N_{\mathscr{V}'}\}$ is a locally finite closed cover of $\tilde{h}(|L|)$, the map from $|L|$ to $\tilde{h}(|L|)$ that coincides with $\tilde{h}$ in every point is proper. Hence, $\tilde{h}$ is proper, because $\tilde{h}(|L|) = h(|N_{\mathscr{V}'}|)$ is closed in $|C|$.

Now, let us show that there is a subset M of X that is full and such that f is injective on M. For every $V \in \mathscr{V}'$ we take a point x_V of V that does not belong to any other element of $\mathscr{V}'$. We define $M = \{x_V \mid V \in \mathscr{V}'\}$. Since M has points in any element of $\mathscr{V}'$, it is full in X. On the other hand, since $\tilde{f}(x_V) = [V]$ and g is injective on the vertices of $N_{\mathscr{V}'}$, f is injective on M.

This concludes step one. In step two we will need the following

Assertion: *The inverse image $g^{-1}(\mathrm{St}(p, L))$ of the simplicial star of any vertex p of L in L is path-connected.*

Note that the image by $\tilde{h}$ of a simplex of L is not a simplex of C but that of its second barycentric subdivision. Hereafter we will use the notation α_σ for a simplex of L, where σ is a simplex of $N_{\mathscr{V}'}$ and α is a simplex of the second barycentric subdivision of C contained in $h(\sigma)$ such that $\tilde{h}(\alpha_\sigma) = \alpha$, although we used the notation α_σ for a simplex α of C in the step one above. Let $\mathrm{St}(p, L)$ be the simplicial star of a vertex L. It is a union of sets of the form $\alpha_\sigma - \beta_\sigma$ such that p is a vertex of α_σ and β_σ is the simplex spanned by the vertices of α_σ different from p. To prove that $g^{-1}(\mathrm{St}(p, L))$ is path-connected, it suffices to prove that $g^{-1}(\alpha_\sigma - \beta_\sigma)$ is path-connected, because these inverse images have non-empty intersection, namely, $g^{-1}(p)$.

First, we see that $g^{-1}(\alpha_\sigma - \beta_\sigma) = g^{-1}(g(h^{-1}(\alpha - \beta) \cap \sigma))$ for any simplices α_σ and β_σ as above. In fact, if $x \in g^{-1}(\alpha_\sigma - \beta_\sigma)$ then $g(x) \in \alpha_\sigma - \beta_\sigma$. Thus $g(x) \in g(\sigma)$ and also $h(x) \in \alpha - \beta$, by taking the images by $\tilde{h}$. This implies that there is a $y \in \sigma$ such that $g(y) = g(x)$ and $h(y) \in \alpha - \beta$. On the other hand, if $x \in h^{-1}(\alpha - \beta) \cap \sigma$ then $g(x) \in g(\sigma)$ and $h(x) \in \alpha - \beta$. Thus, $g(x) \in \alpha_\sigma - \beta_\sigma$.

Next, let us prove that $g^{-1}(g(B))$ is path-connected for any path-connected subset B of $|N_{\mathscr{V}'}|$. Let x and y be two points of $|N_{\mathscr{V}'}|$ such that $x \approx y$. Then, $h(x) = h(y)$ and there is a simplex τ that contains x and y. The points x and y belong to $\tau \cap h^{-1}(h(x))$, which is convex in τ because it is equal to $(h|_\tau)^{-1}(h(x))$ and $h|_\tau$ is linear. Thus x and y can be joined by a linear path formed by points $\approx x$. This implies that the equivalence class of any point of $|N_{\mathscr{V}'}|$ by $\sim$ is path-connected. Thus, since $g^{-1}(g(B))$ is the union of the equivalence classes of the points of B, it is path-connected.

Finally, let α, β and σ be as above. Since $\alpha - \beta$ is convex and $h|_\sigma$ is linear, $h^{-1}(\alpha - \beta) \cap \sigma$ is convex in σ, and thus path-connected. Then, $g^{-1}(\alpha_\sigma - \beta_\sigma)$ is path-connected and hence the inverse image by g of the simplicial star of any vertex of L is path-connected.

Step two: Proof of the isomorphism condition.

Now let us check that $\Pi(f)$ is an isomorphism. We will describe only the proof for the case $\Pi = \mathrm{p}\tilde{\pi}_1^\infty$; the case $\Pi = \mathrm{p}\tilde{\pi}_1$ can be deduced from it.

First we fix the systems of infinity neighbourhoods of X and $|L|$. For X we choose $\mathscr{U}$, which was used to define G_m and C_Γ. The embedding of $|C_\Gamma|$ in $\mathbf{R}_+$ defines an order on $|C_\Gamma|$. For any $m \geq 1$ we define $U_m^\Gamma = \bigcup \mathrm{St}(\Gamma, C_\Gamma)$ where Γ ranges over the vertices of C_Γ such that $\Gamma \geq \{m\}$. These sets define a system of infinity neighbourhoods $\mathscr{U}^\Gamma$ of $|C_\Gamma|$ that satisfy $\overline{U_{m+1}^\Gamma} \subset U_m^\Gamma$ for any m. Let $\tilde{h}_\Gamma$ (resp. h_Γ) be the composition of the projection of $|C|$ onto $|C_\Gamma|$ and $\tilde{h}$ (resp. h). Since $\tilde{h}_\Gamma$ is proper the sets $U_m^L = \tilde{h}_\Gamma^{-1}(U_m^\Gamma)$ define a system of infinity neighbourhoods $\mathscr{U}^L$ of $|L|$. Also, $\tilde{h}_\Gamma$ is continuous and thus $\overline{U_{m+1}^L} \subset \tilde{h}_\Gamma^{-1}(\overline{U_{m+1}^\Gamma}) \subset U_m^L$ for any m.

