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Oscillation and nonoscillation criteria for half-linear second order differential equations

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ABSTRACT. We investigate oscillatory properties of the second order half-linear differential equation

$$(*)\qquad (r(t)\Phi(y'))' + c(t)\Phi(y) = 0, \qquad \Phi(s) := |s|^{p-2}s, \qquad p > 1,$$

viewed as a perturbation of a nonoscillatory equation of the same form

$$(r(t)\Phi(y'))' + \tilde{c}(t)\Phi(y) = 0.$$

Conditions on the difference $c(t) - \tilde{c}(t)$ are given which guarantee that equation $(*)$ becomes oscillatory (remains nonoscillatory).

1. Introduction

In this paper we investigate oscillatory properties of the half-linear second order differential equation

$$(1)\qquad (r(t)\Phi(y'))' + c(t)\Phi(y) = 0, \qquad \Phi(s) := |s|^{p-2}s,$$

where $p > 1$, $t \in I := [T, \infty)$, r, c are real-valued continuous functions and $r(t) > 0$ in I. Oscillation theory of half-linear equations (1) attracted a considerable attention in the recent years, see e.g. [1, 2, 4, 7, 15, 16, 18, 19, 21, 25] and the reference given therein. In these papers it was shown that many of the (non)oscillation criteria for the linear Sturm-Liouville second order differential equation

$$(2)\qquad (r(t)y')' + c(t)y = 0$$

(which is a special case $p = 2$ of (1)) extend to half-linear equation (1).

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In the majority of these oscillation criteria, equation (1) is viewed as a perturbation of the *one-term* differential equation

$$(r(t)\Phi(y'))' = 0,$$

and (non)oscillation criteria are formulated in terms of the integrals

$$\int_t^\infty c(s)ds \quad \text{if} \quad \int^\infty r^{1-q}(t)dt = \infty, \qquad q = \frac{p}{p-1}$$

or

$$\int_t^\infty \left(\int_s^\infty r^{1-q}(\tau)d\tau\right)^p c(s)ds \qquad \text{if} \qquad \int^\infty r^{1-q}(t)dt < \infty,$$

see e.g. [15, 16].

In this paper we use a more general idea, we investigate equation (1) as a perturbation of the *two-term* equation of the same form

$$(r(t)\Phi(y'))' + \tilde{c}(t)\Phi(y) = 0 \tag{3}$$

and the obtained criteria are formulated in terms of the integral

$$\int (c(t) - \tilde{c}(t))\tilde{y}^p(t)dt$$

(limits in this integral depend on the particular situation), where $\tilde{y}$ is the so-called principal solution of (3). In the case $\tilde{c}(t) \equiv 0$ and $\int^\infty r^{1-q}(t)dt = \infty$, this principal solution is $\tilde{y}(t) \equiv 1$, while if $\int^\infty r^{1-q}(t)dt < \infty$, this solution is $\tilde{y}(t) = \int_t^\infty r^{1-q}(s)ds$, i.e., our oscillation criteria reduce to those proved in [15, 16] when $\tilde{c}(t) \equiv 0$.

The idea to compare oscillatory properties of equation (1) with another two-term equation was used for the first time by Elbert [9], where the equation

$$(\Phi(y'))' + c(t)\Phi(y) = 0 \tag{4}$$

is regarded as a perturbation of the half-linear Euler differential equation

$$(\Phi(y'))' + \frac{\gamma_p}{t^p}\Phi(y) = 0, \qquad \gamma_p = \left(\frac{p-1}{p}\right)^p, \tag{5}$$

and it is proved that (4) is oscillatory provided

$$\int^\infty \left(c(t) - \frac{\gamma_p}{t^p}\right) t^{p-1}\, dt = \infty. \tag{6}$$

This result was complemented in [4], where it was shown that if the integral in (6) is convergent, then (4) is oscillatory provided

$$\lim_{t\to\infty} \log t \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, dt > 2\left(\frac{p-1}{p}\right)^{p-1}. \tag{7}$$

In this paper we insert these criteria into a general framework, where the function $\tilde{c}$ is any function such that equation (3) is nonoscillatory. Among others, we prove that the constant $2\left(\frac{p-1}{p}\right)^{p-1}$ in (7) can be replaced by the better one $\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. Our investigation is based on the recently established Picone's identity (cf. [14]) and the "quadratization" of a certain nonlinear term appearing in this identity, coupled with the classical tools as the variational principle and the Riccati technique. An important role is also played by the concept of the principal solution of half-linear equation (1) (cf. [11, 26]).

The paper is organized as follows. In the next section we give some auxiliary results concerning the oscillation theory of half-linear equations (1) and Section 3 is devoted to the main results of our paper—new oscillation and nonoscillation criteria for (1). The last part of Section 3 contains some remarks about possible extensions of the results of this paper.

