Noether's problem and unramified Brauer groups (joint work with M. Kang and B.E. Kunyavskii)

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§1. Introduction: Noether's problem

- k; a field (base field, not necessarily algebraically closed)
- G; a finite group
- G acts on $k(x_g \mid g \in G)$ by $g \cdot x_h = x_{gh}$ for $g, h \in G$
- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem

Emmy Noether (1913) asks whether $k(G)$ is rational over k?
(= purely transcendental over k?; $k(G) = k(\exists t_1, \ldots, \exists t_n)$?)

- the quotient variety $\mathbb{A}^n/G$ is rational over k?

Theorem (Fisher, 1915)

Let A be a finite abelian group of exponent e. Assume that (i) either char $k = 0$ or char $k > 0$ with char $k \nmid e$, and (ii) k contains a primitive e-th root of unity. Then $k(A)$ is rational over k.

- $\mathbb{C}(A)$ is rational over $\mathbb{C}$!

Noether's problem

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(= purely transcendental over k?; $k(G) = k(\exists t_1, \ldots, \exists t_n)$?)

Let A be a finite abelian group.

- (Swan, 1969) $\mathbb{Q}(C_{47})$ is not rational over $\mathbb{Q}$
  He used K. Masuda's method (1968).
- S. Endo, T. Miyata (1973), V.E. Voskresenskii (1973), ...
  e.g. $\mathbb{Q}(C_8)$ is not rational over $\mathbb{Q}$.
- (Lenstra, 1974) $k(A)$ is rational over k $\iff$ certain conditions;
  for example, $\mathbb{Q}(C_{p^r})$ is rational over $\mathbb{Q}$
  $\iff$ $\exists \alpha \in \mathbb{Z}[\zeta_{\varphi(p^r)}]$ such that $| N_{\mathbb{Q}(\zeta_{\varphi(p^r)})/\mathbb{Q}}(\alpha) | = p$
- $h(\mathbb{Q}(\zeta_m)) = 1$ if $m < 23$
  $\Longrightarrow$ $\mathbb{Q}(C_p)$ is rational over $\mathbb{Q}$ for $p \leq 43$. rational also for $61, 67, 71$;
  $\mathbb{Q}(C_p)$ is not rational over $\mathbb{Q}$ for $p = 79$ (Endo-Miyata), and
  $p = 53, 59, 73$. But we do not know when $p = 83, 89, 97, \ldots$
- G; non-abelian case, ..., nilpotent, p-groups, ..., ?

Let G be a finite groups, k be any field.

- (Maeda, 1989) $k(A_5)$ is rational over k;
- (Rikuna, 2003; Plans, 2007)
  $k(GL_2(\mathbb{F}_3))$ and $k(SL_2(\mathbb{F}_3))$ is rational over k;
- (Serre, 2003)
  if 2-Sylow subgroup of $G \simeq C_{8m}$, then $\mathbb{Q}(G)$ is not rational over $\mathbb{Q}$;
  if 2-Sylow subgroup of $G \simeq Q_{16}$, then $\mathbb{Q}(G)$ is not rational over $\mathbb{Q}$;
  e.g. $G = Q_{16}, SL_2(\mathbb{F}_7), SL_2(\mathbb{F}_9)$,
  $SL_2(\mathbb{F}_q)$ with $q \equiv 7$ or $9 \pmod{16}$.

Some examples: monomial actions

- $k(G) := k(x_g \mid g \in G)^G$; invariant field

Noether's problem

Emmy Noether (1913) asks whether $k(G)$ is rational over k?
($=$ purely transcendental over k?; $k(G) = k(\exists t_1, \ldots, \exists t_n)$?)

By Hilbert 90, we have

No-name lemma (e.g. Miyata (1971, Remark 3))

Let G act faithfully on k-vector space V, W be a faithful $k[G]$-submodule of V. Then $K(V)^G$ is rational over $K(W)^G$.

Rationality problem: linear action

Let G act on finite-dimensional k-vector space V and $\rho : G \to GL(V)$ be a representation. Whether $k(V)^G$ is rational over k?

- the quotient variety V/G is rational over k?

