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On Strongly Invertible Knots

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A knot K in the 3-sphere S^3 is said to be *strongly invertible*, if there is an orientation-preserving involution h on S^3, such that

(1) $h(K)=K$,

(2) $\mathrm{Fix}(h)$ is a circle intersecting K in two points.

There are "many" strongly invertible knots; in fact, about 85 percent of the prime knots with 10 crossings or less are strongly invertible (see [13]). Moreover, Sakai [34] has shown that every knot polynomial is the Alexander polynomial of a strongly invertible knot. Other aspects of the strongly invertible knots can be found in [2, 3, 4, 11, 15, 16, 19, 21, 23, 42, 44].

In this paper, we consider the pair (K, h), where K is a knot in an oriented S^3 and h is an involution on S^3 satisfying the above conditions (1) and (2). We use the term "*strongly invertible knot*" to mean such a pair, and two such pairs (K, h) and (K', h') are said to be *equivalent*, denoted by $(K, h)\cong(K', h')$, iff there is an orientation-preserving homeomorphism f on S^3 such that $f(K)=K'$ and $f\circ h\circ f^{-1}=h'$.

In Section 1, we define the equivariant connected sum of strongly invertible knots and present the unique prime decomposition theorem (Theorem I).

In Section 2, we define a polynomial invariant of a pair (K, h), which we call the *η-polynomial*, and denote by the symbol $\eta_{(K,h)}(t)$. The η-polynomial is characterized by the following properties (Theorem II);

(1) $\eta_{(K,h)}(t)=\eta_{(K,h)}(t^{-1})$,

(2) $\eta_{(K,h)}(\pm 1)=0$.

And it takes the value 0 if K is a trivial knot. The calculation of the η-polynomial is very easy (see Section 2), and it does not always vanish even if the Alexander polynomial is trivial (Example 4.2). Moreover, we show that the η-polynomial is an equivariant coborism invariant, and in fact, it gives a homomorphism from the group of all equivariant cobordism classes of the

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strongly invertible knots to the additive group $\mathbf{Z}\langle t\rangle$ (Theorem III).

In the appendix, we give a table of the symmetry groups and the η-polynomials of the prime knots with 9 crossings or less. To do this, we use a certain relationship between strong invertibility and periodicity of a simple knot (Proposition 3.1), which is deudced from the results of Thurston [41]. A relation between strong invertibility and amphicheirality of a simple knot is also observed, by which a certain criterion for an invertible knot to be nonamphicheiral is given in terms of the η-polynomial (Proposition 3.4).

1. Equivariant prime decomposition

For given two strongly invertible knots (K_i, h_i) $(i=1, 2)$, we define their equivariant connected sum $(K_1, h_1)\#(K_2, h_2)$ as follows. Let z_i be a point of $\mathrm{Fix}(h_i)\cap K_i$ and B_i be an equivariant regular neighbourhood of z_i for each $i=1, 2$, and let f be an orientation-reversing equivariant homeomorphism from $\partial(B_1, B_1\cap K_1)$ to $\partial(B_2, B_2\cap K_2)$. Then the manifold pair

$$\{(S^3, K_1)-(\mathring{B}_1, \mathring{B}_1\cap K_1)\}\cup_f\{(S^3, K_2)-(\mathring{B}_2, \mathring{B}_2\cap K_2)\}$$

is homeomorphic to $(S^3, K_1\# K_2)$, and the involutions h_1 and h_2 naturally determine an inverting involution h of $(S^3, K_1\# K_2)$. We want to define $(K_1, h_1)\#(K_2, h_2)$ to be $(K_1\# K_2, h)$. But there remains the following ambiguities in this "definition".

(1) The choice of the point $z_i\in\mathrm{Fix}(h_i)\cap K_i\cong S^0$ for each $i=1, 2$.

(2) The choice of the equivariant homeomorphism f.

To remove these ambiguities, we attach the following additional informations to each strongly invertible knot (K, h).

(i) An orientation of $\mathrm{Fix}(h)$.

(ii) A "base point" ∞ of $\mathrm{Fix}(h)$, which lies in one of the components of $\mathrm{Fix}(h)-K$.

We call these additional informations a *direction* of (K, h), and a strongly invertible knot with a direction is said to be *directed*. We can now define the equivariant connected sum of two directed strongly invertible knots (K_i, h_i) $(i=1, 2)$ as indicated in Fig. 1.1. This indication clearly resolves the ambiguity (1). Concerning the ambiguity (2), it specifies the restriction of f to $\partial B_1\cap\mathrm{Fix}(h_1)$. Let g be another orientation-reversing equivariant homeomorphism from $\partial(B_1, B_1\cap K_1)$ to $\partial(B_2, B_2\cap K_2)$ of which restriction to $\partial B_1\cap\mathrm{Fix}(h_1)$ is equal to that of f. Then g is equivariantly isotopic to either f or $(h_2|_{\partial B_2})\circ f$ rel. $\partial B_1\cap K_1$. [Proof. The equivariant homeomorphism $g\circ f^{-1}$ on $\partial(B_2, B_2\cap K_2)$ induces a homeomorphism ψ on $\partial B_2/h_2$ which is identity on the subset $P=\{\partial B_2\cap(\mathrm{Fix}(h_2)\cup K_2)\}/h_2$. Note that P consists of three