2. Auxiliary results

Let us start with the principal concepts of the half-linear oscillation theory. These concepts are very similar to the linear case $p = 2$. Two points t_1, t_2 are said to be *conjugate* relative to (1) if there exists a nontrivial solution y of this equation such that $y(t_1) = 0 = y(t_2)$. Equation (1) is said to be *disconjugate* in an interval $I \subset \mathbf{R}$ if there exists no pair of points of this interval which are conjugate relative to (1), in the opposite case (1) is said to be *conjugate* in I. It is known, see Elbert [8] and Mirzov [25], that the basic facts of the linear Sturmian theory extend almost verbatim to (1). Consequently, if I is an interval of the form $I = [a, \infty)$, then any solution of (1) has either infinitely many or only a finite number of zeros in I. Hence, similar to the linear case, equations (1) can be classified as *oscillatory* or *nonoscillatory*.

Concerning the unique solvability of (1), given $A, B \in \mathbf{R}$ and $t_0 \in I = (a, b)$, I being an interval where the functions r, c are continuous and $r(t) > 0$, then the initial value problem $y(t_0) = A$, $r(t_0)\Phi(y'(t_0)) = B$ for (1) has the unique solution which is extensible up to a and b. This statement was proved by Elbert [8] using the generalized Prüfer transformation.

Our first result of the next section, Theorem 1, leans on the relationship between disconjugacy of (1) and the positivity of the functional

$$\mathscr{J}_p(y; a, b) = \int_a^b [r(t)|y'|^p - c(t)|y|^p]dt, \qquad y \in W_0^{1,p}(a, b),$$

in particular, (1) is disconjugate in $[a, b]$ if and only if $\mathscr{J}_p(y; a, b) > 0$ for every nontrivial $y \in W_0^{1,p}(a, b)$, for more details see Li and Yeh [19] and also Mařík [22].

The next results, oscillation and nonoscillation criteria given in Theorems 2, 3 are based on the Riccati technique consisting in the fact that if y is a solution of (1) for which $y(t) \neq 0$ in some interval, then the Riccati variable $w = \frac{r\Phi(y')}{\Phi(y)}$ solves the generalized Riccati equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \qquad q = \frac{p}{p-1}, \tag{8}$$

in this interval. In particular, equation (1) is disconjugate in the interval $[a,b]$ if and only if there exists a solution w of (8) which is defined in the whole interval $[a,b]$. Recall also the concept of the so-called *distinguished and principal* solutions of (8) and (1), respectively, introduced in [26] and independently in [11]. An alternative approach (but equivalent) to these concepts can be found in [5]. Suppose that (1) is nonoscillatory. Then there exists at least one solution of (8) which is extensible up to infinity. Among all solutions having this property there exists the minimal one, called (by analogue with the linear case) *distinguished solution* (another terminology is *eventually minimal solution*), which is minimal in the following sense. If $\tilde{w}$ is the eventually minimal solution of (8) and w is any other solution which exists on some interval $[T,\infty)$, then $w(t) > \tilde{w}(t)$ in this interval. Recall that the eventually minimal solution $\tilde{w}$ is constructed as follows. Suppose that (1) is disconjugate on $[T,\infty)$ and $b > T$. Let y_b be the solution of (1) given by the initial condition $y_b(b) = 0$, $y_b'(b) = -1$. Then $y_b(t) > 0$ for $[T,b)$ and denote $w_b = r\Phi(y_b'/y_b)$ the corresponding solution of (8). The eventually minimal solution $\tilde{w}$ is then given by the formula

$$\tilde{w}(t) = \lim_{b\to\infty} w_b(t),$$

where the convergence $w_b \to \tilde{w}$ is uniform on every compact subinterval of $[T,\infty)$.

The *principal solution* of (1) is defined as the solution of this equation which is associated with the distinguished solution w of (8), i.e., it is given by the formula

$$y(t) = \exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(w(s))ds\right\},$$

where w is the distinguished solution of (8) and Φ^{-1} is the inverse function of Φ.

The link line between the Riccati and variational technique is the so-called Picone identity which reads in the half-linear case as follows (here we present this identity in a simplified form since we will not need it in its full generality).

Lemma 1 ([14]). *Suppose that w is a solution of (8) which is defined in the whole interval $I = [a,b]$. Then for any $y \in W^{1,p}(a,b)$ the following identity holds:*

$$\mathscr{I}_p(y;a,b) = w(t)|y|^p|_a^b + p\int_a^b \left[\frac{1}{p}r(t)|y'|^p - w(t)\Phi(y)y' + \frac{1}{q}r^{1-q}(t)|w(t)|^q|y|^p\right]dt$$
$$= w(t)|y|^p|_a^b + p\int_a^b r^{1-q}(t)P(r^{q-1}y', w\Phi(y))dt,$$

where

$$P(u,z) = \frac{|u|^p}{p} - uz + \frac{|z|^q}{q} \geq 0 \tag{9}$$

for any $u, z \in \mathbf{R}$ *with equality if and only if* $z = \Phi(u)$.