Let us show that f is level-preserving. Since $U_m \cap G_i = \varnothing$ for any $i < m$, $U_m \cap \tilde{G}_\Gamma = \varnothing$ for any $\Gamma < \{m\}$. Then $U_m \subset \bigcup_{\Gamma \geq \{m\}} \tilde{G}_\Gamma$, which implies that $f(U_m) \subset U_m^L$. Thus by Lemma 2.17 it suffices to check that $\mathrm{p}\tilde{\pi}_1^\infty(f)$:

$\mathrm{p}\tilde{\pi}_1^\infty(X, M, \mathscr{U}) \to (|L|, f(M), \mathscr{U}^L)$ for any subspace M of X on which f is injective satisfies:

(i) for any m there is an $m' \geq m$ such that $\mathrm{Ker}\, \tilde{\pi}_1(f_{m'}) \subset \mathrm{Ker}\, \tilde{\pi}_1(p_{mm'})$,

(ii) for any m there is an $m' \geq m$ such that $\mathrm{Im}\, \tilde{\pi}_1(q_{mm'}) \subset \mathrm{Im}\, \tilde{\pi}_1(f_m)$,

where $p_{mm'} : U_{m'} \hookrightarrow U_m$ and $q_{mm'} : U^L_{m'} \hookrightarrow U^L_m$ are the inclusion maps and $f_m = f|_{U_m}$.

In the proof of (i) and (ii) we will use repeatedly the following fact. Let $(K, |K|)$ be a simplicial complex, α a path in $|K|$ and $\mathscr{B}$ a family of simplicial stars of vertices of K that covers the image of α. Then there are $\mathrm{St}(p_1, K), \dots, \mathrm{St}(p_k, K) \in \mathscr{B}$ and paths $\alpha_1, \dots, \alpha_k$ in $|K|$ such that $\alpha = \alpha_1 \dots \alpha_k$ and α_i is contained in $\mathrm{St}(p_i, K)$ for any $i \leq k$. In fact, the inverse images by α of the elements of $\mathscr{B}$ form an open cover of $[0, 1]$, and it suffices to take subintervals of length less than the Lebesgue number.

We begin with the condition (ii) and prove that $\mathrm{Im}\, \tilde{\pi}_1(q_{m,m+2}) \subset \mathrm{Im}\, \tilde{\pi}_1(f_m)$. Let α^L be a loop in U^L_{m+2} with base point in $f(M)$. We will construct a loop α^X in U_m such that $f \circ \alpha^X$ is homotopic to α^L in U^L_m. By the above fact, there are vertices $p_1, \dots, p_k$ of L and paths $\alpha^L_1, \dots, \alpha^L_k$ in $|L|$ such that $\alpha^L = \alpha^L_1 \dots \alpha^L_k$ and α^L_i is contained in $\mathrm{St}(p_i, L)$ for any i. Since $\mathrm{St}(p_i, L)$ intersects $\mathrm{St}(p_{i+1}, L)$, $g^{-1}(\mathrm{St}(p_i, L))$ intersects $g^{-1}(\mathrm{St}(p_{i+1}, L))$ for any i. Let us choose a point y_i in this intersection. Since $g^{-1}(\mathrm{St}(p_i, L))$ is path-connected for any i by the assertion written in the last part of step one, there is a path α^N_i in $g^{-1}(\mathrm{St}(p_i, L))$ that connects y_{i-1} and y_i. Let $\alpha^N = \alpha^N_1 \dots \alpha^N_k$. Using the above fact again, we can take simplicial stars $\mathrm{St}([W_1], N_{\mathscr{V}'}), \dots, \mathrm{St}([W_l], N_{\mathscr{V}'})$ that cover α^N such that $\mathrm{St}([W_j], N_{\mathscr{V}'})$ intersects $\mathrm{St}([W_{j+1}], N_{\mathscr{V}'})$ for any j. Then, $[W_j]$ and $[W_{j+1}]$ belong to the same simplex of $N_{\mathscr{V}'}$ and thus W_j intersects W_{j+1}. So, we can define a path $\alpha^X = \alpha^X_1 \dots \alpha^X_l$ in X as before such that α^X_j is contained in W_j. Now, since the intersection of simplicial stars is path-connected and the simplicial stars themselves are contractible, $\tilde{f} \circ \alpha^X$ is homotopic to α^N in $|N_{\mathscr{V}'}|$ and $g \circ \alpha^N$ is homotopic to α^L in $|L|$.

