A generalization of elementary divisor theory

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Contents

Conventions.

Throughout the lecture, we utilize the letter A, $\mathcal{B}$, $\mathcal{C}$, S and X to denote a commutative noetherian ring with unit, an essentially small abelian category, a category, a finite set and the affine sheme associated with A respectively.

For any non-negative integer m, let us denote the totally ordered set of integers k such that $0 \leq k \leq m$ with the usual order by $[m]$.

1 Main theorems and motivation

Main theorems.

1 Abstract Buchsbaum-Eisenbud theorem for multi-complexes.

2 Weak geometric presentation theorem

For any strictly regular closed immersion $Y \hookrightarrow X$, we have a derived equivalence

$$X^{Y}_{w\mathcal{M}} \overset{\mathrm{Tot}}{\to} X^{Y}_{\mathbf{Top}}.$$

3 (In progression work) **Caborn-Fossum, Dutta chow groups problem**

If A is regular local, then $\mathbf{Ch}_d(X) = 0$ for any $d < \dim X$.

The statement $\boxed{1}$ is an assertion about objects and morphisms in $\mathcal{B}$ involves a natural number n.

- $n = 1$

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$$

If ba is a monomorphism, then a is a monomorphism.

- $n = 2$

$$\begin{array}{ccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\downarrow & \mathbf{I} & \downarrow & \mathbf{II} & \downarrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\downarrow & \mathbf{III} & \downarrow & \mathbf{IV} & \downarrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet,
\end{array}$$

If the big square **I**+**II**+**III**+**IV** *is a Cartesian square and if all morphisms in the diagram above are monomorphisms, then the square* **I** *is Cartesian.*

• The CDF problem is a variant of the following classical theorem.

Theorem.

If A is a unique factorization domain, then $\operatorname{Pic} X = 0$*.*

• The CDF problem is known for various A by utilizing

structure theorems over a base

or

weight argument of Adams operations.

Ideally, the CDF problem should be proven from the following conjectural statement which I call

an absolute geometric presentation theorem.

If A is regular local, then the canonical map

$$X^p_{\mathcal{M}} \to X^p_{\mathbf{Top}}$$

is a derived universally homeomorphism for any $0 \leq p \leq \dim X$.

Compare with the weak geometric presentation theorem with an absolute geometric presentation theorem.

2 Spirit of elemetary divisor theory

Sorting out modules

by Systems of matrices

with equivalence relations.

Syzygy.

Systems of matrices = complexes of finitely generated free modules.

Equivalence relation = quasi-isomorphism.

Bourbaki-Iwasawa-Serre theory.

Equivalence relation = pseudo-isomorphism.

Here a homomorphism of A-modules $f : M \to N$ is a **pseudo-isomorphism** if $\mathrm{Codim} \ker f \geq 2$ and $\mathrm{Codim}\,\mathrm{Coker}\, f \geq 2$.

Today's lecture.

Modules= TT-pure weight modules.

Systems of matrices = Koszul cubes.

Equivalence relation = totalized quasi-isomorphism. ($\mathbb{A}^1$-homotopy equivalence, generic isomorphism).

What makes a comlex exact?

Buchsbaum-Eisenbud theorem.

For a complex of free A-modules of finite rank.

$$F_\bullet : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \xrightarrow{\phi_{s-1}} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0,$$

set $r_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank} F_j$*. Then the following are equivalent:*

(1) $F_\bullet$ *is a resolution of* $\mathrm{H}_0(F_\bullet)$*.*

(2) $\operatorname{grade} I_{r_i}(\phi_i) \geq i$ *for any* $1 \leq i \leq s$ *where* $I_{r_i}(\phi_i)$ *is the* r_i*-th Fitting ideal of* ϕ_i*.*