We will also need the following refinement of the relationship between nonoscillation of (1) and solvability of (8).

Lemma 2. *Suppose that equation* (3) *is nonoscillatory and* h *is an eventually positive solution of this equation. Further suppose that the integral* $\int^\infty (c(s) - \tilde{c}(s))h^p(s)ds$ *converges and denote*

$$C(t) = \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)ds.$$

Then equation (1) *is nonoscillatory provided there exists a differentiable function* v *which for large* t *satisfies the Riccati-type inequality*

$$v' \leq -p\left[\frac{1}{q}\left|\frac{v + C(t)}{h(t)}\right|^q r^{1-q}(t) - h'(t)\left(\frac{v + C(t)}{h(t)}\right)\right] - \tilde{c}(t)h^p(t). \tag{10}$$

Proof. Suppose that v satisfies (10) and exists on some interval $[T, \infty)$. Then by the standard theory of differential inequalities (see e.g. [17]) there exists a solution $\tilde{v} : [T, \infty) \to \mathbf{R}$ of the differential equation

$$\tilde{v}' = -p\left[\frac{1}{q}\left|\frac{\tilde{v} + C(t)}{h(t)}\right|^q r^{1-q}(t) - h'(t)\left(\frac{\tilde{v} + C(t)}{h(t)}\right)\right] - \tilde{c}(t)h^p(t)$$

satisfying the inequality $\tilde{v}(t) \geq v(t)$ for large t, i.e., $\tilde{v}$ also exists on the whole interval $[T, \infty)$ (the fact that $\tilde{v}$ cannot blow up to infinity can be proved using the same argument as in the linear case, see e.g. [31]). Now, if $w = \frac{\tilde{v}+C}{h^p}$, one can verify directly that this function satisfies (8) and hence (1) is nonoscillatory. $\square$

3. Main results

In this section we present the main results of the paper. We investigate oscillatory properties of equation (1) viewed as a perturbation of (nonoscillatory) equation (3). We start with a statement proved using the *variational principle* which requires (in contrast to other statements of this section) *no additional restrictions* on the functions r, c in (1).

Theorem 1. *Suppose that h is the principal solution of (nonoscillatory) equation* (3) *and*

$$\int^{\infty}(c(t)-\tilde{c}(t))h^{p}(t)dt := \lim_{b\to\infty}\int^{b}(c(t)-\tilde{c}(t))h^{p}(t)dt = \infty. \tag{11}$$

Then equation (1) *is oscillatory.*

Proof. According to the relationship between disconjugacy of (1) and positivity of the functional $\mathscr{I}_p$ mentioned in Section 2, to prove that (1) is oscillatory, it suffices to find for any $T \in \mathbf{R}$ a function $y \in W^{1,p}(T,\infty)$, with a compact support in (T,∞), such that $\mathscr{I}_p(y;T,\infty) \le 0$. Hence, let $T \in \mathbf{R}$ be arbitrary and $T < t_0 < t_1 < t_2 < t_3$ (these points will be specified later). Define the test function y as follows.

$$y(t) = \begin{cases} 0 & T \le t \le t_0, \\ f(t) & t_0 \le t \le t_1, \\ h(t) & t_1 \le t \le t_2, \\ g(t) & t_2 \le t \le t_3, \\ 0 & t_3 \le t < \infty, \end{cases}$$

where f, g are solutions of (3) given by the boundary conditions $f(t_0) = 0$, $f(t_1) = h(t_1)$, $g(t_2) = h(t_2)$, $g(t_3) = 0$. Recall that these solutions exist if t_0 is sufficiently large. Indeed, disconjugacy of (3) for large t implies that its solution y given by $y(t_0) = 0$, $y'(t_0) > 0$ satisfies $y(t) > 0$ for $t > t_0$ and by the homogeneity property of the solution space of (3) $f(t) = y(t)(h(t_1)/y(t_1))$. The existence of the solution g is proved using the same argument. Denote

$$w_f := \frac{r\Phi(f')}{\Phi(f)}, \qquad \tilde{w} := \frac{r\Phi(h')}{\Phi(h)}, \qquad w_g := \frac{r\Phi(g')}{\Phi(g)},$$

i.e., w_f, w_g, $\tilde{w}$ are solutions of (8) generated by f, g, h respectively. Note that nonoscillation of (3) implies that $f(t) > 0$, $g(t) > 0$ on $(t_0, t_1]$, $[t_2, t_3)$, respectively, if t_0 is sufficiently large. Using integration by parts we have

$$\begin{aligned} \mathscr{I}_p(f;t_0,t_1) &= \int_{t_0}^{t_1}\{r(t)|f'(t)|^p - \tilde{c}(t)|f(t)|^p\}dt - \int_{t_0}^{t_1}(c(t)-\tilde{c}(t))|f(t)|^p dt \\ &= r(t)\Phi(f'(t))f(t)|_{t_0}^{t_1} - \int_{t_0}^{t_1}(c(t)-\tilde{c}(t))f^p(t)dt \\ &= f^p(t_1)w_f(t_1) - \int_{t_0}^{t_1}(c(t)-\tilde{c}(t))f^p(t)dt. \end{aligned}$$