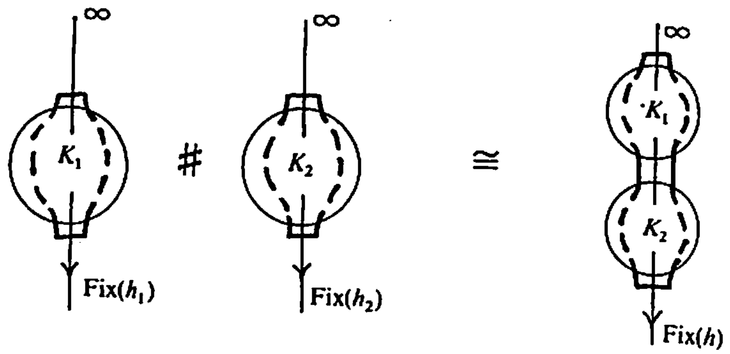


Fig. 1.1

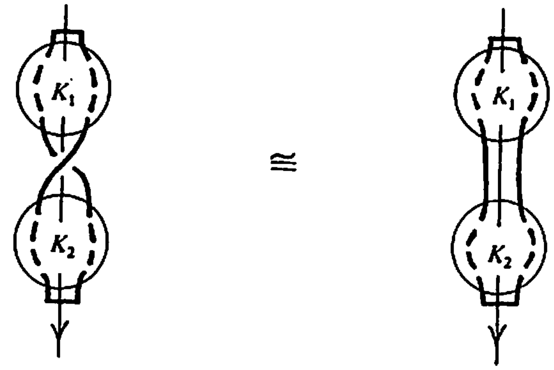


Fig. 1.2

points. Then, by Theorem 4.5 of [1], ψ is isotopic to the identity map rel. P. This isotopy lifts to an equivariant isotopy between $g\circ f^{-1}$ and either the identity map or $h_2|_{\partial B_2}$.] Moreover, as shown in Fig. 1.2, f and $(h_2|_{\partial B_2})\circ f$ determine the equivalent strongly invertible knots. Thus the equivariant connected sum is well-defined for directed strongly invertible knots.

Definition 1.1. (1) (K, h) is said to be *trivial*, if K is a trivial knot and h is the standard inverting involution.

(2) (K, h) is said to be *prime*, if it is not trivial, and is not equivalent to a sum of two nontrivial strongly invertible knots.

(3) For an oriented knot $k=(S^3, k)$, $D(k)$ denotes the strongly invertible knot $(k\#-k, h)$, where $-k$ denotes the knot $(S^3, -k)$ and h is the inverting involution which interchanges the factors k and $-k$.

(4) For a finite sequence $\{(K_i, h_i)\,|\,1\leqq i\leqq n\}$ of directed strongly invertible knots, $\#_{i=1}^n(K_i, h_i)$ denotes the directed strongly invertible knot $(((K_1, h_1)\#(K_2, h_2))\#(K_3, h_3))\#\cdots)\#(K_n, h_n)$.

The set $\tilde{\mathscr{S}}$ of all directed strongly invertible knots together with the operation # forms a non-commutative semi-group. It is easily seen that $D(k)$ (with any direction) belongs to the center of the semi-group. (Furthermore, by the unique decomposition theorem stated below, the center consists only of $D(k)$'s.) We have the followings.

Lemma 1.2. (1) (*Marumoto* [21] *Proposition* 2) *(K, h) is trivial, iff K is trivial.*

(2) *(K, h) is prime, iff K is prime or $(K, h) \cong D(k)$ for some prime knot k.*

Theorem I. (1) *Any nontrivial, directed, strongly invertible knot (K, h) has an equivariant prime decomposition. Any prime decomposition is equivalent to a decomposition $(K, h) \cong \{\#_{i=1}^{r}(K_i, h_i)\} \# \{\#_{j=1}^{s} D(k_j)\}$ where K_i $(1 \leqq i \leqq r)$ and k_j $(1 \leqq j \leqq s)$ are prime knots.*

(2) *Let $\{\#_{i=1}^{r}(K_i, h_i)\} \# \{\#_{j=1}^{s} D(k_j)\}$ and $\{\#_{i=1}^{r'}(K'_i, h'_i)\} \# \{\#_{j=1}^{s'} D(k'_j)\}$ be prime decompositions of a directed strongly invertible knot. Then the followings hold.*

(a) *$r = r'$ and $(K_i, h_i) \cong (K'_i, h'_i)$ for each i $(1 \leqq i \leqq r)$.*

(b) *$s = s'$ and after a permutation $D(k_j) \cong D(k'_j)$ for each j $(1 \leqq j \leqq s)$.*

Here, $\cong$ denotes the equivalence as directed strongly invertible knots.