Assume that
$\rho : G \to GL(V)$; monomial, i.e. the corresponding matrix representatin of g has exactly one non-zero entry in each row and each column for $\forall g \in G$.
$k(V) = k(w_1, \ldots, w_n)$ where $\{w_1, \ldots, w_n\}$; a basis of $V^* = \mathrm{Hom}(V, k)$.

Then G acts on $k(\mathbb{P}(V)) = k(\frac{w_1}{w_n}, \ldots, \frac{w_{n-1}}{w_n})$ by monomial action.
By Hilbert 90, we obtain

Lemma (e.g. Miyata (1971, Lemma))

$k(V)^G$ is rational over $k(\mathbb{P}(V))^G$ (i.e. $k(V)^G = k(\mathbb{P}(V))^G(t)$).

$V/G \approx \mathbb{P}(V)/G \times \mathbb{P}^1$ (birational equivalent)

Example: $\mathrm{GL}(2,\mathbb{F}_3)$ and $\mathrm{SL}(2,\mathbb{F}_3)$

$G=\mathrm{GL}(2,\mathbb{F}_3)=\langle A,B,C,D\rangle\subset GL_4(\mathbb{Q})$,
$H=\mathrm{SL}(2,\mathbb{F}_3)=\langle A,B,C\rangle\subset GL_4(\mathbb{Q})$ where

$$A=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix},\ B=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix},\ C=\begin{bmatrix}0&0&1&0\\-1&0&0&0\\0&-1&0&0\\0&0&0&1\end{bmatrix},$$

$$D=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}.\ (\#G=48,\#H=24)$$

The actions of G and H on $\mathbb{Q}(V)=\mathbb{Q}(w_1,w_2,w_3,w_4)$ are:

$$A: w_1\mapsto -w_2\mapsto -w_1\mapsto w_2\mapsto w_1,\ w_3\mapsto -w_4\mapsto -w_3\mapsto w_4\mapsto w_3,$$

$$B: w_1\mapsto -w_3\mapsto -w_1\mapsto w_3\mapsto w_1,\ w_2\mapsto w_4\mapsto -w_2\mapsto -w_4\mapsto w_2,$$

$$C: w_1\mapsto -w_2\mapsto w_3\mapsto w_1,\ w_4\mapsto w_4,\quad D: w_1\mapsto w_1,\ w_2\mapsto -w_2,\ w_3\leftrightarrow w_4.$$

$\mathbb{Q}(\mathbb{P}(V))=\mathbb{Q}(x,y,z)$ where $x=w_1/w_4$, $y=w_2/w_4$, $z=w_3/w_4$.
G and H act on $\mathbb{Q}(x,y,z)$ as $G/Z(G)\simeq S_4$ and $H/Z(H)\simeq A_4$ by

$$A: x\mapsto \frac{y}{z},\ y\mapsto\frac{-x}{z},\ z\mapsto\frac{-1}{z},\ B: x\mapsto\frac{-z}{y},\ y\mapsto\frac{-1}{y},\ z\mapsto\frac{x}{y},$$

$$C: x\mapsto y\mapsto z\mapsto x,\ D: x\mapsto\frac{x}{z},\ y\mapsto\frac{-y}{z},\ z\mapsto\frac{1}{z}.$$

Definition (monomial action)

A k-automorphism σ of $k(x_1,\ldots,x_n)$ is called monomial if

$$\sigma(x_j) = c_j(\sigma)\prod_{i=1}^{n} x_i^{a_{i,j}}, \quad 1 \le j \le n$$

where $[a_{i,j}]_{1\le i,j\le n} \in \mathrm{GL}(n,\mathbb{Z})$ and $c_j(\sigma) \in k^\times := k \setminus \{0\}$.

If $c_j(\sigma) = 1$ for any $1 \le j \le n$ then σ is called purely monomial.

A group action on $k(x_1,\ldots,x_n)$ by monomial k-automorphisms is also called monomial.

Theorem (Hajja,1987)

Let k be a field, G be a finite group acting on $k(x_1,x_2)$ by monomial k-automorphisms. Then $k(x_1,x_2)^G$ is rational over k.

Theorem (Hajja-Kang 1994, H.-Rikuna 2008)

Let k be a field, G be a finite group acting on $k(x_1, x_2, x_3)$ by purely monomial k-automorphisms. Then $k(x_1, x_2, x_3)^G$ is rational over k.