3 What makes a multi-complex exact?

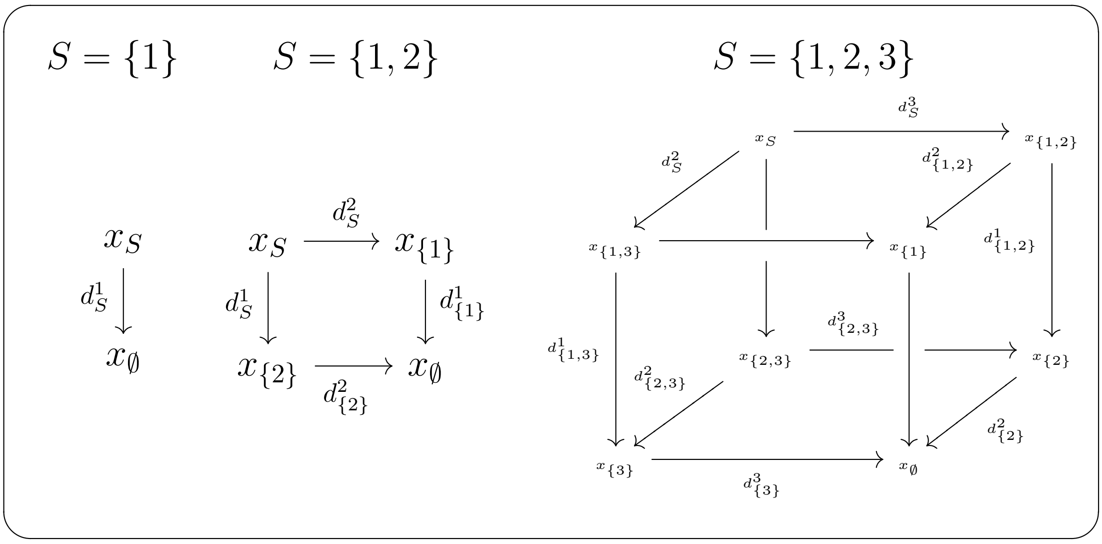
Definitions.

An S**-cube** in $\mathcal{C}$ is a contravariant functor

$$\mathcal{P}(S)^{\mathrm{op}}(\xrightarrow{\sim}[1]^{S\mathrm{op}}) \to \mathcal{C}\,.$$

For any $U \in \mathcal{P}(S)$ and $k \in U$,

- $x_U(:= x(U))$ vertex of x at U.
- $d_U^{k,x}(= d_U^k) := x(U \setminus \{k\} \hookrightarrow U)$ k-boundary map at U.



Example 1

For a family of morphisms $\mathfrak{x} := \{x_s \overset{d_s}{\to} x\}_{s \in S}$ in $\mathcal{B}$, we put

$$\operatorname{Fib} \mathfrak{x}_U := \begin{cases} x & \text{if } U = \emptyset \\ x_s & \text{if } U = \{s\} \\ x_{t_1} \times_x x_{t_2} \times_x \cdots \times_x x_{t_r} & \text{if } U = \{t_1, \cdots, t_r\} \end{cases}$$

Definition.

An S-cube x in $\mathcal{B}$ is **fibered** if the canonical map

$$x \to \operatorname{Fib}\{x_{\{s\}} \overset{d^s_{\{s\}}}{\to} x_\emptyset\}_{s \in S}$$

is an isomorphism.

Example 2

For a family of elements $\mathfrak{f}_S = \{f_s\}_{s \in S}$ in A, we put

$$\operatorname{Typ}(\mathfrak{f}_S)_U := A \ \text{ and } \ d^s_U = f_s$$

for any $U \in \mathcal{P}(S)$ and $s \in U$.

(Notice that $\operatorname{Tot} \operatorname{Typ}(\mathfrak{f}_S)$ is the Koszul complex associated with $\mathfrak{f}_S$.)

Faces and homology of cubes

Definitions.

$k \in S$, x: S-cube

$$B^k,\ F^k : \mathcal{P}(S \smallsetminus \{k\}) \to \mathcal{P}(S)$$

$$F^k : U \mapsto U$$

$$B^k : U \mapsto U \sqcup \{k\}$$

- xB^k: backside k-face of x.
- xF^k: frontside k-face of x.
- $\mathrm{H}_0^k(x) := \mathrm{Coker}(xB^k \to xF^k)$: k-direction 0-th homology of x.

$S = \{1, 2\}$

$$\begin{array}{ccccc} x_S & \longrightarrow & x_{\{1\}} & & \mathrm{H}_0^2(x)_{\{1\}} \\ \downarrow & & \downarrow & \Rightarrow & \downarrow \\ x_{\{2\}} & \longrightarrow & x_{\emptyset} & & \mathrm{H}_0^2(x)_{\emptyset} \end{array}$$

$$\Downarrow$$

$$\mathrm{H}_0^1(x)_{\{2\}} \longrightarrow \mathrm{H}_0^2(x)_{\emptyset}$$

Admissibility

Definition.