Note that $h_\Gamma : N_{\mathscr{V}'} \to C_\Gamma$ is simplicial and thus $h_\Gamma(\mathrm{St}[V], N_{\mathscr{V}'}))$ is contained in $\mathrm{St}(h_\Gamma([V]), C_\Gamma)$ for any vertex $[V]$ of $N_{\mathscr{V}'}$. Let us show that $\tilde{h}_\Gamma(\mathrm{St}(p, L)) \subset \mathrm{St}(\tilde{h}_\Gamma(p), C_\Gamma)$ for any vertex p of L. Let ε be a simplex of L that contains p. Then there is a simplex σ of $N_{\mathscr{V}'}$ such that $\tilde{h}(\mathring{\varepsilon}) \subset h(\mathring{\sigma})$. Then $\tilde{h}_\Gamma(p) \in h_\Gamma(\sigma)$ and $\tilde{h}_\Gamma(\mathring{\varepsilon}) \subset h_\Gamma(\mathring{\sigma})$. Since h_Γ is simplicial, $h_\Gamma(\sigma)$ is a simplex of C_Γ and $h_\Gamma(\mathring{\sigma})$ is its interior, which implies that $\tilde{h}_\Gamma(\mathring{\varepsilon}) \subset \mathrm{St}(\tilde{h}_\Gamma(p), C_\Gamma)$. Thus $\tilde{h}_\Gamma(\mathrm{St}(p, L)) \subset \mathrm{St}(\tilde{h}_\Gamma(p), C_\Gamma)$.

Let us prove that $f \circ \alpha^X$ is homotopic to α^L in U^L_m. Since $\tilde{f} \circ \alpha^X$ is homotopic to α^N in $\mathrm{St}([W_1], N_{\mathscr{V}'}) \cup \dots \cup \mathrm{St}([W_l], N_{\mathscr{V}'})$ and $g \circ \alpha^N$ is homotopic to α^L in $\mathrm{St}(p_1, L) \cup \dots \cup \mathrm{St}(p_k, L)$, it suffices to check that $g(\mathrm{St}([W_j], N_{\mathscr{V}'}))$ and $\mathrm{St}(p_i, L)$ are contained in U^L_m for any j and i. Since $\mathrm{St}(p_i, L)$ intersects U^L_{m+2}, $\mathrm{St}(\tilde{h}_\Gamma(p_i), C_\Gamma)$ intersects U^Γ_{m+2}. If we denote by d the Euclidean distance in $|C_\Gamma| \subset \mathbf{R}$, then $d(x, y) \leq 1$ for any two points $x, y \in |C_\Gamma|$ with $\mathrm{St}(x, C_\Gamma) \cap \mathrm{St}(y, C_\Gamma) \neq \emptyset$. So, $d(\tilde{h}_\Gamma(p_i), \Gamma) \leq 1$ for some vertex $\Gamma \geq \{m+2\}$. Then $\tilde{h}_\Gamma(p_i) \geq \{m+1, m+2\}$ and thus $\mathrm{St}(\tilde{h}_\Gamma(p_i), C_\Gamma) \subset U^\Gamma_{m+1}$. Hence, $\mathrm{St}(p_i, L) \subset U^L_{m+1} \subset U^L_m$. On the other hand, since α^N intersects $\mathrm{St}([W_j], N_{\mathscr{V}'})$, there is an i' such that $\mathrm{St}([W_j], N_{\mathscr{V}'})$ intersects $g^{-1}(\mathrm{St}(p_{i'}, L))$. Then $\mathrm{St}(h_\Gamma([W_j]), C_\Gamma)$ intersects $\mathrm{St}(\tilde{h}_\Gamma(p_{i'}), C_\Gamma)$ and thus $h_\Gamma([W_j]) \geq \{m+1\}$, because $\tilde{h}_\Gamma(p_{i'}) \geq \{m+1, m+2\}$. Then $\mathrm{St}(h_\Gamma([W_j]), C_\Gamma) \subset U^\Gamma_{m+1}$ and hence $g(\mathrm{St}([W_j], N_{\mathscr{V}'})) \subset U^L_{m+1} \subset U^L_m$.

Finally, since $h_\Gamma([W_j]) \geq \{m+1\}$, we have $W_j \subset \bigcup_{i \geq m+1} G_i \subset U_m$ and thus α^X is contained in U_m.

Now, let us prove the condition (i). There is an $m'' \geq m$ such that any loop in $\bar{A}_i \cap U_{m''}$ with base point in M is null-homotopic in U_m for any i. We will prove that $\mathrm{Ker}\, \tilde{\pi}_1(f_{m''+2}) \subset \mathrm{Ker}\, \tilde{\pi}_1(p_{m,m''+2})$. Let α^X be a loop in $U_{m''+2}$ with base point in M verifying that the closed path $\alpha^L = f \circ \alpha^X$ is contractible to a point in $U^L_{m''+2}$. We need to check that α^X is contractible to a point in U_m.