Theorem (Prokhorov, 2010)

Let G be a finite group acting on $\mathbb{C}(x_1, x_2, x_3)$ by monomial k-automorphisms. Then $\mathbb{C}(x_1, x_2, x_3)^G$ is rational over $\mathbb{C}$.

Theorem (Kang-Prokhorov, 2010)

Let G be a finite 2-group and k be a field of char $k \neq 2$ and $\sqrt{a} \in k$ for any $a \in k$. If G acts on $k(x_1, x_2, x_3)$ by monomial k-automorphisms, then $k(x_1, x_2, x_3)^G$ is rational over k.

However negative solutions exist for some (k, G) in dimension 3 case, e.g. $\mathbb{Q}(x_1, x_2, x_3)^{\langle\sigma\rangle}$, $\sigma : x_1 \mapsto x_2 \mapsto x_3 \mapsto \frac{-1}{x_1 x_2 x_3}$, is not $\mathbb{Q}$-rational (Hajja,1983).

Theorem (Saltman, 2000)

Let k be a field of char $k \neq 2$, σ be a monomial k-automorphism action of $k(x_1, x_2, x_3)$ by $x_1 \mapsto \frac{a_1}{x_1}$, $x_2 \mapsto \frac{a_2}{x_2}$, $x_3 \mapsto \frac{a_3}{x_3}$.
If $[k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : k] = 8$, then $k(x_1, x_2, x_3)^{\langle\sigma\rangle}$ is not retract rational over k, hence not rational over k.

Theorem (Kang, 2004)

Let k be a field, σ be a monomial k-automorphism acting on $k(x_1, x_2, x_3)$ by $x_1 \mapsto x_2 \mapsto x_3 \mapsto \frac{c}{x_1 x_2 x_3} \mapsto x_1$. Then $k(x_1, x_2, x_3)^{\langle\sigma\rangle}$ is rational over k if and only if at least one of the following conditions is satisfied:
(i) char $k = 2$; (ii) $c \in k^2$; (iii) $-4c \in k^4$; (iv) $-1 \in k^2$.
If $k(x, y, z)^{\langle\sigma\rangle}$ is not rational over k, then it is not retract rational over k.

- rational over $k \Longrightarrow$ "retract rational" over k;
  not rational over $k \Longleftarrow$ not retract rational over k
  (we will recall the definition later)

Lemma (Kang-Prokhrov, 2010, Lemma 2.8)

Let k be a field, G be a finite group acting on $k(x_1,\ldots,x_n)$ by monomial k-automorphism. Then there is a normal subgroup H of G such that
(i) $k(x_1,\ldots,x_n)^H = K(z_1,\ldots,z_n)$;
(ii) G/H acts on $k(z_1,\ldots,z_n)$ by monomial k-automorphisms;
(iii) $\rho : G/H \to GL_n(\mathbb{Z})$ is injective.

Hence we may assume that $\rho : G \to GL_3(\mathbb{Z})$ is injective.

$\exists G \leq GL_3(\mathbb{Z})$; 73 finite subgroups (up to conjugacy).

Theorem (Yamasaki, arXiv:0909.0586)

Let k be a field of char $k \neq 2$. $\exists\ 8$ groups $G \leq GL_3(\mathbb{Z})$ such that $k(x_1, x_2, x_3)^G$ is not retract rational over k, hence not rational over k. Moreover, we may give the necessary and sufficient conditions.

Two of 8 groups are Saltman's and Kang's cases.

Theorem (Yamasaki-H.-Kitayama, 2011)

Let k be a field of char $k \neq 2$, $G \leq GL_3(\mathbb{Z})$ act on $k(x_1, x_2, x_3)$ by monomial k-automorphisms. Then $k(x_1, x_2, x_3)^G$ is rational over k except for the Yamasaki's 8 cases and one case of A_4.
The exceptional case of A_4, it is rational over k if $[k(\sqrt{a}, \sqrt{-1}) : k] \leq 2$.

Corollary

$\exists L = k(\sqrt{a})$ with $a \in k^\times$ such that $L(x_1, x_2, x_3)^G$ is rational over L.