We say that an S-cube x in $\mathcal{B}$ is **admissible** if

(1) its boundary morphism(s) is (are) monomorphism(s) and

(2) if for every k in S, $\mathrm{H}_0^k(x)$ is admissible.

- We can prove that any admissible cube is fibered.
- We can prove that an S-cube x in $\mathcal{B}$ is admissible iff

(1) all faces of the S-cube x are admissible and

(2) $\mathrm{H}_k(\mathrm{Tot}\, x) = 0$ for any $k > 0$.

Example 1.

For a commutative diagram of monomorphisms in $\mathcal{B}$

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \big\downarrow & & \big\downarrow \\ \bullet & \longrightarrow & \bullet, \end{array}$$

the diagram above is admissible iff it is Cartesian.

Example 2.

For any family of elements $\mathfrak{f}_S = \{f_s\}_{s \in S}$ in A, $\mathrm{Typ}(\mathfrak{f}_S)$ is admissible iff $\mathfrak{f}_S$ is a regular sequence in any order.

Admissibility

= a higher analogue of the notion about Cartesian squares

= a categorical variant of the notion about regular sequences.

Double cubes

Definitions.

A **double S-cube** in $\mathcal{C}$ is a contravariant functor

$$x : [2]^{S\,\mathrm{op}} \to \mathcal{C}\,.$$

For any $\mathfrak{i} \in [1]^S$, we put

$$e_{\mathfrak{i}} : [1]^S \to [2]^S,\ \mathfrak{j} \mapsto \mathfrak{i} + \mathfrak{j} \text{ and}$$

$$\mathrm{Out} : [1]^S \to [2]^S,\ \mathfrak{j} \mapsto 2\,\mathfrak{j}\,.$$

Example.

$$\begin{array}{ccccc}
x_{(2,2)} & \longrightarrow & x_{(2,1)} & \longrightarrow & x_{(2,0)} \\
\downarrow & \mathbf{I} & \downarrow & \mathbf{II} & \downarrow \\
x_{(1,2)} & \longrightarrow & x_{(1,1)} & \longrightarrow & x_{(1,0)} \\
\downarrow & \mathbf{III} & \downarrow & \mathbf{IV} & \downarrow \\
x_{(0,2)} & \longrightarrow & x_{(0,1)} & \longrightarrow & x_{(0,0)}
\end{array}$$

I$= xe_{(1,1)}$, **II**$= xe_{(1,0)}$, **III**$= xe_{(0,1)}$, **IV**$= xe_{(0,0)}$ and

I+**II**+**III**+**IV** $= x\,\mathrm{Out}$.

ABE theorem

Theorem.

Let x be a double S-cube in $\mathcal{B}$. We assume that the following conditions hold.

• The S-cube $x\,\mathrm{Out}$ is admissible.

• All boundary morphisms of the double S-cube x are monomorphisms.

• If $\#S \geq 3$, all faces of the S-cube xe_T are admissible for any proper subset T of S.

Then the S-cube xe_S is also an admissible S-cube.

Adjugates of cubes

From now on, let $\mathcal{B}$ be the category of A-modules.

Definitions.

An **adjugate of an S-cube** x in $\mathcal{B}$ is a pair $(\mathfrak{a}, \mathfrak{d}^*)$ consisting of a family of elements $\mathfrak{a} = \{a_s\}_{s \in S}$ in A and a family of morphisms $\mathfrak{d}^* = \{d_T^{t*} : x_{T \smallsetminus \{t\}} \to x_T\}_{T \in \mathcal{P}(S), t \in T}$ in $\mathcal{B}$ which satisfies the following two conditions.

• We have the equalities $d_T^t d_T^{t*} = (a_t)_{x_{T \smallsetminus \{t\}}}$ and $d_T^{t*} d_T^t = (a_t)_{x_T}$ for any $T \in \mathcal{P}(S)$ and $t \in T$.