To prove Lemma 1.2 (2) and Theorem I, we need the following: For a strongly invertible knot (K, h), let $\theta(K, h) = p(\mathrm{Fix}(h)) \cup p(K)$, where p is the projection $S^3 \to S^3/h \cong S^3$. We call it the *θ-curve* associated with (K, h). $\theta(K, h)$ is said to be *prime*, if it is "nontrivial", and if every 2-sphere intersecting $\theta(K, h)$ transversely at three points bounds a 3-ball B such that $(B, B \cap \theta(K, h))$ is homeomorphic to the cone over $\partial(B, B \cap \theta(K, h))$. $\theta(K, h)$ is said to be *irreducible*, if it is prime, and does not contain a local knot. Note that the 2-fold branched cover $\Sigma(K)$ of K is homeomorphic to the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ branched cover of $\theta(K, h)$. Let τ be the covering transformation of $\Sigma(K)$, and $\tilde{h}$ be a lift of h to $\Sigma(K)$. Then, by the results of [21], [22], and [43], we obtain the following equivalences.

(i) K is trivial. $\rightleftarrows \Sigma(K) \cong S^3$.
$\rightleftarrows \theta(K, h)$ is trivial. (Cf. [24])

(ii) K is prime. $\rightleftarrows \Sigma(K)$ has no essential 2-sphere.
$\rightleftarrows \theta(K, h)$ is irreducible.

(iii) (K, h) is prime. $\rightleftarrows \Sigma(K)$ does not have $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ invariant essential 2-sphere.
$\rightleftarrows \theta(K, h)$ is prime.

(iv) (K, h) is prime, but K is not prime.
$\rightleftarrows \Sigma(K)$ contains an essential 2-sphere S, such that

$\tau(S)=S$ and $\tilde{h}(S)\cap S=\emptyset$.
$\rightleftarrows\theta(K,h)$ is prime, but not irreducible.
$\rightleftarrows(K,h)\cong D(k)$ for some prime knot k.

In particular, we obtain Lemma 1.2 (2). The firtst half of Theorem I follows from Lemma 1.2 and the existence of the prime decomposition of a knot [37]. The above observations say that we have only to show the uniqueness of the "prime decomposition" of a θ-curve to prove the latter half of Theorem I. But this can be done by a standard cut and paste method; so, we omit it. We note that the prime decomposition of a θ-curve mentioned here may be considered as the prime decomposition of it as an orbifold [*A* θ-curve is viewed as the branch line of a $Z_2\oplus Z_2$ branched cover.], and the existence and the uniqueness of the prime decómoposition of a "pseudo-good" orbifold are claimed by Bonahon-Siebenmann [5].

2. η-polynomial

For a strongly invertible knot (K,h), let N be an equivariant tubular neighbourhood of K, and $l\subset\partial N$ be a preferred longitude of N (cf. [32]), such that $h(l)\cap l=\emptyset$. Put $O=p(\mathrm{Fix}(h))$ and $L=p(l)$, where p is the projection $S^3\to S^3/h\cong S^3$. Then $L(K,h)\equiv O\cup L$ is a 2-component link in S^3 and $\mathrm{lk}(O,L)=0$ (cf. Fig. 2.1). Let $\eta_{(K,h)}(t)$ be the η-function of the link $L(K,h)$ defined by Kojima-Yamasaki [18] (cf. [8, 9]); that is, the Laurent polynomial with integral coefficients defined by the equality

$$\eta_{(K,h)}(t)=\sum_{i=-\infty}^{\infty}\mathrm{lk}(\tilde{L}',t^i\tilde{L})t^i,$$

where $\tilde{L}$ is a lift of L to the infinite cyclic cover $\widetilde{E(O)}$ of $E(O)\equiv S^3-O$, $\tilde{L}'$ is