However $\exists$ monomial action of $C_2 \times C_2$ such that $\mathbb{C}(x_1, x_2, x_3, x_4)^{C_2 \times C_2}$ is not retract rational, hence not rational over $\mathbb{C}$!

§2. Main theorem: Noether's problem over $\mathbb{C}$

Let G be a p-group. $\mathbb{C}(G) := \mathbb{C}(x_g \mid g \in G)^G$.

- (Fisher, 1915) $\mathbb{C}(A)$ is rational over $\mathbb{C}$ if A; finite abelian group.
- (Saltman, 1984)
  For $\forall p$; prime, $\exists$ meta-abelian p-group G of order p^9
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.
- (Bogomolov, 1988)
  For $\forall p$; prime, $\exists$ meta-abelian p-group G of order p^6
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

Indeed they showed $B_0(G) \neq 0$; unramified Brauer group

- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "$B_0(G) = 0$"

not rational $\Leftarrow$ not stably rational $\Leftarrow$ not retract rational $\Leftarrow$ $B_0(G) \neq 0$

where $B_0(G)$ is the unramified Brauer group $H^2_{\mathrm{nr}}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$
We will give the precise definition later.

Noether's problem over $\mathbb{C}$

Let G be a p-group.

- (Chu-Kang, 2001)
  Let G be a p-group of order $\leq p^4$. Then $\mathbb{C}(G)$ is rational over $\mathbb{C}$.
- (Chu-Hu-Kang-Prokhorov, 2008)
  Let G be a group of order $2^5 = 32$. Then $\mathbb{C}(G)$ is rational over $\mathbb{C}$.
- (Chu-Hu-Kang-Kunyavskii, 2010) If G is a group of order $2^6 = 64$, then $B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{16}.
  In particular, $\exists$ 9 groups G of order $2^6 = 64$
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$. (by $B_0(G) \neq 0$)
- $\exists 267$ groups of order 64. $(\Phi_1, \ldots, \Phi_{27})$
- (Moravec, to appear in Amer. J. Math.) If G is a group of order $3^5 = 243$, then $B_0(G) \neq 0 \iff G = G(243, i)$ with $28 \leq i \leq 30$.
  In particular, $\exists$ 3 groups G of order $3^5 = 243$
  such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.
- $\exists 67$ groups of order 243. $(\Phi_1, \ldots, \Phi_{10})$

Main theorem

Theorem (H.-Kang-Kunyavskii, arXiv:1202.5812)

Let p be an odd prime and G be a group of order p^5. Then
$B_0(G) \neq 0 \iff G$ belongs to the isoclinism family Φ_{10}.
In particular, $\exists$ $\gcd(4, p-1)+$ $\gcd(3, p-1) + 1$ (resp. $\exists 3$) groups G of order p^5 $(p \geq 5)$ (resp. $p = 3$) such that $\mathbb{C}(G)$ is not retract rational over $\mathbb{C}$.

- $\exists 15$ (14) groups of order $p^4 (p \geq 3)$ $(p = 2)$.
- $\exists 2p + 61+$ $\gcd(4, p-1) + 2$ $\gcd(3, p-1)$ groups of order $p^5 (p \geq 5)$. $(\Phi_1, \ldots, \Phi_{10})$

Definition (isoclinic)

Two p-groups G_1 and G_2 are called isoclinic if there exist group isomorphisms $\theta : G_1/Z(G_1) \to G_2/Z(G_2)$ and $\phi : [G_1, G_1] \to [G_2, G_2]$ such that $\phi([g,h]) = [g', h']$ for any $g, h \in G_1$ with $g' \in \theta(gZ(G_1))$, $h' \in \theta(hZ(G_1))$.

$$
\begin{array}{ccc}
G_1/Z(G_1) \times G_1/Z(G_1) & \xrightarrow[\simeq]{(\theta,\theta)} & G_2/Z(G_2) \times G_2/Z(G_2) \\
{\scriptstyle [\ ,\ ]}\downarrow & \circlearrowleft & \downarrow{\scriptstyle [\ ,\ ]} \\
[G_1, G_1] & \xrightarrow[\simeq]{\phi} & [G_2, G_2]
\end{array}
$$

- Let $G_n(p)$ be the set of all non-isomorphic groups of order p^n.
  equivalence relation $\sim \iff$ they are isoclinic.
  Each equivalence class is called an isoclinism family.