Since α^L is null-homotopic in $U^L_{m''+2}$ there is a map ψ_L from the unit 2-disk D^2 to $U^L_{m''+2}$ such that $\psi_L|_{\partial D^2} = \alpha^L$. It suffices to define a map $\psi_X : D^2 \to U_m$ such that $\psi_X|_{\partial D^2} = \alpha^X$. Since the simplicial stars of the vertices of L form an open cover of $|L|$, their inverse images by ψ_L form an open cover of D^2 that is compact. We take a triangulation Q of D^2 such that the diameter of each simplex of Q is less than the Lebesgue number of this cover. Then, the image by ψ_L of any simplex of Q is contained in the simplicial star of a vertex of L. We define ψ_X on ∂Q by α^X. Let τ be a 1-simplex and σ a 2-simplex of Q not contained in ∂Q. We define E_τ by the closed simplicial star of the barycenter of τ in the second barycentric subdivision of Q. We define E_σ by a 2-disk in the interior of σ that does not intersect E_ρ for any 1-simplex ρ of $Q - \partial Q$. We will define ψ_X on $\overline{D^2 - \bigcup E_\tau - \bigcup E_\sigma}$ so that the paths $\psi_X|_{\partial E_\tau}$ and $\psi_X|_{\partial E_\sigma}$ are null-homotopic in U_m, where τ and σ range over all the 1- and 2-simplices of Q not contained in ∂Q.

Let u be a vertex of Q not lying in ∂Q. Since g is onto, $g^{-1}(\psi_L(u))$ is not empty and we can define a point $\psi_N(u) \in g^{-1}(\psi_L(u))$. Let ε be the simplex of $N_{\mathscr{V}'}$ that contains $\psi_N(u)$ in its interior. Then, the intersection of the elements of $\mathscr{V}'$ which correspond to the vertices of ε is not empty, and we define $\psi_X(u)$ by a point of this intersection. Note that $\tilde{f}(\psi_X(u))$ lies in the interior of a simplex of $N_{\mathscr{V}'}$ that contains ε. Moreover, if v is a vertex of Q and τ'' is a 1-simplex of the second barycentric subdivision of Q containing v and contained in the 1-skeleton of Q but not in ∂Q, we define $\psi_X(x) = \psi_X(v)$ for any $x \in \tau''$. Also, we define $\psi_N(u') = \tilde{f}(\psi_X(u'))$ for any vertex $u' \in \partial Q$.

We define $\tilde{h}_S$ (resp. h_S) by the composition of the projection of $|C|$ onto $|C_S|$ and $\tilde{h}$ (resp. h). Note that h_S is a simplicial map that sends a vertex

$[V_\Gamma^{S'}(x)]$ of $N_{\mathscr{V}'}$ to the vertex $\langle S'\rangle$ of C_S. Let p be a vertex of L. We define η_p by the unique simplex of C_S that contains $\tilde{h}_S(p)$ in its interior. Let us prove that for any path α' in $g^{-1}(\mathrm{St}(p,L))$ and any vertex $\langle S'\rangle$ of η_p we can cover α' by simplicial stars $\mathrm{St}([W],N_{\mathscr{V}'})$ such that $h_S([W])=\langle S'\rangle$. Fix a point $\alpha'(t)$ of α'. Then, $\alpha'(t)$ belongs to the interior of some simplex δ of $N_{\mathscr{V}'}$. Since δ intersects $g^{-1}(\mathrm{St}(p,L))$, $h_S(\delta)$ intersects $\tilde{h}_S(\mathrm{St}(p,L))$. But $\tilde{h}_S(\mathrm{St}(p,L))$ is contained in $\mathrm{St}(\tilde{h}_S(p),C_S)$, by an argument similar to that of h_Γ. So, there is a simplex ε of C_S containing $\tilde{h}_S(p)$ such that $h_S(\delta)\cap\mathring{\varepsilon}\neq\emptyset$. Then ε is a face of $h_S(\delta)$ and since η_p is a face of ε, η_p is also a face of $h_S(\delta)$. Hence there is a vertex $[W]$ of δ such that $h_S([W])=\langle S'\rangle$ for any vertex $\langle S'\rangle$ of η_p. Finally, since $[W]$ is a vertex of δ, $\alpha'(t)\in\mathrm{St}([W],N_{\mathscr{V}'})$.