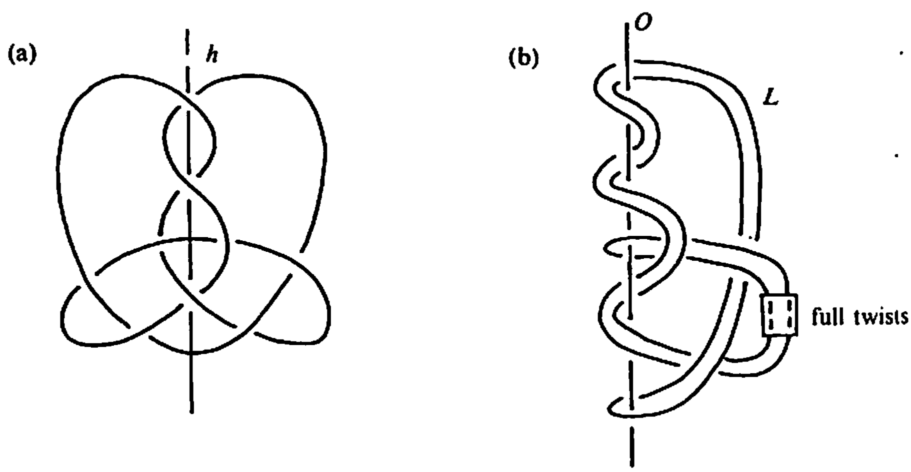


Fig. 2.1

the lift of a preferred longitude L' of L near $\tilde{L}$, and t is a generator of the covering transformation group of $\widetilde{E(O)}$. Clearly $\eta_{(K,h)}(t)$ is an invariant of the equivalence class of (K, h), and we call it the *η-polynomial* of (K, h). We remark that this polynomial was originally introduced in the joint paper of S. Furusawa and the author [7] for a different purpose.

Example 2.1. *Let* $K=9_{49}$ and h be the involution as illustrated in Fig. 2.1 (a). Then $\eta_{(K,h)}(t)=[-6, 2, 2, -2, 1$. Here, $[a_0, a_1, \cdots, a_n$ represents the polynomial $a_0+\sum_{i=1}^{n} a_i(t^{-i}+t^i)$.

The folloiwng theorem characterizes the η-polynomial.

Theorem II. (1) $\eta_{(K,h)}(t)$ *satisfies the following conditions*:

(i) $\eta_{(K,h)}(t)=\eta_{(K,h)}(t^{-1})$,

(ii) $\eta_{(K,h)}(1)=0$,

(iii) $\eta_{(K,h)}(-1)=0$.

(2) *Conversely, for any Laurent polynomial* $\eta(t)$ *with integral coefficients satisfying the above conditions, there exists a strongly invertible knot* (K, h) *such that* $\eta_{(K,h)}(t)=\eta(t)$.

Proof. (1) The first and the second conditions are found in Proposition 2 of [18]. [The second condition follows from the equality $\eta_{(K,h)}(1)=\mathrm{lk}(L', L)=0$.] The third condition follows from the second condition and the equality

$$\sum_{i=\mathrm{odd}}(\text{the coefficient of } t^i)=\mathrm{lk}(l, h(l))=0\,.$$

(2) is proved at the end of this section.

We now give a convenient method for calculating the η-polynomial. The method consists of the following six steps.

Step 1. Draw the projection of $\theta(K, h)$ so that (i) $O=p(\mathrm{Fix}(h))$ is represented by a straigt line, and (ii) the two vertices of $\theta(K, h)$ are on the top and the bottom respectively (see Fig. 2.2 (b))

Step 2. From the above projection, construct the "pseudo-fundamental region" of the infinite cyclic cover $\widetilde{E(O)}$ as indicated in Fig. 2.2 (c).

Step 3. Assign an index and an orientation to each arc in the pseudo-fundamental region as follows (see Fig. 2.2 (c)).

(i) The top arc has index 0, and is oriented downward.

(ii) Suppose an arc α is already indexed and directed. Let A be the end point of α, and B be the point opposite to A. Then the arc β which contains B is oriented so that B is the starting point of β, and index(β) is defined to be index $(\alpha)+1$ or index$(\alpha)-1$ according as B is on the right side or on the left side.

Step 4. Assign each double point P a signature $\varepsilon_P \in \{+, -\}$ and an integer $d_P \in \mathbf{Z}$ as follows. Let α and β be the over-pass and the under-pass at P respectively. Then $d_P = \text{index}(\alpha) - \text{index}(\beta)$, and $\varepsilon_P = +$ or $-$ according as β crosses α from left to right or from right to left.

Step 5. Let $\tilde{\eta}$ be the integral polynomial of $\{x_i \mid i \in \mathbf{Z}\}$ define by the equality $\tilde{\eta} = \sum_P \varepsilon_P x_{d_P}$, where P ranges over all double points in the pseudo-fundamental region.

Step 6. Let $\eta'(t)$ be the Laurent polynomial obtained from $\tilde{\eta}$ by putting $x_i = t^{i-1} - 2t^i + t^{i+1}$. $\eta'(t)$ is symmetric, and theorefore it takes the form of $[b_0, b_1, \cdots, b_n$. Then, we have

$$\eta_{(K,h)}(t) = [a_0, a_1, \cdots, a_n,$$

where

$$a_i = \begin{cases} -2\sum_{j \geqq 1} b_{2j} & (i=0) \\ -\sum_{j \geqq 1} b_{2j+1} & (i=1) \\ b_i & (i \geqq 2). \end{cases}$$

Example 2.2. The θ-curve and the pseudo-fundamental region of the strongly invertible knot (K, h) given in Example 2.1 are illustrated in Fig. 2.2. Thus $\tilde{\eta} = x_{-3} + x_{-1} + x_0 + x_1 + x_3$, and $\eta'(t) = [0, -1, 2, -2, 1,$ and therefore $\eta_{(K,h)}(t) = [-6, 2, 2, -2, 1$.

