Differences between Galois representations in outer-automorphisms of π_1 and those in automorphisms, implied by topology of moduli spaces

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2010/5/12, au Séminaire de Géométrie Arithmétique PARIS-TOKYO.

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This study is supported in part by JSPS Grant-In-Aid #19204002, and JSPS Core-to-Core Program No.18005. Thanks to Richard Hain for essential mathematical ingredient.

- Recall the monodromy representation on π_1 of curves.
- Galois monodromy often contains geometric monodromy.
- Using this connection, obtain implications from topology to Galois monodromy.

1. Monodromy on π_1.

- K: a field $\subset \overline{K} \subset \mathbb{C}$.
- A family of (g, n)-curves $C \to B$: $\overset{def}{\Leftrightarrow}$

  B: smooth noetherian geometrically connected scheme/ K.

  $F^{cpt} : C^{cpt} \to B$: proper smooth family of genus g curves (with geometrically connected fibers).

  $s_i : B \to C^{cpt}$ $(1 \le i \le n)$ disjoint sections,

  $F : C \to B$: complement $C^{cpt} \setminus \cup s_i(B) \to B$.
- We assume hyperbolicity $2g - 2 + n > 0$.
- $\Pi_{g,n}$: (classical) fundamental group of n-punctured genus g Riemann surface (referred to as *surface group*)
- $\Pi_{g,n}^{\wedge}$, $\Pi_{g,n}^{(\ell)}$: its profinite, resp. pro-ℓ, completion.

$$
\begin{array}{ccc}
\bar{x} & & \\
\downarrow & & \\
C_{\bar{b}} & \longrightarrow & C \\
\downarrow & \Box & \downarrow \\
\bar{b} & \longrightarrow & B
\end{array}
$$

$\bar{b}, \bar{x}$: (geometric) base points.

Gives a short exact sequence of arithmetic(=etale) π_1:

$$1 \longrightarrow \pi_1(C_{\bar{b}}, \bar{x}) \longrightarrow \pi_1(C, \bar{x}) \longrightarrow \pi_1(B, \bar{b}) \longrightarrow 1$$

The short exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(C_{\bar{b}}, \bar{x}) & \to & \pi_1(C, \bar{x}) & \to & \pi_1(B, \bar{b}) & \to & 1 \\ & & \Vert_{\text{GAGA}} & & & & & & \\ & & \Pi_{g,n}^{\wedge} & & & & & & \end{array}$$

gives the pro-ℓ outer monodromy representation:

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(If $B = \operatorname{Spec} K$, we have $\rho_{O,C} : G_K = \pi_1(B) \to \operatorname{Out} \Pi_{g,n}^{(\ell)}$.)

2. Universal monodromy. Grothendieck, Takayuki Oda, ...

- $\mathcal{M}_{g,n}$: the moduli stack of (g,n)-curves over $\mathbb{Q}$.
- $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ be the universal family of (g,n)-curves.

Applying the previous construction, we have:

$$\begin{array}{ccccccc} 1 \to & \pi_1(C_{\bar{b}}, \bar{x}) & \to & \pi_1(\mathcal{C}_{g,n}, \bar{x}) & \to & \pi_1(\mathcal{M}_{g,n}, \bar{b}) & \to 1 \\ & \downarrow & & \downarrow \rho_{A,univ,\bar{x}} & & \downarrow \rho_{O,univ} & \\ 1 \to & \operatorname{Inn}\Pi_{g,n}^{(\ell)} & \to & \operatorname{Aut}\Pi_{g,n}^{(\ell)} & \to & \operatorname{Out}\Pi_{g,n}^{(\ell)} & \to 1 \end{array}$$

This representation is universal, since any (g,n)-family $C \to B$ has classifying morphism,

$$\begin{array}{ccc} C & \to & \mathcal{C}_{g,n} \\ \downarrow & \square & \downarrow \\ B & \to & \mathcal{M}_{g,n}, \end{array}$$