Similarly we have

$$\mathcal{J}_p(h; t_1, t_2) = h^p(t)\tilde{w}(t)|_{t_1}^{t_2} - \int_{t_1}^{t_2} (c(t) - \tilde{c}(t))h^p(t)dt$$

and

$$\mathcal{J}_p(g; t_2, t_3) = -g^p(t_2)w_g(t_2) - \int_{t_2}^{t_3} (c(t) - \tilde{c}(t))g^p(t)dt.$$

Consider the integral $\int_{t_2}^{t_3}(c - \tilde{c})g^p\,dt$. The function g/h is monotonically decreasing on (t_2, t_3) with $(g/h)(t_2) = 1$, $(g/h)(t_3) = 0$. Indeed, if $(g/h)'(t) = 0$ for some $t \in (t_2, t_3)$, then $(g'h - gh')(t) = 0$ and hence $\tilde{w}(t) = w_g(t)$. But this contradicts the unique solvability of (8), hence $(g/h)'(t) < 0$ for $t \in (t_2, t_3)$. Therefore, by the second mean value of integral calculus, there exists a $\xi \in (t_2, t_3)$ such that

$$\int_{t_2}^{t_3} (c(t) - \tilde{c}(t))g^p(t)dt = \int_{t_2}^{t_3} (c(t) - \tilde{c}(t))h^p(t)\left(\frac{g(t)}{h(t)}\right)^p dt$$
$$= \int_{t_2}^{\xi} (c(t) - \tilde{c}(t))h^p(t)dt.$$

Now, let $T < t_0 < t_1$ be fixed (and sufficiently large), and denote

$$K = h^p(t_1)[w_f(t_1) - \tilde{w}(t_1)] - \int_{t_0}^{t_1} (c(t) - \tilde{c}(t))f^p(t)dt.$$

Then, using the fact that $f(t_1) = h(t_1)$, $h(t_2) = g(t_2)$, we have

$$\mathcal{J}_p(y; T, \infty) = K - \int_{t_1}^{\xi} (c(t) - \tilde{c}(t))h^p(t)dt + h^p(t_2)[\tilde{w}(t_2) - w_g(t_2)]. \tag{12}$$

Now, if $\varepsilon > 0$ is arbitrary, then according to (11) t_2 can be chosen in such a way that $\int_{t_1}^{t}(c(s) - \tilde{c}(s))h^p(s)ds > K + \varepsilon$ whenever $t > t_2$. Finally, since h is the principal solution of (3), i.e., $\tilde{w}$ is the eventually minimal solution of (8), by its construction described above Lemma 1 we have (observe that w_g actually depend also on t_3)

$$\lim_{t_3 \to \infty} h^p(t_2)[\tilde{w}(t_2) - w_g(t_2)] = 0,$$

hence the last summand in (12) is less than ε if t_3 is sufficiently large. Consequently, $\mathcal{J}_p(y; t_0, t_3) < 0$ if t_0, t_1, t_2, t_3 are chosen in the above specified way. □

Remark 1. As we have already mentioned in the introductory section, the previous theorem is a generalization of the oscillation criterion of Elbert [9] which claims that the equation

$$(\Phi(y'))' + c(t)\Phi(y) = 0 \tag{13}$$

is oscillatory provided

$$\int^{\infty}\left(c(t)-\frac{\gamma_p}{t^p}\right)t^{p-1}\,dt=\infty,\qquad \gamma_p=\left(\frac{p-1}{p}\right)^p. \tag{14}$$

Indeed, if $r(t)\equiv 1$ and $\tilde{c}(t)=\gamma_p t^{-p}$, i.e., (3) reduces to the generalized Euler equation (5) with the so-called *critical coefficient* γ_p, then $h(t)=t^{(p-1)/p}$ is the principal solution of (5) (see e.g. [10]) and (11) reduces to (14). A detailed investigation of (13) viewed as a perturbation of Euler equation (5) can be found in [12].

Now we turn our attention to the criteria proved using the Riccati technique. Compared with the previous theorem, these statements apply also to the case when $\int^{\infty}(c-\tilde{c})h^p$ is convergent, on the other hand, some additional technical assumptions are needed.