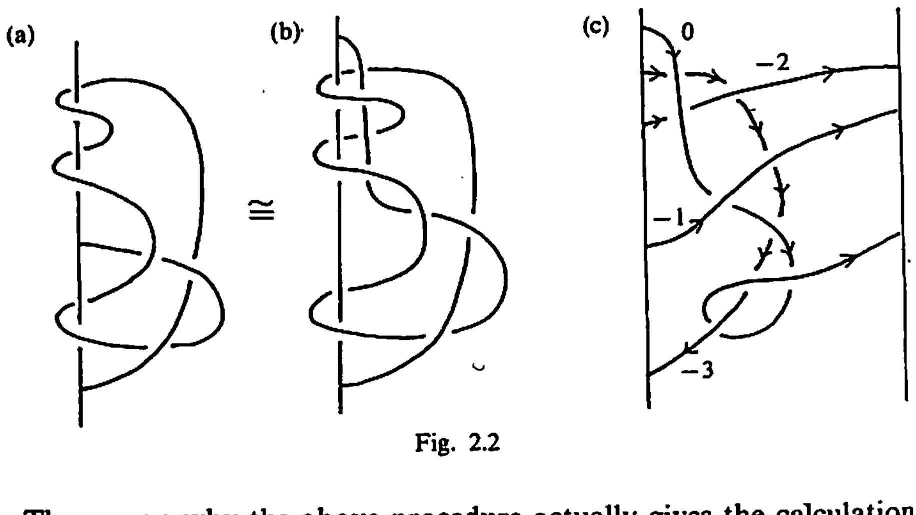


Fig. 2.2

The reason why the above procedure actually gives the calculation of the η-polynomial can be seen by comparing Fig. 2.2 (c) with Fig. 2.3 (b) which gives a fundamental region of $\widetilde{E(O)}$. Note that each crossing in the pseudo-fundamental region corresponds to a set of four crossings in the fundamental region as shown in Fig. 2.4. This suggests the correspondence between x_i and $t^{i-1}-2t^i+t^{i+1}$.

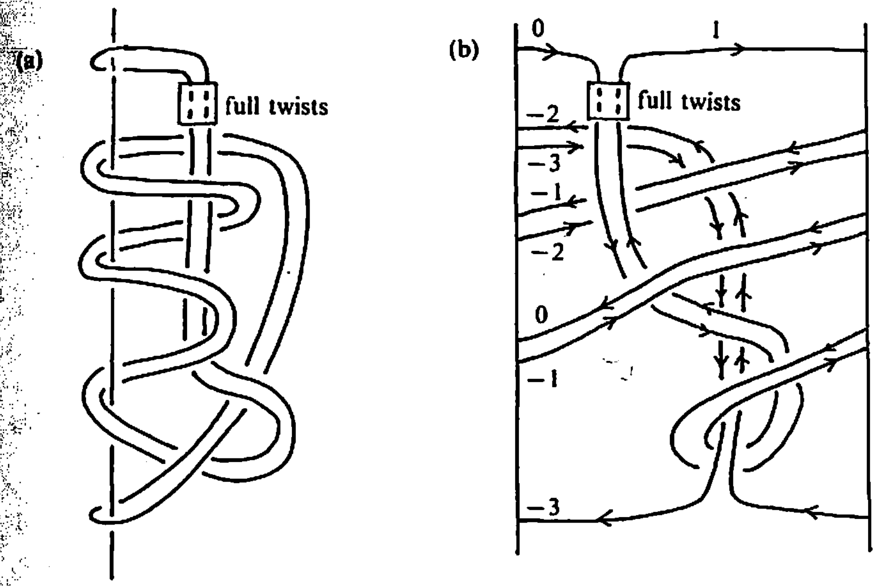


Fig. 2.3

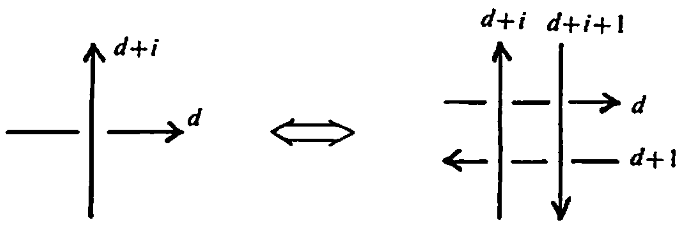


Fig. 2.4

It is well-known that every 2-bridge knot is strongly invertible, and as is proved in the next section (Proposition 3.6), a strongly invertible knot (K, h) obtained from a 2-bridge knot is equivalent to the strongly invertible knot $I_1(\alpha_1, \cdots, \alpha_n; c_1, \cdots, c_n)$ or $I_2(a_1, \cdots, a_n)$ which are given in Fig. 2.5 (a) and (b) respectively. Here, α_i, c_i, and a_i are integers, and we may assume that a_i $(1 \leq i \leq n)$ are even. Their η-polynomials are calculated through the method given in this section. We state them without proof.