Let τ be a 1-simplex of Q not contained in ∂Q. We will define ψ_X on ∂E_τ. Let σ and σ' be the 2-simplices of Q that contain τ. Let p (resp. p') be a vertex of L such that $\mathrm{St}(p,L)$ (resp. $\mathrm{St}(p',L)$) contains $\psi_L(\sigma)$ (resp. $\psi_L(\sigma')$). Also, let u and v be the vertices of τ. Since $\psi_N(u)$ and $\psi_N(v)$ belong to $g^{-1}(\mathrm{St}(p,L))$ that is path-connected, there is a path β in $g^{-1}(\mathrm{St}(p,L))$ from $\psi_N(u)$ to $\psi_N(v)$. Similarly, there is a path β' in $g^{-1}(\mathrm{St}(p',L))$ from $\psi_N(u)$ to $\psi_N(v)$. Let $\tilde{\beta}$ be the loop $\beta\cdot(\beta')^{-1}$ with base point $\psi_N(u)=\tilde{\beta}(0)$. Since $\mathrm{St}(p,L)\cap\mathrm{St}(p',L)\neq\emptyset$, p and p' span a 1-simplex ρ of L. Since $\tilde{h}(\rho)$ is a simplex of $\mathrm{sd}^2\,C$, there is a simplex ε of C such that $\tilde{h}(\mathring{\rho})\subset\mathring{\varepsilon}$. By the definition of barycentric subdivision, $\tilde{h}(p)$ or $\tilde{h}(p')$ belongs to $\mathring{\varepsilon}$. There is a simplex μ of $N_{\mathscr{V}'}$ such that $\mathring{\varepsilon}\subset h(\mathring{\mu})$. Thus $\tilde{h}_S(p)$ or $\tilde{h}_S(p')$ belongs to $h_S(\mathring{\mu})$. After exchanging p' with p if necessary, we may assume that $\tilde{h}_S(p)$ belongs to $h_S(\mathring{\mu})$. Then, $\eta_p=h_S(\mu)$ by the definition of η_p and $\eta_{p'}\subset h_S(\mu)$. So, $\eta_p\cap\eta_{p'}=\eta_{p'}$ and hence $\eta_p\cap\eta_{p'}$ is shown to be non-empty.

Let $\langle S'\rangle$ be a vertex of $\eta_p\cap\eta_{p'}$. We can cover the path $\tilde{\beta}$ by simplicial stars $\mathrm{St}([W],N_{\mathscr{V}'})$ such that $h_S([W])=\langle S'\rangle$, because $\tilde{\beta}$ is contained in $g^{-1}(\mathrm{St}(p,L))\cup g^{-1}(\mathrm{St}(p',L))$. Thus there are simplicial stars $\mathrm{St}([W_1],N_{\mathscr{V}'})$, $\dots,\mathrm{St}([W_k],N_{\mathscr{V}'})$ that cover $\tilde{\beta}$ and such that $h_S([W_i])=\langle S'\rangle$ for any i, and also there are paths $\tilde{\beta}_i$ in $\mathrm{St}([W_i],N_{\mathscr{V}'})$ for any i such that $\tilde{\beta}=\tilde{\beta}_1\dots\tilde{\beta}_k$. Since $\mathrm{St}([W_i],N_{\mathscr{V}'})$ intersects $\mathrm{St}([W_{i+1}],N_{\mathscr{V}'})$, W_i intersects W_{i+1} and then we can define a path $\tilde{\gamma}_i$ in W_i such that $\tilde{\gamma}_i(1)=\tilde{\gamma}_{i+1}(0)$. Since $\psi_N(u)=\tilde{\beta}(0)$ belongs to $\mathrm{St}([W_1],N_{\mathscr{V}'})$, $\tilde{f}(\psi_X(u))\in\mathrm{St}([W_1],N_{\mathscr{V}'})$ and then $\psi_X(u)\in W_1$. Similarly, $\psi_X(v)\in W_l$ for some $l\leq k$. There are a path γ_0 in W_1 from $\psi_X(u)$ to $\tilde{\gamma}_1(0)$ and a path γ_l in W_l from $\tilde{\gamma}_l(1)$ to $\psi_X(v)$. Let $\tilde{\gamma}=\gamma_0\tilde{\gamma}_1\dots\tilde{\gamma}_l\gamma_l(\gamma_l)^{-1}\tilde{\gamma}_{l+1}\dots\tilde{\gamma}_k(\gamma_0)^{-1}$. We can define ψ_X on ∂E_τ by the loop $\tilde{\gamma}$.

Thus, we have defined ψ_X on ∂E_τ for any 1-simplex τ of $Q-\partial Q$. If τ is a 1-simplex of ∂Q we define $E_\tau=\emptyset$. Let σ be a 2-simplex of Q and τ_1, τ_2 and τ_3 its 1-faces. Since $\overline{\sigma-(E_\sigma\cup E_{\tau_1}\cup E_{\tau_2}\cup E_{\tau_3})}$ is homeomorphic to a cylinder over S^1, and ψ_X is already defined on one of the components of its boundary $\partial(\overline{\sigma-(E_{\tau_1}\cup E_{\tau_2}\cup E_{\tau_3})})$, we can extend ψ_X to this cylinder in the obvious way.

Let us prove that $\psi_X(\partial E_\tau) \subset U_{m''}$ and $\psi_X(\partial E_\sigma) \subset U_{m''}$ for any 1-simplex τ and 2-simplex σ of Q not contained in ∂Q. Let $[W_1], \dots, [W_k]$ be the vertices of $N_{\mathscr{V}'}$ and p and p' the vertices of L used to define ψ_X on ∂E_τ. For any i, $g(\mathrm{St}([W_i], N_{\mathscr{V}'}))$ intersects $\mathrm{St}(p, L)$ or $\mathrm{St}(p', L)$, and these simplicial stars intersect $U^L_{m''+2}$, then $W_i \subset U_{m''}$, by the same argument as in (ii). Thus $\psi_X(\partial E_\tau) \subset U_{m''}$. On the other hand, for a 2-simplex σ we see that $\psi_X(\partial E_\sigma) \subset \psi_X(\partial E_{\tau_1} \cup \partial E_{\tau_2} \cup \partial E_{\tau_3} \cup \partial Q) \subset U_{m''}$, where τ_1, τ_2 and τ_3 are the 1-faces of σ. As a consequence, we have defined a map $\psi_X : \overline{D^2 - \bigcup E_\tau - \bigcup E_\sigma} \to U_{m''}$ that coincides with α^X on ∂D^2.