Theorem 2. *Let* $\int^{\infty} r^{1-q}(t)dt=\infty$,

$$\int^{\infty} c(t)dt \ \text{converges and} \ \int_t^{\infty} c(s)ds\geq 0 \qquad \text{for large } t. \tag{15}$$

Further suppose that equation (3) *is nonoscillatory and possesses a positive solution* h *satisfying*

(i) *The derivative* $h'(t)>0$ *for large* t,

(ii) *It holds*

$$\int^{\infty} r(t)(h'(t))^p dt=\infty,$$

(iii) *There exists a finite limit*

$$\lim_{t\to\infty} r(t)h(t)\Phi(h'(t)) =: L>0. \tag{16}$$

Denote by

$$G(t)=\int^t \frac{ds}{r(s)h^2(s)(h'(s))^{p-2}}$$

and suppose that the integral

$$\int^{\infty}(c(t)-\tilde{c}(t))h^p(t)dt=\lim_{b\to\infty}\int^b (c(t)-\tilde{c}(t))h^p(t)dt \tag{17}$$

is convergent. If

$$\liminf_{t\to\infty} G(t)\int_t^{\infty}(c(s)-\tilde{c}(s))h^p(s)ds>\frac{1}{2q} \tag{18}$$

then equation (1) *is oscillatory.*

Proof. Suppose, by contradiction, that (1) is nonoscillatory, i.e., there exists an eventually positive principal solution y of this equation. Denote by $w := r(t)\frac{\Phi(y')}{\Phi(y)}$. Then w satisfies the Riccati equation (8) and $w(t) \geq 0$ for large t. This follows from the half-linear version of the Hartman-Wintner theorem (cf. [20] or [27]), by this theorem w is also a solution of the integral equation

$$w(t) = \int_t^\infty c(s)ds + (p-1)\int_t^\infty r^{1-q}(s)|w(s)|^q ds$$

and hence $w(t) \geq 0$ for large t according to (15). Using the Picone identity for half-linear equations given in Lemma 1 we have

$$\begin{aligned}
&\int_T^t [r(s)|x'|^p - c(s)|x|^p]ds \\
&\quad = w(s)|x|^p|_T^t + \int_T^t \{r(s)|x'|^p - px'w(s)\Phi(x) + (p-1)r^{1-q}(s)|w(s)|^q|x|^p\}ds \\
&\quad = w(s)|x|^p|_T^t + p\int_T^t r^{1-q}(s)P(r^{q-1}x', w\Phi(x))ds
\end{aligned}$$

for any differentiable function x, where P is given by (9), and integration by parts yields

$$\begin{aligned}
\int_T^t [r(s)|x'|^p - c(s)|x|^p]ds &= \int_T^t [r(s)|x'|^p - \tilde{c}(s)|x|^p]ds - \int_T^t (c(s) - \tilde{c}(s))|x|^p ds \\
&= r(s)x\Phi(x')|_T^t - \int_T^t x[(r(s)\Phi(x'))' + \tilde{c}(s)\Phi(x)]ds \\
&\quad - \int_T^t (c(s) - \tilde{c}(s))|x|^p ds.
\end{aligned}$$

Substituting $x = h$ into the last two equalities (h being a solution of (3) satisfying the assumptions (i)–(iii) of theorem), we get

$$h^p(\tilde{w} - w)|_T^t = \int_T^t (c(s) - \tilde{c}(s))h^p\, ds + p\int_T^t r^{1-q}(s)P(r^{q-1}h', w\Phi(h))ds, \tag{19}$$

where $\tilde{w} = \frac{r\Phi(h')}{\Phi(h)}$. Since $w(t) \geq 0$ for large t, we have

$$\begin{aligned}
&h^p(t)\tilde{w}(t) + h^p(T)(w(T) - \tilde{w}(T)) \\
&\qquad \geq \int_T^t (c(s) - \tilde{c}(s))h^p\, ds + p\int_T^t r^{1-q}(s)P(r^{q-1}h', w\Phi(h))ds.
\end{aligned}$$

Letting $t \to \infty$, since $P(u, v) \geq 0$ and using (17), this means

$$\int^{\infty} r^{1-q}(t)P(r^{q-1}(t)h'(t), w(t)\Phi(h(t))dt < \infty. \tag{20}$$

Now, since (16), (17), (20) hold, from (19) it follows that there exists the limit

$$\lim_{t\to\infty} h^p(t)(w(t) - \tilde{w}(t)) =: \beta \tag{21}$$

and also the limit

$$\lim_{t\to\infty} \frac{w(t)}{\tilde{w}(t)} = \lim_{t\to\infty} \frac{h^p(t)w(t)}{h^p(t)\tilde{w}(t)} = \frac{L+\beta}{L}.$$