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$$\begin{array}{ccccc} C_{\bar{b}} & \to & C & \to & \mathcal{C}_{g,n} \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \bar{b} & \to & B & \to & \mathcal{M}_{g,n}, \end{array}$$

then universality as follows.

$$
\begin{array}{ccccccccc}
1 & \to & \pi_1(C_{\bar b},\bar x) & \to & \pi_1(C,\bar x) & \to & \pi_1(B,\bar b) & \to & 1\\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \to & \pi_1(C_{\bar b},\bar x) & \to & \pi_1(\mathcal{C}_{g,n},\bar x) & \to & \pi_1(\mathcal{M}_{g,n},\bar b) & \to & 1\\
 & & \downarrow & & \downarrow \rho_{A,univ,\bar x} & & \downarrow \rho_{O,univ} & & \\
1 & \to & \operatorname{Inn}\Pi^{(\ell)}_{g,n} & \to & \operatorname{Aut}\Pi^{(\ell)}_{g,n} & \to & \operatorname{Out}\Pi^{(\ell)}_{g,n} & \to & 1
\end{array}
$$

where the vertical composition is $\rho_{A,C,x}$ (middle), $\rho_{O,C}$ (right). In particular, if $C \to B = b = \operatorname{Spec} K$, we have

$$\rho_{O,C} : G_K = \pi_1(b,\bar b) \to \pi_1(\mathcal{M}_{g,n}/K,\bar b) \overset{\rho_{O,univ}}{\to} \operatorname{Out}\Pi^{(\ell)}_{g,n}$$

and hence

$$\rho_{O,C}(G_K) \subset \rho_{O,univ}(\pi_1(\mathcal{M}_{g,n}/K)) \subset \operatorname{Out}\Pi^{(\ell)}_{g,n}.$$

Definition If the equality holds for the left inclusion, the curve $C \to b$ is called *monodromically full*.

Theorem (Tamagawa-M, 2000) The set of closed points in $\mathcal{M}_{g,n}$ corresponding to monodromically full curves is infinite, and dense in $\mathcal{M}_{g,n}(\mathbb{C})$ with respect to the complex topology.

Remark As usual, the π_1 of $\mathcal{M}_{g,n}$ is an extension

$$1 \to \pi_1(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}) \to \pi_1(\mathcal{M}_{g,n}) \to G_{\mathbb{Q}} \to 1.$$

The left hand side is isomorphic to the profinite completion of the mapping class group $\Gamma_{g,n}$. (Topologists studied a lot.)
Monodromically full $\Leftrightarrow$ Galois image contains $\Gamma_{g,n}$.

Scketch of Proof of Theorem goes back to Serre, Terasoma, ...
Hilbert's irreducibility + almost pro-ℓ ness.

Proposition If P is a finitely generated pro-ℓ group, then take $H := [P, P]P^\ell \triangleleft P$. Then P/H is a finite group (flattini quotient).
If a morphism of profinite groups $\Gamma \to P$ is surjective modulo H, namely

$$\Gamma \to P \to P/H$$

is surjective, then $\Gamma \to P$ is surjective.

Definition A profinite group G is *almost pro-ℓ* if it has a pro-ℓ open subgroup P.

Corollary Suppose in addition G is finitely generated.
Put $H := [P, P]P^\ell$. Then $[G : H] < \infty$.
If $\Gamma \to G \to G/H$ is surjective, so is $\Gamma \to G$.

Claim $C \to B$ be a family of (g,n)-curves over a smooth variety B over a NF K. Then the image of

$$\pi_1(B) \to \mathrm{Out}\,\Pi_{g,n}^{(\ell)}$$

is a finitely generated almost pro-ℓ group.