Let τ be a 1-simplex of Q not contained in ∂Q. Let $[W_j]$ be the vertices of $N_{\mathscr{V}'}$ and $\langle S' \rangle$ the vertex of C_S used to define ψ_X on ∂E_τ. Then $\bigcup_j W_j \subset \tilde{A}_{S'}$, because $h_S([W_j]) = \langle S' \rangle$ for any j. So, $\psi_X(\partial E_\tau) \subset A_l$ for any $l \in S'$ and thus $\psi_X|_{\partial E_\tau}$ is null-homotopic in U_m.

Let us check that for any 2-simplex σ of Q there is an A_r that contains $\psi_X(\partial E_\sigma)$, and thus $\psi_X|_{\partial E_\sigma}$ is null-homotopic in U_m. There is a simplicial star $\mathrm{St}(p, L)$ such that $\psi_L(\sigma) \subset \mathrm{St}(p, L)$. Let τ_1, τ_2 and τ_3 be the 1-faces of σ. If $\tau_i \not\subset \partial Q$, let $[W^i_j]$ be the vertices of $N_{\mathscr{V}'}$ and $\langle S'_i \rangle$ the vertex of C_S used to define ψ_X on ∂E_{τ_i}. If $\tau_i \subset \partial Q$, since the path $\tilde{f} \circ \psi_X|_{\tau_i}$ is contained in $g^{-1}(\mathrm{St}(p, L))$, then for any vertex $\langle S'_i \rangle$ of η_p there are simplicial stars $\mathrm{St}([W^i_1], N_{\mathscr{V}'}), \dots, \mathrm{St}([W^i_k], N_{\mathscr{V}'})$ that cover $\tilde{f} \circ \psi_X|_{\tau_i}$ and such that $h_S([W^i_j]) = \langle S'_i \rangle$. Let $\langle S_1 \rangle, \dots, \langle S_t \rangle$ be the vertices of η_p. Then, $\bigcup_{i,j} W^i_j \subset \tilde{A}_{S_1} \cup \dots \cup \tilde{A}_{S_t}$. Since η_p is the image of a simplex of $N_{\mathscr{V}'}$ by h_S, there is a permutation $\{S_{i_1}, \dots, S_{i_t}\}$ of $\{S_1, \dots, S_t\}$ such that $S_{i_1} \subset \dots \subset S_{i_t}$. So, $\tilde{A}_{S_1} \cup \dots \cup \tilde{A}_{S_t} \subset A_r$ for any r in S_{i_1}. Thus, $\psi_X(\partial E_\sigma) \subset A_r$ for any $r \in S_{i_1}$.

Step three: The paracompact case.

We have proved the result for a σ-compact space X. Now we will study the case when X is paracompact. Since X is a Hausdorff, locally compact, and paracompact space, it can be decomposed as a disjoint union of σ-compact subspaces (see [3], I, p. 70, Theorem 5). Let us denote them by X_α with $\alpha \in \mathscr{A}$. Each X_α is σ-compact, locally compact, locally pathwise-connected, and Hausdorff. Also, since X_α is open and closed in X, an open cover $\{A_k \mid 0 \le k \le n\}$ of X by Π-categorical sets induces an open cover $\{A_k \cap X_\alpha \mid 0 \le k \le n\}$ of X_α by Π-categorical sets. Thus, Π-cat $X_\alpha \le n$, and applying the previous steps we obtain complexes $(L_\alpha, |L_\alpha|)$ and maps $f_\alpha : X_\alpha \to |L_\alpha|$ for every α. We define the complex $(L, |L|)$ by the disjoint union of the complexes $(L_\alpha, |L_\alpha|)$, and the map $f : X \to |L|$ by the union of the f_α.

For any α let $\mathscr{U}^\alpha$ be the system of infinity neighbourhoods of X_α and $\mathscr{V}'_\alpha$ the covering of X_α by connected sets used to define $(L_\alpha, |L_\alpha|)$. We define the cover $\mathscr{V}' = \bigcup_\alpha \mathscr{V}'_\alpha$ of X and using $\mathscr{V}'$ we define $N_{\mathscr{V}'}$, C, $\tilde{f}$, g, h and $\tilde{h}$ as in the σ-compact case. The $(L, |L|)$ and f defined in this way coincide with the simplicial complex and map obtained in the last paragraph. Now we define a system of infinity neighbourhoods $\mathscr{U}$ of X given by the sets $U = \bigcup_\alpha U^\alpha_{m_\alpha}$ where $U^\alpha_{m_\alpha} \in \mathscr{U}^\alpha$ and all but a finite number of indices m_α are equal to 1. Using this system of infinity neighbourhoods $\mathscr{U}$ and cover $\mathscr{V}'$ we can prove the isomorphism condition for f in a similar way to the σ-compact case. It suffices to replace U_m with $\bigcup_\alpha U^\alpha_{m_\alpha}$ and then in the condition (ii) U_{m+2} with $\bigcup_\alpha U^\alpha_{k_\alpha}$, where $k_\alpha = m_\alpha + 2$ if $m_\alpha \neq 1$ and 1 otherwise, and in the condition (i) we replace $U_{m''}$ and $U_{m''+2}$ by $\bigcup_\alpha U^\alpha_{r_\alpha}$ and $\bigcup_\alpha U^\alpha_{s_\alpha}$, respectively, where $s_\alpha = r_\alpha + 2$ if $r_\alpha \neq 1$ and 1 otherwise.