Invariants

- lower central series
- $\#$ of conj. classes with precisely p^i members
- $\#$ of irr. complex rep. of G of degree p^i

- $\#G = p^4 (p > 2)$. $\exists 15$ groups (Φ_1, Φ_2, Φ_3)
- $\#G = 2^4 = 16$. $\exists 14$ groups (Φ_1, Φ_2, Φ_3)
- $\#G = p^5 (p > 3)$. $\exists 2p + 61 + (4, p-1) + 2 \times (3, p-1)$ groups $(\Phi_1, \ldots, \Phi_{10})$

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
$\#$	7	15	13	$p+8$	2	$p+7$	5	1
$(p=3)$						7		

	Φ_9	Φ_{10}
$\#$	$2 + (3, p-1)$	$1 + (4, p-1) + (3, p-1)$
$(p=3)$		3

Question 1.11 in [HKK] (arXiv:1202.5812)

Let G_1 and G_2 be isoclinic p-groups.
Is it true that the fields $k(G_1)$ and $k(G_2)$ are stably isomorphic, or, at least, that $B_0(G_1)$ is isomorphic to $B_0(G_2)$?

- $G_1 \sim G_2 \Longrightarrow B_0(G_1) = B_0(G_2)$
  proved by Moravec (arXiv:1203.2422)
- $G_1 \sim G_2 \Longrightarrow k(G_1) \approx k(G_2)$
  proved by Bogomolov-Böhning (arXiv: 1204.4747)

§3. Unramified Brauer groups & retract rationality

Definition (stably rational)

L is called stably rational over k if $L(y_1, \ldots, y_m)$ is rational over k.

Definition (retract rational) $\leftrightarrow$ "projective" object by Saltman (1984)

Let k be an infinite field, and $k \subset L$ be a field extension.
L is retract rational over k if $\exists k$-algebra $R \subset L$ such that
(i) L is the quotient field of R;
(ii) $\exists f \in k[x_1, \ldots, x_n]$ $\exists k$-algebra hom. $\varphi : R \to k[x_1, \ldots, x_n][1/f]$ and $\psi : k[x_1, \ldots, x_n][1/f] \to R$ satisfying $\psi \circ \varphi = 1_R$.

Definition (unirational)

L is unirational over k if L is a subfield of rational field extension of k.

- Let L_1 and L_2 be stably isomorphic fields over k.
  If L_1 is retract rational over k, then so is L_2 over k.
- "rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational"

Retract rationality

Theorem (Saltman, DeMeyer)

Let k be an infinite field and G be a finite group.
The following are equivalent:
(i) $k(G)$ is retract k-rational.
(ii) There is a generic G-Galois extension over k;
(iii) There exists a generic G-polynomial over k.

- related to Inverse Galois Problem (IGP). (i) $\Longrightarrow$ IGP(G/k): true

Definition (generic polynomial)

A polynomial $f(t_1,\ldots,t_n;X) \in k(t_1,\ldots,t_n)[X]$ is generic for G over k if
(1) $\mathrm{Gal}(f/k(t_1,\ldots,t_n)) \simeq G$;
(2) $\forall L/M \supset k$ with $\mathrm{Gal}(L/M) \simeq G$,
$\exists a_1,\ldots,a_n \in M$ such that $L = \mathrm{Spl}(f(a_1,\ldots,a_n;X)/M)$.

- By Hilbert's irreducibility theorem, $\exists L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq G$.

"rational" $\Longrightarrow$ "stably rational" $\Longrightarrow$ "retract rational" $\Longrightarrow$ "unirational".