- Out(fin.gen.pro-ℓ) is almost pro-ℓ.
- a closed subgroup of almost pro-ℓ group is again so.
- finitely generatedness: $\pi_1(B \otimes \bar{K})$ is finitely generated. G_K not. But take $L \supset K$ so that $C(L) \neq \emptyset$ and $G_L \to \mathrm{Out}\,\Pi^{(\ell)}$ has pro-ℓ image. Only finite number of places of O_L ramifies, and class field theory tells that $\mathrm{Im}(G_L)$ has finite flattini quotient.

Corollary $\exists H < \mathrm{Im}(\pi_1(B))$ such that

$$\Gamma \twoheadrightarrow \mathrm{Im}(\pi_1(B))/H \qquad \text{implies} \qquad \Gamma \twoheadrightarrow \mathrm{Im}(\pi_1(B)).$$

Corollary Take a subgroupthe above H for the image of $\pi_1(B)$. H' the inverse image in $\pi_1(B)$. Let $B' \to B$ be the etale cover corresponding to H'. If $b \in B$ has a connected fiber (i.e. one point) in B', Then the composition

$$G_{k(b)} \to \mathrm{Im}(\pi_1(B)) \to \mathrm{Im}(\pi_1(B))/H$$

is surjective, hence the left arrow is surjective.

Last Claim

Existence of many such b follows from Hilbertian property:
Take a quasi finite dominating ratl. map $B \to \mathbb{P}_K^{\dim B}$.
Apply Hilbertian property to $B' \to B \to \mathbb{P}_K^{\dim B}$.

3. Aut and Out. Again consider $C \to b = \operatorname{Spec} K$. Take a closed point x in C, and $\bar{x}$ a geometric point. This yields

$$\begin{array}{ccccccccc} & & & & G_{k(x)} & & & & \\ & & & & \downarrow x_* & & & & \\ 1 & \to & \pi_1(C_{\bar{b}}, \bar{x}) & \to & \pi_1(C, \bar{x}) & \to & G_K & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \operatorname{Inn} \Pi_{g,n}^{(\ell)} & \to & \operatorname{Aut} \Pi_{g,n}^{(\ell)} & \to & \operatorname{Out} \Pi_{g,n}^{(\ell)} & \to & 1. \end{array}$$

Vertical composition gives

$$\begin{array}{rccc} \rho_{A,x} : & G_{k(x)} & \to & \operatorname{Aut} \Pi_{g,n}^{(\ell)} \\ & \cap & & \downarrow \\ \rho_O : & G_K & \to & \operatorname{Out} \Pi_{g,n}^{(\ell)} \end{array}$$

Question: Is the map $AO(C, x) : \rho_{A,x}(G_{k(x)}) \to \rho_O(G_K)$ injective? (Do we lose information in $\operatorname{Aut} \to \operatorname{Out}$?)

Definition
$I(C,x) :=$ the statement "$AO(C,x)$ is injective."

Remark If $C = \mathbb{P}^1 - \{0,1,\infty\}/\mathbb{Q}$ and x is a canonical tangential base point, then $AO(C,x)$ is an isom (hence $I(C,x)$ holds: Belyi, Ihara, Deligne, 80's).

Main Theorem (M, 2009) Suppose $g \geq 3$ and ℓ divides $2g-2$. Let $C \to \operatorname{Spec} K$ be a monodromically full $(g,0)$-curve ($[K:\mathbb{Q}] < \infty$).
Then, for every closed point x in C such that $\ell \not| [k(x):K]$, $I(C,x)$ does not hold.
In this case, the kernel of $AO(C,x)$ is infinite.

A topological result.

Proof reduces to a topological result.

$\Gamma_{g,n} := \pi_1^{orb}(\mathcal{M}_{g,n}^{an})$.

$\Gamma_g := \Gamma_{g,0}$, $\Pi_{g,0} = \Pi_g$.