This concludes the proof of the theorem. $\square$

Example 4.3. There is a 4-dimensional manifold X with $\mathrm{p}\tilde{\pi}_1\text{-cat}\, X = \mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\, X = 3$ such that there is no proper map $f : X \to |L|$ for any locally finite simplicial complex $(L, |L|)$ of dimension 3 that induces an isomorphism of fundamental pro-groups.

Let A be an aspherical homology 3-sphere, which can be constructed by Jørgensen-Thurston's hyperbolic Dehn surgery theory, see [15]. We define X by the product of A and the half-line. We choose the system of infinity neighbourhoods of X formed by $U_m = A \times (m, \infty)$ for any integer $m \geq 0$. The space X is Hausdorff, locally compact, paracompact and locally path-connected.

The L-S π_1-category of A is 3, because its fundamental group is not free (see [10]). Let $\{A_i \mid 0 \leq i \leq 3\}$ be an open cover of A by π_1-contractible subsets. Since A is normal, it is easy to get an open cover $\{B_i \mid 0 \leq i \leq 3\}$ of A such that the closure $\overline{B_i} \subset A_i$ and hence $\overline{B_i}$ is π_1-contractible in A for any i. The sets $X_i = B_i \times [0, \infty)$ form an open cover of X that is $\mathrm{p}\tilde{\pi}_1$-categorical in X. Indeed, since the inclusion map from $\overline{X_i}$ to X is the product of the inclusion map from $\overline{B_i}$ to A and the identity map of the half-line, $\overline{X_i}$ is $\tilde{\pi}_1$-contractible in X. Analogously, for each $m \geq 0$ the inclusion map from $\overline{X_i} \cap U_m = \overline{B_i} \times (m, \infty)$ to U_m is the product of the inclusion map from $\overline{B_i}$ to A and the identity map of $[m, \infty)$, and then X_i is $\mathrm{p}\tilde{\pi}_1^\infty$-categorical in X. Thus $\mathrm{p}\tilde{\pi}_1^\infty\text{-cat}\, X \leq \mathrm{p}\tilde{\pi}_1\text{-cat}\, X \leq 3$.

If there were a 3-dimensional simplicial complex $(L, |L|)$, a map $f : X \to |L|$ and a subset M of X verifying the conditions of Theorem 4.1 for $\mathrm{p}\tilde{\pi}_1^\infty$, there will be a morphism η of pro-groups such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{p}\tilde{\pi}_1^\infty(X, M) \arrow[rr, "="] \arrow[dr, "\mathrm{p}\tilde{\pi}_1^\infty(f)"'] & & \mathrm{p}\tilde{\pi}_1^\infty(X, M) \\
& \mathrm{p}\tilde{\pi}_1^\infty(|L|, f(M)) \arrow[ur, "\eta"'] &
\end{tikzcd}$$

We may assume that M is countable. Now we will define a proper map from the underlying space of a subcomplex of L to X. Let $\mathscr{G}'$ be the category of groups and tow-$\mathscr{G}'$ the category of towers of groups, that is, the category of inverse systems of groups for which the index set is $\mathbf{N}$. We can define an equivalence relation in the set of proper maps from an infinity neighbourhood to another in the following way: two maps defined in infinity neighbourhoods are equivalent if they coincide in a smaller infinity neighbourhood. We call the equivalence classes *germs*. Two germs are *homotopic at infinity* if two representatives are properly homotopic in an infinity neighbourhood. These definitions can be extended to properly based spaces. In this case the germs must preserve the base ray and the homotopies must be homotopies relative to the base ray. In [4] it is proved (Proposition 3.5) the following:

Lemma 4.4. *Let (P, α) be a properly based connected locally compact one-ended polyhedron and (Q, β) a properly based space that is properly aspherical at infinity, that is, the pro-groups* pro-$\pi_n(Q, \beta)$ *are trivial in* tow-$\mathscr{G}'$ *for any $n \geq 2$. Then, the fundamental pro-group functor induces a natural bijection*

$$[P, Q]_{\infty}^{\mathbf{R}_+} \cong \mathrm{Hom}(\text{pro-}\pi_1(P, \alpha), \text{pro-}\pi_1(Q, \beta))$$

where "Hom" stands for the morphism set in tow-$\mathscr{G}'$ *and $[P, Q]_{\infty}^{\mathbf{R}_+}$ is the set of proper homotopy classes of germs relative to the base ray.*