- The direction of the implication cannot be reversed.
- (Lüroth's problem) "unirational" $\Longrightarrow$ "rational" ? YES if trdeg$= 1$
- (Castelnuovo, 1894)
  L is unirational over $\mathbb{C}$ and $\mathrm{trdeg}_{\mathbb{C}} L = 2 \Longrightarrow L$ is rational over $\mathbb{C}$.
- (Zariski, 1958) Let k be an alg. closed field and $k \subset L \subset k(x,y)$. If $k(x,y)$ is separable algebraic over L, then L is rational over k.
- (Zariski cancellation problem) $V_1 \times \mathbb{P}^n \approx V_2 \times \mathbb{P}^n \Longrightarrow V_1 \approx V_2$?
  Inparticular, "stably rational" $\Longrightarrow$ "rational"?
- $L = \mathbb{Q}(x,y,t)$ with $x^2 + 3y^2 = t^3 - 2$
  $\Longrightarrow L$ is not rational over $\mathbb{Q}$ and $L(y_1,y_2,y_3)$ is rational over $\mathbb{Q}$.
  (Beauville, J.-L. Colliot-Thélène, Sansuc Swinnerton-Dyer, 1985)
- $L(y_1,y_2)$ is rational over $\mathbb{Q}$ (Shepherd-Barron).
- $\mathbb{Q}(C_{47})$ is not stably rational over $\mathbb{Q}$ but retract rational over $\mathbb{Q}$.
- $\mathbb{Q}(C_8)$ is not retract rational over $\mathbb{Q}$ but unirational over $\mathbb{Q}$.

Unramified Brauer group

Definition (Unramified Brauer group) Saltman (1984)

Let $k \subset K$ be an extension of fields.
$\mathrm{Br}_{v,k}(K) = \cap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$ where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R uns over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R.

- If k is infinite field and K is retract rational over k, then natural map $\mathrm{Br}(k) \to \mathrm{Br}_{v,k}(K)$ is an isomorphism. In partidular, if k is an algebraically closed field and K is retract rational over k, then $\mathrm{Br}_{v,k}(K) = 0$.
- "retract rational" $\Longrightarrow B_0(G) = 0$ where $B_0(G) = \mathrm{Br}_{v,k}(k(G))$.

Theorem (Bogomolov 1988, Saltman 1990)

Let G be a finite group, k be an algebraically closed field with $\gcd\{|G|, \mathrm{char}\, k\} = 1$. Let μ denote the multiplicative subgroup of all roots of unity in k. Then $\mathrm{Br}_{v,k}(k(G))$ is isomorphic

$$B_0(G) = \bigcap_A \mathrm{Ker}\{\mathrm{res}_G^A : H^2(G,\mu) \to H^2(A,\mu)\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

- "retract rational" $\Longrightarrow B_0(G) = 0$ where $B_0(G) = \mathrm{Br}_{v,k}(k(G))$.
  $B_0(G) \neq 0 \Longrightarrow$ not retract rational over $k \Longrightarrow$ not rational over k.
- $B_0(G)$ is a subgroup of the Schur multiplier
  $H_2(G,\mathbb{Z}) \simeq H^2(G,\mathbb{Q}/\mathbb{Z})$, which is called Bogomolov multiplier.

§4. Proof (Φ_{10}): $B_0(G) \neq 0$

We give a sketch of the proof of

Theorem 1 (the case Φ_{10})

Let p be an odd prime and G be a group of order p^5 belonging to the isoclinism family Φ_{10}. Then $B_0(G) \neq 0$.

We may obtain the following two lemmas:

Lemma 1

Let G be a finite group, N be a normal subgroup of G. Assume that (i) $\mathrm{tr}\colon H^1(N, \mathbb{Q}/\mathbb{Z})^G \to H^2(G/N, \mathbb{Q}/\mathbb{Z})$ is not surjective where tr is the transgression map, and (ii) for any bicyclic subgroup A of G, the group AN/N is a cyclic subgroup of G/N. Then $B_0(G) \neq 0$.

Lemma 2

Let $p \geq 3$ and G be a p-group of order p^5 generated by f_i where $1 \leq i \leq 5$. Suppose that, besides other relations, the generators f_i satisfy the following conditions:

(i) $f_4^p = f_5^p = 1$, $f_5 \in Z(G)$,

(ii) $[f_2, f_1] = f_3$, $[f_3, f_1] = f_4$, $[f_4, f_1] = [f_3, f_2] = f_5$, $[f_4, f_2] = [f_4, f_3] = 1$, and

(iii) $\langle f_4, f_5 \rangle \simeq C_p \times C_p$, $G/\langle f_4, f_5 \rangle$ is a non-abelian group of order p^3 and of exponent p.