Topological version of universal family yields

$$1 \to \Pi_g \to \Gamma_{g,1} \to \Gamma_g \to 1$$

and by putting $H := \Pi_g^{ab} =: \Pi_g/\Pi_g'$

$$1 \to H \to \Gamma_{g,1}/\Pi_g' \to \Gamma_g \to 1$$

Theorem (S. Morita 98, Hain-Reed 00). Let $g \geq 3$. The cohomology class of the above extension

$$[e] \in H^2(\Gamma_g, H)$$

has the order $2g - 2$.

Proof of Main Theorem. Suppose $\ell | (2g-2)$, $x \in C$ with $\ell \not| [k(x) : K]$. Suppose $I(C,x)$, namely the image of $G_{k(x)}$ in the middle

$$G_{k(x)} \to \operatorname{Aut} \Pi_g^{(\ell)} \to \operatorname{Out} \Pi_g^{(\ell)}$$

is same with the image in the third. Let S be this image. This gives a restricted section from S to the middle group:

$$\begin{array}{ccccccccc} 1 & \to & \operatorname{Inn} \Pi_g^{(\ell)} & \to & \operatorname{Aut} \Pi_g^{(\ell)} & \to & \operatorname{Out} \Pi_g^{(\ell)} & \to & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \to & \operatorname{Inn} \Pi_g^{(\ell)} & \to & \operatorname{Im} \rho_{A,univ,x} & \to & \operatorname{Im} \rho_{O,univ} & \to & 1 \\ & & & & & & \vee & & \\ & & & & & & S & & \end{array}$$

By taking the quotient by the commutator ${\Pi_g^{(\ell)}}'$, we have the top short exact sequence in the following:

$$\begin{array}{ccccccccc}
1 & \to & H^{(\ell)} & \to & \operatorname{Im}\rho_{A,univ,x}/\Pi^{(\ell)\prime} & \to & \operatorname{Im}\rho_{O,univ} & \to & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \to & H^{(\ell)} & \to & \widetilde{\Gamma_g} & \to & \Gamma_g & \to & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \to & H & \to & \Gamma_{g,1}/\Pi' & \to & \Gamma_g & \to & 1.
\end{array}$$

The middle row is the pullback along $\Gamma_g \to \operatorname{Im}(\rho_{O,univ})$.
The bottom row is the classic topological one.
Let $[e_{univ}] \mapsto [e_\ell] \leftarrow [e]$ be the corresponding elements in

$$H^2(\operatorname{Im}\rho_{O,univ}, H^{(\ell)}) \to H^2(\Gamma_g, H^{(\ell)}) \leftarrow H^2(\Gamma_g, H).$$

order: (a multiple of ℓ^ν or ∞), ℓ^ν, $2g-2$, resp., where $\ell^\nu || 2g-2$.

By assuming $I(C,x)$, a section restricted to S exists for

$$\begin{array}{ccccccccc} & & & & & & S & & \\ & & & & & & \wedge & & \\ 1 & \to & H^{(\ell)} & \to & \operatorname{Im}\rho_{A,univ,x}/\Pi^{(\ell)\prime} & \to & \operatorname{Im}\rho_{O,univ} & \to & 1. \end{array}$$

Now monodromically fullness implies

$$\operatorname{Im}\rho_{O,univ} = \rho_{O,C}(G_K),$$

and $S = \rho_{O,C}(G_{k(x)})$ is a finite index subgroup with index dividing $[k(x) : K]$, hence coprime to ℓ. This implies that the restriction of $[e_{univ}]$ by

$$H^2(\operatorname{Im}\rho_{O,univ}, H^{(\ell)}) \to H^2(S, H^{(\ell)})$$

does not vanish, hence there should be no restricted section from S, a contradiction.

Remark Recently Yuichiro Hoshi (RIMS) proved

- For any (g, n)-curve C over number field, $\exists$ ∞-many closed points x such that $I(C, x)$ does not hold.
- There are examples where $I(C, x)$ holds for (not tangential, usual) closed point x for proper / affine curves.

THIS IS THE END : Thank you for listening