Let L_0 be the subcomplex of L formed by the simplices that intersect $f(X)$ and their faces. Since A is path-connected, our infinity neighbourhoods of X are also path-connected, and thus X is one-ended. This implies that $|L_0|$ is one-ended, because $f : X \to |L_0|$ is proper. Also, $|L_0|$ is path-connected, because $f(X)$ is path-connected. If $M = \{x_n \mid n \geq 0\}$ then there is a ray γ in X such that $\gamma(n) = x_n$ for any n. The image of this ray by f is a ray γ_0 in $|L_0|$. Since $\overline{U_m} = A \times [m, \infty)$ for any m, $\pi_n(\overline{U_m}, (p, q)) = \pi_n(A, p) \oplus \pi_n([m, \infty), q)$ for any $(p, q) \in \overline{U_m}$ and $n \geq 1$. Thus, X is properly aspherical at infinity. Hence, $(|L_0|, \gamma_0)$ and (X, γ) satisfy the hypothesis of the Lemma 4.4. Since the isomorphisms of $\mathrm{p}\tilde{\pi}_1^{\infty}$ fundamental pro-groups of the above diagram induce a corresponding diagram of isomorphisms of pro-π_1 fundamental pro-groups, applying this lemma we obtain a germ g making the following diagram of germs commutative up to homotopy at infinity relative to the base rays:

$$\begin{tikzcd}
(X, \gamma) \arrow[rr, "="] \arrow[dr, "f"'] & & (X, \gamma) \\
& (|L_0|, \gamma_0) \arrow[ur, "g"'] &
\end{tikzcd}$$

where the $\nrightarrow$ arrows denote germs. There is a corresponding diagram for the end homology. This homology is defined for locally finite CW-complexes as follows. If $C_m(Y)$ and $C_m^\infty(Y)$ denote the chain complexes of the cellular finite chains and cellular infinite chains with coefficients in $\mathbf{Z}$ over a locally finite CW-complex Y, there is an exact sequence:

$$0 \to C_m(Y) \to C_m^\infty(Y) \to C_m^\infty(Y)/C_m(Y) \to 0$$

that induces a long exact sequence in homology:

$$\cdots \to H_m Y \to H_m^\infty Y \to H_m(C_*^\infty(Y)/C_*(Y)) \to \cdots$$

The homology $H_{m-1}^e Y = H_m(C_*^\infty(Y)/C_*(Y))$ is called the *end homology* and is an invariant of homotopy at infinity type. The homology $H_m^\infty Y$ is called the *infinite homology* and is a proper homotopy invariant, see [9] or [13] for details. As we wrote above, there is a corresponding diagram for the end homology:

$$\begin{tikzcd} H_3^e\overline{U_1} \arrow[rr, "="] \arrow[dr, "f_*"'] & & H_3^e\overline{U_1} \\ & H_3^e|L_0| \arrow[ur, "g_*"'] & \end{tikzcd}$$

Note that $\overline{U_1}$ is a locally finite polyhedron and thus $H_3^e\overline{U_1}$ is well-defined. Since $\dim|L_0| \le 3$, $H_3^e|L_0| = H_4(C_*^\infty(|L_0|)/C_*(|L_0|)) = 0$ and thus $H_3^e\overline{U_1} = 0$. On the other hand, there is an exact sequence:

$$0 \to \varprojlim{}^1 H_4\overline{U_i} \to H_3^e\overline{U_1} \to \varprojlim H_3\overline{U_i} \to 0$$

where the limits are taken on the index $i \in \mathbf{N}$ (see [9]). Since $\overline{U_i}$ is of the same homotopy type as A, $H_4\overline{U_i} = 0$ for any i, and the first derived limit is 0. Thus, $H_3^e\overline{U_1}$ is isomorphic to $\varprojlim H_3\overline{U_i}$. But, since $H_3\overline{U_i} = H_3 A = \mathbf{Z}$, and the bonding maps are the identity, this limit is $\mathbf{Z}$, which is a contradiction.

As a byproduct we obtain that $\mathrm{p}\tilde{\pi}_1^\infty$-cat $X = 3$ and thus $\mathrm{p}\tilde{\pi}_1$-cat $X = 3$, because if it is ≤ 2 then there are a simplicial complex $(L, |L|)$ of dimension $d \le 3$ and a map f satisfying the conditions of Theorem 4.1, and using the 3-dimensional complex $L \times [0,1]^{3-d}$ we reach a contradiction as above.

Moreover, we may consider a contractible space $CA \cup X$ instead of X, where CA is the topological cone of A, and prove that $\mathrm{p}\tilde{\pi}_1^\infty$-cat $CA \cup X =$ $\mathrm{p}\tilde{\pi}_1$-cat $CA \cup X = 3$. Furthermore, we can take a compact contractible (topological) manifold Y with $\partial Y = A$ instead of CA to get an open contractible 4-manifold with $\mathrm{p}\tilde{\pi}_1^\infty$-cat $= \mathrm{p}\tilde{\pi}_1$-cat $= 3$.

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