Then $B_0(G) \neq 0$.

Proof of Lemma 2.

Choose $N = \langle f_4, f_5 \rangle \simeq C_p \times C_p$. Then we may check that Lemma 1 is satisfied. Thus $B_0(G) \neq 0$.

Proof of Theorem 1.

All groups which belong to Φ_{10} satisfy the conditions as in Lemma 2.

Lemma 1

Let G be a finite group, N be a normal subgroup of G. Assume that (i) $\mathrm{tr}\colon H^1(N,\mathbb{Q}/\mathbb{Z})^G \to H^2(G/N,\mathbb{Q}/\mathbb{Z})$ is not surjective where tr is the transgression map, and (ii) for any bicyclic subgroup A of G, the group AN/N is a cyclic subgroup of G/N. Then $B_0(G) \neq 0$.

Proof. Consider the Hochschild–Serre 5-term exact sequence

$$0 \to H^1(G/N,\mathbb{Q}/\mathbb{Z}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^1(N,\mathbb{Q}/\mathbb{Z})^G \xrightarrow{\mathrm{tr}} H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z})$$

where ψ is the inflation map.

Since tr is not surjective (the first assumption (i)), we find that ψ is not the zero map. Thus $\mathrm{Image}(\psi) \neq 0$.

We will show that $\mathrm{Image}(\psi) \subset B_0(G)$. By the definition, it suffices to show that, for any bicyclic subgroup A of G, the composite map $H^2(G/N,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(A,\mathbb{Q}/\mathbb{Z})$ becomes the zero map.

Consider the following commutative diagram:

$$\begin{array}{ccccc} H^2(G/N,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\psi} & H^2(G,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\mathrm{res}} & H^2(A,\mathbb{Q}/\mathbb{Z}) \\ \psi_0 \downarrow & & & & \uparrow \psi_1 \\ H^2(AN/N,\mathbb{Q}/\mathbb{Z}) & & \overset{\widetilde{\psi}}{\simeq} & & H^2(A/A\cap N,\mathbb{Q}/\mathbb{Z}) \end{array}$$

where ψ_0 is the restriction map, ψ_1 is the inflation map, $\widetilde{\psi}$ is the natural isomorphism.

Since AN/N is cyclic (the second assumption (ii)), write $AN/N \simeq C_m$ for some integer m.

It is well-known that $H^2(C_m, \mathbb{Q}/\mathbb{Z}) = 0$.

Hence ψ_0 is the zero map. Thus $\mathrm{res} \circ \psi \colon H^2(G/N, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})$ is also the zero map.

By $\mathrm{Image}(\psi) \subset B_0(G)$ and $\mathrm{Image}(\psi) \neq 0$, we get that $B_0(G) \neq 0$. □

§5. Proof (Φ_6): $B_0(G) = 0$

- $G = \Phi_6(211)a = \langle f_1, f_2, f_0, h_1, f_2 \rangle, f_1^p = h_1, f_2^p = h_2,$
  $Z(G) = \langle h_1, h_2 \rangle, f_0^p = h_1^p = h_2^p = 1$
  $[f_1, f_2] = f_0, [f_0, f_1] = h_1, [f_0, f_2] = h_2$

$$0 \to H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})^G \xrightarrow{\mathrm{tr}} H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z})$$

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$$\downarrow$$

$$\mathrm{Ker}\{H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(N, \mathbb{Q}/\mathbb{Z})\} =: H^2(G, \mathbb{Q}/\mathbb{Z})_1$$

$$\downarrow$$

$$H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z}))$$

$$\lambda \downarrow$$

$$H^3(G/N, \mathbb{Q}/\mathbb{Z})$$

- Explicit formula for λ is given
  by Dekimpe-Hartl-Wauters (arXiv:1103.4052)
- $N := \langle f_1, f_0, h_1, h_2 \rangle \Longrightarrow G/N \simeq C_p \Longrightarrow H^2(G/N, \mathbb{Q}/\mathbb{Z}) = 0$
- $B_0(G) \subset H^2(G, \mathbb{Q}/\mathbb{Z})_1$
- We should show $H^2(G, \mathbb{Q}/\mathbb{Z})_1 = 0$ ($\Longleftrightarrow$ λ: injective)