Proposition 2.3. (1) *The η-polynomial $\eta(t)$ of $I_1(\alpha_1, \cdots, \alpha_n; c_1, \cdots, c_n)$ is as follows. Let c_i' and δ_i be the integers determined by the equalities $c_i = 2c_i' + \delta_i$, $\delta_i = 0$ or 1, and put $\varepsilon_i = \prod_{j=i}^{n}(-1)^{\delta_j}$. Then*

$$\eta(t) = -\sum_{i=1}^{n}\{c_i' D(\sum_{j=1}^{i}\varepsilon_j\alpha_j) + \delta_i W(\varepsilon_i \sum_{j=1}^{i}\varepsilon_j\alpha_j)\} + r(t) .$$

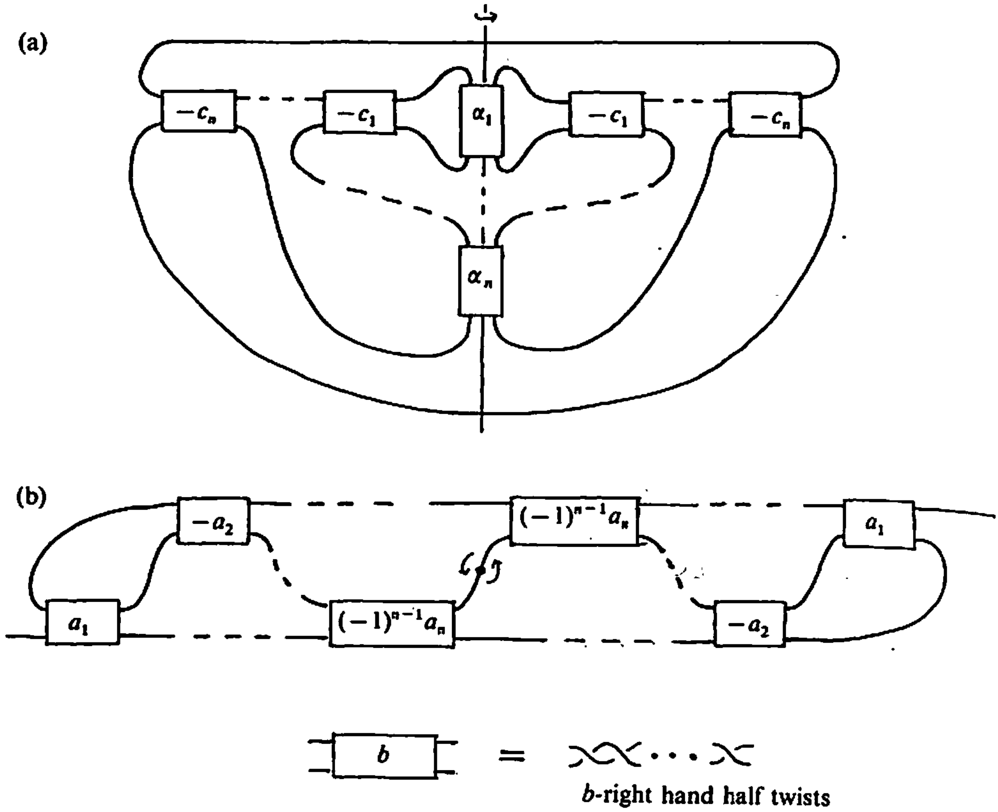


Fig. 2.5

Here, $D(\alpha)=x_{-\alpha}+x_\alpha$,

$$W(\alpha)=\begin{cases}\sum_{i=1}^{\alpha}D(i) & (\alpha\geqq 1)\,,\\ 0 & (\alpha=0,\ -1)\,,\\ -\sum_{i=1}^{|\alpha|-1}D(i) & (\alpha\leqq -2)\,,\end{cases}$$

$$x_\alpha=t^{\alpha-1}-2t^\alpha+t^{\alpha+1}\,,$$

and $r(t)$ *is the symmetric Laurent polynomial of degree* $\leqq 2$ *which is uniquely determined by the equality* $\eta(1)=\eta(-1)=0$.

(2) *Suppose that* a_i $(1\leqq i\leqq n)$ *are even integers. Then the* η*-polynomial* $\eta(t)$ *of* $I_2(a_1, \cdots, a_n)$ *is as follows.*

$$\eta(t)=\frac{1}{2}\sum_{j\geqq 0}a_{2j+1}(2-(t^{-2}+t^2))\,.$$

In particular, the η-polynomial of $I_1(-1, \cdots, -1, -2b_2, \cdots, -2b_n)$ is

$$r'(t)+b_2(t^{-2}+t^2)+\sum_{i=3}^{n}b_i(t^{-i}-2t^{-i+1}+t^{-i+2}+t^{i-2}-2t^{i-1}+t^i)\,,$$

where $r'(t)$ is a symmetric polynomial of degree 2. Hence, by choosing b_i $(2 \leqq i \leqq n)$ suitably, we can realize every polynomial which satisfies the conditions given by Theorem II (1) (cf. [33]). This proves the latter half of Theorem II.