We have

$$\int^{\infty} r^{1-q}(t)P(r^{q-1}(t)h'(t), w(t)\Phi(h(t)))dt = \int^{\infty} r(t)(h'(t))^p P(1, w(t)/\tilde{w}(t))dt$$

$$= \int^{\infty} r(t)(h'(t))^p Q(w(t)/\tilde{w}(t))dt,$$

where

$$Q(\lambda) := \frac{1}{q}|\lambda|^q - \lambda + \frac{1}{p}.$$

Since $\int^{\infty} r(t)(h'(t))^p dt = \infty$ and $Q(\lambda) \ge 0$ with equality if and only if $\lambda = 1$, we have (in view of (20)) $\beta = 0$ in (21). Therefore, letting $t \to \infty$ in (19) and then replacing T by t, we have

$$h^p(t)(w(t) - \tilde{w}(t)) = C(t) + p\int_t^{\infty} r^{1-q}(s)P(r^{q-1}h', w\Phi(h))ds,$$

where $C(t) = \int_t^{\infty}(c(s) - \tilde{c}(s))h^p(s)ds$.

Concerning the function $P(u,z)$, we have for $u, z > 0$

$$P(u,z) = \frac{u^p}{p} - uz + \frac{z^q}{q} = u^p\left(\frac{1}{q}\frac{z^q}{u^p} - zu^{1-p} + \frac{1}{p}\right) = u^p Q(zu^{1-p}), \tag{22}$$

and

$$\lim_{\lambda\to 1} \frac{Q(\lambda)}{(\lambda-1)^2} = \frac{q-1}{2}. \tag{23}$$

Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$P(u,z) \ge \left(\frac{q-1}{2} - \varepsilon\right)u^p\left(\frac{z}{u^{p-1}} - 1\right)^2, \tag{24}$$

whenever $|zu^{1-p} - 1| < \delta$. If we denote

$$f(t) := h^p(t)(w(t) - \tilde{w}(t)), \qquad H(t) := \frac{1}{r(t)h^2(t)(h'(t))^{p-2}}$$

then using (22), (24) and the fact that $\beta = 0$ in (21), i.e., $\left|\frac{w(t)}{\tilde{w}(t)} - 1\right| < \delta$ for large t, we have

$$(25) \qquad f(t) \geq C(t) + \left(\frac{p(q-1)}{2} - \tilde{\varepsilon}\right)\int_t^\infty r(s)(h'(s))^p\left(\frac{w(s)}{\tilde{w}(s)} - 1\right)^2 ds$$
$$= C(t) + \left(\frac{q}{2} - \tilde{\varepsilon}\right)\int_t^\infty H(s)f^2(s)ds,$$

for large t, where $\tilde{\varepsilon} = p\varepsilon$. Multiplying (25) by $G(t)$ we get

$$(26) \qquad G(t)f(t) \geq G(t)C(t) + \left(\frac{q}{2} - \tilde{\varepsilon}\right)G(t)\int_t^\infty H(s)f^2(s)ds.$$

Inequality (26) together with (18) imply that there exists a $\tilde{\delta} > 0$ such that

$$(27) \qquad G(t)f(t) \geq \frac{1}{2q} + \tilde{\delta} + \left(\frac{q}{2} - \tilde{\varepsilon}\right)G(t)\int_t^\infty \frac{H(s)}{G^2(s)}[G(s)f(s)]^2 ds$$

for large t.

Suppose first that $\liminf_{t\to\infty} G(t)f(t) =: c < \infty$. According to (27) $c > 0$, and for every $\bar{\varepsilon} > 0$ we have $[G(t)f(t)]^2 > (1-\bar{\varepsilon})^2c^2$ for large t according to the definition of the number c. By (27) and the fact that $G(t)\int_t^\infty \frac{H(s)}{G^2(s)}\,ds = 1$, we have

$$(1+\bar{\varepsilon})c \geq \frac{1}{2q} + \tilde{\delta} + \left(\frac{q}{2} - \tilde{\varepsilon}\right)(1-\bar{\varepsilon})^2c^2.$$

Now, letting $\tilde{\varepsilon}, \bar{\varepsilon} \to 0$ we obtain

$$c \geq \frac{1}{2q} + \tilde{\delta} + \frac{q}{2}c^2 \Leftrightarrow \frac{q}{2}\left(c - \frac{1}{q}\right)^2 + \tilde{\delta} \leq 0,$$

a contradiction.

Finally, if

$$(28) \qquad \liminf_{t\to\infty} G(t)f(t) = \infty,$$

denote by $m(t) = \inf_{t\leq s}\{G(s)f(s)\}$. Then m is nondecreasing and (27) implies that

$$G(t)f(t) \geq K + \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(t),$$

where $K = \frac{1}{2q} + \tilde{\delta}$. Since m is nondecreasing, we have for $s > t$

$$G(s)f(s) \geq K + \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(s) \geq K + \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(t), \qquad t \leq s,$$

and hence

$$m(t) \geq K + \left(\frac{q}{2} - \tilde{\varepsilon}\right) m^2(t)$$

which contradicts (28). The proof is complete. □

When (3) reduces to Euler-type equation (5) and we take $h(t) = t^{(p-1)/p}$, technical assumptions (i)–(iii) in the previous theorem are satisfied and this theorem is simplified as follows.