• For any $T \in \mathcal{P}(S)$ and any distinct elements a and $b \in T$, we have the equality $d_T^b d_T^{a*} = d_{T \smallsetminus \{b\}}^{a*} d_{T \smallsetminus \{a\}}^b$. Namely, the following diagram is commutative.

$$\begin{array}{ccc} x_{T \smallsetminus \{a\}} & \xrightarrow{d_T^{a*}} & x_T \\ {\scriptstyle d_{T \smallsetminus \{a\}}^b} \downarrow & & \downarrow {\scriptstyle d_T^b} \\ x_{T \smallsetminus \{a,b\}} & \xrightarrow[d_{T \smallsetminus \{b\}}^{a*}]{} & x_{T \smallsetminus \{b\}} \end{array} .$$

(2) An adjugate of an S-cube $(\mathfrak{a}, \mathfrak{d}^*)$ is **regular** if $\mathfrak{a}$ forms x_T-regular sequence in any order for any $T \in \mathcal{P}(S)$.

Example.

X: $n \times n$ matrix whose coefficients are in A, $x := [A^{\oplus n} \overset{X}{\to} A^{\oplus n}]$

Then a pair $(\operatorname{adj} X, \det X)$ is an adjugate of x.

BE theorem for cubes

Corollary.

If an S-cube x in $\mathcal{B}$ admits a regular adjugate, then x is admissible.

Proof.

For $S = \{1, 2\}$, we apply the ABE theorem to the double cube below.

$$\begin{array}{ccccc}
x_S & \xrightarrow{d_S^1} & x_{\{2\}} & \xrightarrow{d_S^{1*}} & x_S \\
{\scriptstyle d_S^2}\downarrow & & {\scriptstyle d_{\{2\}}^2}\downarrow & & \downarrow{\scriptstyle d_S^2} \\
x_{\{1\}} & \xrightarrow[d_{\{1\}}^1]{} & x_\emptyset & \xrightarrow{d_{\{1\}}^{1*}} & x_{\{1\}} \\
{\scriptstyle d_S^{2*}}\downarrow & & \downarrow{\scriptstyle d_{\{2\}}^{2*}} & & \downarrow{\scriptstyle d_S^{2*}} \\
x_S & \xrightarrow[d_S^1]{} & x_{\{2\}} & \xrightarrow[d_S^{1*}]{} & x_S.
\end{array}$$

□

4 Weak absolute geometric presentation theorem

Koszul cubes

In this section, let $\mathfrak{f}_S = \{f_s\}_{s \in S}$ be a family of elements in A which forms a regular sequence in any order and x an S-cube in $\mathcal{B}$.

Definition.

x is **Koszul** (**associated with** $\mathfrak{f}_S$) if for any $T \in \mathcal{P}(S)$ and any $k \in T$,

(1) x_T is a finitely generated projective A-module and

(2) d_T^k is injective and

(3) There exists a non-negative integer m_k such that $f_k^{m_k} \operatorname{Coker} d_T^k = 0$.

We denote the category of Koszul cubes associated with $\mathfrak{f}_S$ by $\mathbf{Kos}_A^{\mathfrak{f}_S}$.

• We can prove that any Koszul cube is admissible by BE theorem for cubes.

A morphism between koszul cubes (associated with $\mathfrak{f}_S$) $a : x \to y$ is a **totalized quasi-isomorphism** if $\mathrm{H}_0 \operatorname{Tot} a$ is an isomorphism.

Example.

$\operatorname{Typ}(\mathfrak{f}_S)$ is a Koszul cube.

Solid devices

Definition.

A **solid device** $\mathbf{E} = (\mathcal{E}, w)$ is a pair of a category $\mathcal{E}$ and a class of morphisms w in $\mathcal{E}$ which satisfies the following axioms.

- $\mathcal{E}$ is an exact category.
- $(\mathcal{E}, w)$ and $(\mathcal{E}^{\mathrm{op}}, w^{\mathrm{op}})$ are categories with cofibrations and weak equivalences.
- **(Extensional axiom).** For any commutative diagrams admissible exact sequences in $\mathcal{E}$,

$$\begin{array}{ccccc} x & \rightarrowtail & y & \twoheadrightarrow & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ x' & \rightarrowtail & y' & \twoheadrightarrow & z' \end{array},$$

if a and c are in w, then b is also in w. Let us write $\mathcal{E}^w$ for the full subcategory of $\mathcal{E}$ consisting of those object x such that the canonical morphism $0 \to x$ is in w. Then $\mathcal{E}^w$ naturally becomes an exact category.