3. Invertible simple knots

Kojima [17] showed that any nontrivial link admits only finitely many symmetries, using the results of Thurston [41]. Thus each knot admits only finitely many inverting involutions. The following observation refines this result for simple knots. (See [36], for other refinements.)

Proposition 3.1. (1) *A torus knot admits exactly one inverting involution.*

(2) *Let K be an invertible hyperbolic knot. Then K admits exactly two or one inverting involutions according to whether K has (cyclic or free) period 2 or not.*

Remark 3.2. The assumption that K is simple is essential. In fact, for any positive integer n, we can easily construct a composite knot which has more than n inverting involutions by using Theorem I. It is also possible to construct such a prime knot (see [36]).

Proof. (1) can be seen by using the fact that an involution on the complement of a torus knot preserves a Seifert fibration on it (see [38]).

(2) Let K be an invertible hyperbolic knot, $E(K)$ be its complement, and $\mathrm{Isom}^+E(K)$ be the group of all orientation-preserving isometries of $E(K)$. Then we have the following.

Lemma 3.3. (Riley [30] p. 124) $\mathrm{Isom}^+E(K)$ *is isomorphic to the dihedral group* D_n *of order* $2n$ *for some positive integer* n.

Proof of Lemma 3.3. By Proposition 5.1 of [40], $\mathrm{Isom}^+E(K)$ naturally acts on (S^3, K). Let ψ: $\mathrm{Isom}^+E(K) \to \mathrm{Diff}(K)$ denote the restriction of the action. Then, ψ is injective by the Smith conjecture [25]. Noting that $\mathrm{Isom}^+E(K)$ is finite and K is invertible, we obtain the desired result.

So we may assume that $\mathrm{Isom}^+E(K) = \langle f, h \mid f^n = h^2 = 1,\ hfh^{-1} = f^{-1} \rangle$, and the orientation of K is preserved by f and reversed by h. An element g of $\mathrm{Isom}^+E(K)$ gives an inverting involution, iff $g = f^i h$ for some i $(0 \leqq i \leqq n-1)$ (see [3, 16]); and two such elements $g = f^i h$ and $g' = f^{i'} h$ represent the equivalent involutions, iff they are conjugate in $\mathrm{Isom}^+E(K)$, that is, $i \equiv i'$ modulo the greatest common divisor of 2 and n. Hence K admits exactly two or one "isometric" inverting involutions according to whether n is even or odd. So the proposition follows from the results of Thurston [41] and the fact that n is even iff K has (cyclic or free) period 2.

The following gives a relation between amphicheirality and strong invertibility of a hyperbolic knot, and provides a criterion for proving nonamphicheirality of a hyperbolic knot.

Proposition 3.4. *Let K be an invertible hyperbolic knot which is amphicheiral.*

(1) *Assume that K does not have period 2, and let h be the unique inverting involution. Then $(K, h) \cong (K, h)^*$, where $(K, h)^*$ is the strongly invertible knot obtained from (K, h) by reversing the orientation of S^3. In particular, $\eta_{(K,h)}(t) = 0$.*

(2) *Assume that K has period 2, and let h_1 and h_2 be the inequivalent inverting involutions. Then $(K, h_1) \cong (K, h_2)^*$. In particular, $\eta_{(K,h_1)}(t) = -\eta_{(K,h_2)}(t)$.*

Proof. (1) is a direct consequence of Proposition 3.2.

(2) We may assume that

$$\mathrm{Isom}^+ E(K) = \langle f, h \mid f^n = h^2 = 1,\ hfh^{-1} = f^{-1} \rangle$$

with n even, and $h_1 = h$ and $h_2 = fh$. Since K is amphicheiral, $\mathrm{Isom}\, E(K)$ is an exension of $\mathrm{Isom}^+ E(K)$ by $\mathbf{Z}_2$, and it naturally acts on (S^3, K). Let $\tilde{\psi}: \mathrm{Isom}\, E(K) \to \mathrm{Diff}(K)$ be the restriction of the action to K. Suppose that $\tilde{\psi}$ is not injective. Then, by the proof of Lemma 3.3, there is an orientation-reversing isometry γ such that $K \subset \mathrm{Fix}(\gamma)$. By Smith theory [6], $\mathrm{Fix}(\gamma) \cong S^2$, and therefore K is a trivial knot; a contradiction. Thus $\tilde{\psi}$ is injective. So we have $\mathrm{Isom}\, E(K) = \langle \gamma, h \mid \gamma^{2n} = h^2 = 1,\ h\gamma h^{-1} = \gamma^{-1} \rangle \cong D_{2n}$, where γ is orientation-reversing and $\gamma^2 = f$. Then $\gamma h_1 \gamma^{-1} = \gamma h \gamma^{-1} = fh = h_2$. Hence $(K, h_1) \cong (K, h_2)^*$.