COROLLARY 1. *Suppose that* (15) *holds. Equation* (13) *is oscillatory provided*

$$\liminf_{t\to\infty} \log t \int_t^\infty \left[c(s) - \frac{\gamma_p}{s^p}\right] s^{p-1}\, ds > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}. \tag{29}$$

REMARK 2. As we have already mentioned in the introductory section, the previous corollary improves the main result of [4], where it is proved (using the variational method) that (13) is oscillatory provided (7) holds. Actually, a closer examination of the proof of Theorem 3.1 in [4] reveals that (7) can be replaced by a slightly more general condition

$$\liminf_{t\to\infty} \log t \int_t^\infty \left[c(s) - \frac{\gamma_p}{s^p}\right] s^{p-1}\, ds > 2\left(\frac{p-1}{p}\right)^{p-1}.$$

Now we turn our attention to a nonoscillation criterion which is proved under *no sign restriction* on the integral $\int_t^\infty c(s)ds$ and also under no assumption concerning the divergence of the integral $\int^\infty r^{1-q}(t)dt$ (compare Theorem 2).

THEOREM 3. *Suppose that equation* (3) *is nonoscillatory and possesses a solution h satisfying* (i), (iii) *of Theorem* 2. *Moreover, suppose that integral* (17) *is convergent and*

$$\int^\infty \frac{dt}{r(t)h^2(t)(h'(t))^{p-2}} = \infty. \tag{30}$$

If $G(t)$ *is the same as in Theorem* 2 *and*

$$\limsup_{t\to\infty} G(t) \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)ds < \frac{1}{2q} \tag{31}$$

and

$$\liminf_{t\to\infty} G(t) \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)ds > -\frac{3}{2q} \tag{32}$$

then (1) *is nonoscillatory.*

PROOF. Denote again

$$C(t)=\int_t^\infty (c(s)-\tilde{c}(s))h^p(s)ds.$$

To prove that (1) is nonoscillatory, according to Lemma 2 it suffices to find a differentiable function v which verifies the differential inequality (10) for large t. This inequality can be written in the form

$$\begin{aligned}
v' &\le -p\left[\frac{1}{q}\left|\frac{v+C}{h}\right|^q r^{1-q}-h'\left(\frac{v+C}{h}\right)+\frac{r(h')^p}{p}\right]+r(h')^p-\tilde{c}(t)h^p\\
&= -pr^{1-q}\left[\frac{1}{q}\left|\frac{v+C}{h}\right|^q - r^{q-1}h'\left(\frac{v+C}{h}\right)+\frac{1}{p}r^q(h')^p\right]+r(h')^p-\tilde{c}h^p\\
&= -pr^{1-q}P\left(r^{q-1}h',\frac{v+C}{h}\right)+r(h')^p-\tilde{c}h^p,
\end{aligned}$$

where the function P is introduced in Lemma 1. We will show that the function

$$v(t)=r(t)h(t)\Phi(h'(t))+\frac{1}{2qG(t)}$$

satisfies this inequality for large t. To this end, let $z=\frac{v+C}{h}$, $u=r^{q-1}h'$. First consider the term $r^{1-q}P\big(r^{q-1}h',\frac{v+C}{h}\big)$. We have

$$\frac{z}{\Phi(u)}=\frac{v(t)+C(t)}{h(t)r(t)\Phi(h'(t))}=1+\frac{1+2qC(t)G(t)}{2qG(t)r(t)h(t)\Phi(h'(t))}.$$

Since (16), (30) hold and $G(t)C(t)$ is bounded by (31), (32), we have $z/\Phi(u)\to 1$ as $t\to\infty$. Hence, using (22) and the same argument as in the proof of Theorem 2, for any $\varepsilon>0$, we have (with Q satisfying (23))

$$\begin{aligned}
&pr^{1-q}\left[\frac{1}{q}\left|\frac{v+C}{h}\right|^q - h'r^{q-1}\left(\frac{v+C}{h}\right)+\frac{r^q(h')^p}{p}\right]\\
&= pr^{1-q}r^q(h')^pQ\left(\frac{v+C}{hr\Phi(h')}\right)\\
&\le p\left(\frac{q-1}{2}+\varepsilon\right)r(h')^p\frac{(1+2qGC)^2}{4q^2r^2h^2(h')^{2p-2}G^2}\\
&= \left(\frac{q}{2}+p\varepsilon\right)\frac{1}{rh^2(h')^{p-2}}\frac{(1+2qGC)^2}{4G^2q^2}
\end{aligned}$$

for large t.