- **(Solid axiom).** For any morphism $f : x \to y$ in w, the complex $\mathrm{Cone}\, f = [x \xrightarrow{f} y]$ is connected to a bounded complexes on $\mathcal{E}^w$ by a zig-zag of quasi-isomorphisms.
- **(Fibrational axiom).** The canonical inclusion functors $\mathcal{E}^w \to \mathcal{E}$ and $(\mathcal{E}, i) \to (\mathcal{E}, w)$ induce the sequence having a homotopy type of fibration sequence

$$i\,\mathcal{S}_\bullet\,\mathcal{E}^w \to i\,\mathcal{S}_\bullet\,\mathcal{E} \to w\,\mathcal{S}_\bullet\,\mathcal{E}$$

where i means the class of all isomorphisms and $\mathcal{S}$ means the Segal-Waldhausen $\mathcal{S}$-construction.

• We will functorially associate a solid device $\mathbf{E}$ with

(1) the *non-connective K-spectrum* $\mathbb{K}(\mathbf{E})$ and

(2) the *bounded derived categories* $\mathcal{D}_b(\mathbf{E})$ which is a triangulated category.

A morphism between solid devices $f : \mathbf{E} \to \mathbf{F}$ is a **derived equivalence** if it induces an equivalence of triangulated categories

$$\mathcal{D}_b(\mathbf{E}) \to \mathcal{D}_b(\mathbf{F}).$$

Example 1.

$Y \hookrightarrow X$:closed subset. We put

$$X^Y_{\mathbf{Top}} := (\mathrm{Perf}^Y_X, \mathrm{qis}),$$
$$X^p_{\mathbf{Top}} := (\mathrm{Perf}^p_X, \mathrm{qis})$$

where Perf^Y_X is the category of perfect complexes whose cohomological support are in Y and $\mathrm{Perf}^p_X := \bigcup_{\mathrm{Codim}\, Y \geq p} \mathrm{Perf}^Y_X$. $X^Y_{\mathbf{Top}}$ and $X^p_{\mathbf{Top}}$ are solid devices.

Example 2.

$Y \hookrightarrow X$:regular closed immersion of codimension r.

A perfect $\mathcal{O}_X$-module $\mathcal{F}$ on X is a **TT-weight** r (supported on Y) if $\mathrm{Supp}\, \mathcal{F} \subset Y$ and $\mathrm{Tor\text{-}dim}\, \mathcal{F} \leqq r$.

Let us denote the exact category of TT-weight r modules supported on Y by $\mathbf{Wt}^Y_X$.

We put

$$X^Y_{\mathrm{TT}} := (\mathbf{Wt}^Y_X, \mathrm{isom}).$$

X^Y_{TT} is a solid device.

Example 3.

$Y = V(\mathfrak{f}_S)$. We put

$$X^Y_{w\,\mathcal{M}} := (\mathbf{Kos}^{\mathfrak{f}_S}_A, \mathrm{tq})$$

where tq is the class of totalized quasi-isomorphisms in $\mathbf{Kos}^{\mathfrak{f}_S}_A$. $X^Y_{w\,\mathcal{M}}$ is a solid device.

WGP theorem

Theorem.

For $Y = V(\mathfrak{f}_S)$, the canonical map

$$X^Y_{w\,\mathcal{M}} \overset{\mathrm{Tot}}{\to} X^Y_{\mathbf{Top}}$$

is derived equivalence.

Proof.

$$\begin{array}{ccc} X^Y_{w\,\mathcal{M}} & \xrightarrow{\quad\mathrm{Tot}\quad} & X^Y_{\mathbf{Top}} \\ & {\scriptstyle \mathrm{H}_0\,\mathrm{Tot}} \searrow \quad \nearrow {\scriptstyle \mathsf{I}} & \\ & X^Y_{\mathrm{TT}} & \end{array}$$

In the derived commutative diagram above, the map **I** is a derived equivalence by Hiranouchi-M.

To prove that $\mathrm{H}_0\,\mathrm{Tot}$ is a derived equivalence, one of the key ingredient is giving an algorithm about inductive resolution process of pure weight modules by Koszul cubes. □

5 $\mathbb{A}^1$-homotopy invariances, generic isomorphisms

Regularity and $\mathbb{A}^1$-homotopy invariance

In this section, let A be a regular local ring.

Lemma.