Now, since (31), (32) hold, there exists $\delta > 0$ such that

$$\frac{-3+\delta}{2q} < G(t)C(t) < \frac{1-\delta}{2q} \Leftrightarrow |1 + 2qG(t)C(t)| < 2 - \delta$$

for large t, hence $\varepsilon > 0$ can be chosen in such a way that

$$\left(\frac{q}{2} + p\varepsilon\right) \frac{(1 + 2qG(t)C(t))^2}{4q^2} < \frac{1}{2q}$$

for large t. Consequently (using the fact that h solves (3)), we have

$$-pr^{1-q}\left[\frac{1}{q}\left|\frac{v+C}{h}\right|^q - r^{q-1}h'\left(\frac{v+C}{h}\right) + \frac{r^q(h')^p}{p}\right] + r(h')^p - \tilde{c}(t)h^p$$

$$\geq -\left(\frac{q}{2} + p\varepsilon\right) \frac{1}{G^2 rh^2(h')^{p-2}} \frac{(1+2qGC)^2}{4q^2} + r(h')^p - \tilde{c}(t)h^p$$

$$> -\frac{1}{2q} \frac{1}{G^2 rh^2(h')^{p-2}} + [rh\Phi(h')]' = \left[rh\Phi(h') + \frac{1}{2qG}\right]' = v'.$$

The proof is complete. □

Applying the previous theorem to (13) viewed as a perturbation of (5) gives the following nonoscillation criterion.

Corollary 2. *Equation* (13) *is nonoscillatory provided*

(33) $$\limsup_{t\to\infty} \log t \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds < \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

and

(34) $$\liminf_{t\to\infty} \log t \int_t^\infty \left(c(s) - \frac{\gamma_p}{s^p}\right) s^{p-1}\, ds > -\frac{3}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

Remark 3. (i) One of the results of the classical linear oscillation theory for equation (2) is that under the assumption $\int^\infty r^{-1}(t)dt = \infty$, equation (2) is oscillatory provided

$$\liminf_{t\to\infty}\left(\int^t r^{-1}(s)ds\right)\left(\int_t^\infty c(s)ds\right) > \frac{1}{4}.$$

If $\int^\infty r^{-1}(t)dt < \infty$ then (2) is known to be oscillatory provided

(35) $$\liminf_{t\to\infty}\left(\int_t^\infty r^{-1}(s)ds\right)\left(\int^t c(s)ds\right) > \frac{1}{4}.$$

In particular, if the equation

$$y'' + c(t)y = 0 \tag{36}$$

is viewed as a perturbation of the Euler equation

$$y'' + \frac{1}{4t^2}y = 0, \tag{37}$$

i.e., (36) is written in the form

$$y'' + \frac{1}{4t^2}y + \left(c(t) - \frac{1}{4t^2}\right)y = 0, \tag{38}$$

the transformation $y = h(t)u$ with $h(t) = \sqrt{t}\log t$ (which is a nonprincipal solution of (37)) transforms equation (38) into the equation

$$(t\log^2 tu')' + \left(c(t) - \frac{1}{4t^2}\right)t\log^2 tu = 0$$

and applying (35) to this equation (36) is oscillatory provided

$$\liminf_{t\to\infty} \frac{1}{\log t}\int^t \left(c(s) - \frac{1}{4s^2}\right)s\log^2 s\,ds > \frac{1}{4}. \tag{39}$$

Elbert [10] proved that generalized Euler equation (5) has a nonprincipal solution (i.e., linearly independent of the principal solution $y_0 = t^{(p-1)/p}$) which is asymptotically equivalent to $y = t^{(p-1)/p}(\log t)^{2/p}$. This fact and (39) offer the conjecture that (13) is oscillatory (perhaps under some technical restriction on the function c) provided

$$\liminf_{t\to\infty} \frac{1}{\log t}\int^t \left(c(s) - \frac{\gamma_p}{s^p}\right)s^{p-1}\log^2 s\,ds > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}. \tag{40}$$

This conjecture is a subject of the present investigation. Note that a weaker form of this conjecture is proved in the recent paper [28] (using the variational principle), where it is shown that (13) is oscillatory provided the lower limit in (40) is greater than (the four-times greater constant) $2((p-1)/p))^{p-1}$.

(ii) The investigation of qualitative properties of solutions of the partial differential equation with p-Laplacian (which describes several physical phenomena, see e.g. [1, 3, 7])

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \tag{41}$$

is one of the motivations for the research in the oscillation theory of (1). In the linear case $p = 2$, Hille-type oscillation criteria are established in the paper of Schminke [29]. These criteria are based on modification of the Riccati technique consisting in the fact that if u is a nonzero solution of (41) then $w = \frac{\nabla u}{u}$ satisfies the (Riccati type) equation

$$\operatorname{div} w + c(x) + (p-1)\|w(x)\|^q = 0.$$

The ideas of [29] can be combined with the method used in the proof of statements of the previous section in order to extend oscillation criteria of [29] to p-Laplace equation (41). Partial results along this line are achieved in the papers of Mařík [23, 24].

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