Let $\mathfrak{f}_S = \{f_s\}_{s \in S}$ *be a regular sequence in* A *and we put* $Y = V(\mathfrak{f}_S)$. *Then the canonical map* $A \to A[t]$ *induces an isomorphism of* K*-groups*

$$K_n(X^Y_{w\,\mathcal{M}}) \xrightarrow{\sim} K_n(X[t]^Y_{w\,\mathcal{M}}).$$

- In particular, for any $z(t) \in X[t]^Y_{w\,\mathcal{M}}$, $[z(0)] = [z(1)]$ in $K_0(X^Y_{w\,\mathcal{M}})$.

Generic isomorphisms

Let $0 \leq p \leq \dim A$ be an integer. We propose the following assertions.

(α_p) *For any regular sequence $f_1, \cdots, f_p$ in A, the canonical inclusion functor*

$$X_{\mathbf{Top}}^{V(f_1,\cdots,f_p)} \hookrightarrow X_{\mathbf{Top}}^{V(f_1,\cdots,f_{p-1})}$$

induces the zero map on their Grothendieck groups.

(β_p) *For any regular sequence $\mathfrak{f}_S$ such that $\#S = p$, the Grothendieck group $K_0(X_{w\mathcal{M}}^{\mathfrak{f}_S})$ is generated by Koszul cubes of rank one.*

Lemma.

(1) *The assertion* (β_{p}) *implies* (α_{p}).

(2) *The assertions* (α_{p}) $(0 \leq p \leq \dim A)$ *imply the CDF problem.*

(3) *Let g be an element in A and $\mathfrak{f}_S := \{f_s\}_{s \in S}$ a regular sequence of A such that $\mathfrak{f}_S$ is still a regular sequence in A_g and we put $X_g := \operatorname{Spec} A_g$ and $Y = V(\mathfrak{f}_S)$. Assume $(\alpha_{\#S+1})$, then the canonical localization map $A \to A_g$ induces an isomorphism of Grotheindieck groups*

$$K_0(X_{w\mathcal{M}}^{Y}) \xrightarrow{\sim} K_0(X_{g_{w\mathcal{M}}}^{Y}).$$

• We will prove (β_p) by descending induction of p.

How to prove (β_p)?

Notations.

• For an element λ in A, and for $1 \leq i \neq j \leq n$, $e^n_{ij}(\lambda)$ or simply $e_{ij}(\lambda)$ will denote the $n \times n$ matrix such that the diagonal entries are one, the (i,j) entry is λ and the other entries are zero.

• For elements $a_1, \cdots, a_n$ in A, we write $\mathrm{diag}(a_1, \cdots, a_n)$ for the diagonal matrix whose (i,i) entry is a_i.

Let $x \in X^Y_{w\mathcal{M}}$. Let us put $n := \mathrm{rank}\, x$. By fixing the bases of all vertexes of x, we write $A^s = (a^s_{ij})$ for the matrix description of $d^x_s : x_{\{s\}} \to x_{\emptyset}$ for each $s \in S$. We put $d^s := \gcd\{a^s_{1j}; 1 \leq j \leq n\}$ and $b^s_{1j} = a^s_{1j}/d^s$. Then for any s in S, there exists a subset $T_s \subset \{2, \cdots, n-1, n\}$ such that $c^s_{11} = b^s_{11} + \sum_{j \in T_s} b^s_{1j}$ is prime to f_s. Now we put $g := \prod_{s \in S} c^s_{11}$, $c^s_{j1} := a^s_{j1} + \sum_{l \in T_s} a^s_{jl}$ $(2 \leq j \leq n)$ and

$$B(t)^s = \mathrm{diag}(d^s, 1, 1, \cdots, 1) \prod_{j=2}^{n} e_{1j}\left(-\frac{c^s_{j1}}{c^s_{11}} t\right) \begin{pmatrix} b^s_{11} & b^s_{12} & \cdots & b^s_{1n} \\ a^s_{21} & a^s_{22} & \cdots & a^s_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a^s_{n1} & a^s_{n2} & \cdots & a^s_{nn} \end{pmatrix} \prod_{j \in T_s} e_{j1}(t).$$

Notice that $B(0)^s = A^s$ and the first column of $B(1)^s$ is of the form $\begin{pmatrix} d^s c^s_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. We use $\mathrm{Fib}\{A[t]^{\oplus n} \overset{B(t)^s}{\to} A[t]^{\oplus n}\}_{s \in S}$.