Example 3.5. Though the knots 10_{104} and 10_{155} in the table of [32] have trivial signatures, we can confirm their nonamphicheirality as follows (cf. [29]).

(1) 10_{104} does not have period 2 by [12, 20, 26], and it has a unique inverting involution (see Fig. 3.1 (a)). But, its η-polynomial is [2, −1, 1, −1.

(2) 10_{155} has free period 2 by [12], and it has two inverting involutions (see Fig. 3.1 (b)). But, their η-polynomials are 0 and [−4, 0, 2. (For a relation between amphicheirality and free periodicity, see [35].)

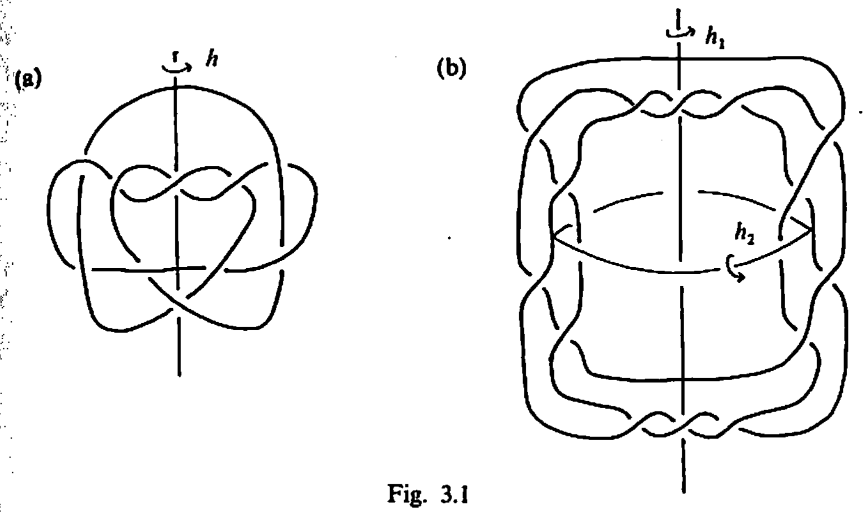


Fig. 3.1

Let $K(p, q)$ be the 2-bridge knot of type (p, q). It is well-known that $K(p, q)$ is simple, invertible, and has cyclic period 2. So, if $K(p, q)$ is not a torus knot (that is, if $q \not\equiv 1$ mod. p), then $K(p, q)$ admits exactly two inverting involutions. We now describe them. Since p is odd, we may assume that q is even and $1 < |q| < p$. Then p/q has the unique continued fraction expansion

$$\frac{p}{q} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n},$$

where a_i $(1 \leqq i \leqq n)$ and n are non-zero even integers (see [39]). We denote it by the symbol $[a_1, a_2, \cdots, a_n]$.

Proposition 3.6. (1) *Assume that* $q^2 \not\equiv 1$ (mod. p). *Then the strongly invertible knots obtained from* $K(p, q)$ *are*

$$I_1(a_1, a_3, \cdots, a_{n-1}; a_2/2, a_4/2, \cdots, a_n/2)$$

and

$$I_1(-a_n, -a_{n-2}, \cdots, -a_2; -a_{n-1}/2, -a_{n-3}/2, \cdots, -a_1/2)\,.$$

(2) *Assume that* $q^2 \equiv 1$ (mod. p) [*and* $q \not\equiv 1$ (mod. p)]. *Then the strongly invertible knots obtained from* $K(p, q)$ *are*

$$I_1(a_1, a_3, \cdots, a_{n-1}; a_2/2, a_4/2, \cdots, a_n/2) \quad \textit{and} \quad I_2(a_1, a_2, \cdots, a_{n/2})\,.$$

Proof. According to Coway's calculation of the outer-automorphism group of the 2-bridge knot groups (see [10]), $\mathrm{Isom}^+ E(K(p, q)) = \langle f, h \mid f^s = h^2 = 1, hfh^{-1} = f^{-1} \rangle$, where $s = 2$ or 4 according to whether $q^2 \not\equiv 1$ (mod. p) or not. By a similar argument as that of [14] Lemma 2 (cf. [39]), we can see that

$q^2 \equiv 1$ (mod. p), iff $a_i = -a_{n-i}$ for each i $(1 \leq i \leq n)$. Thus the action of $\mathrm{Isom}^+ E(K(p, q))$ on $(S^3, K(p, q))$ is as illustrated in Fig. 3.2. Noting that h and fh give inequivalent inverting involutions, we obtain the desired results.

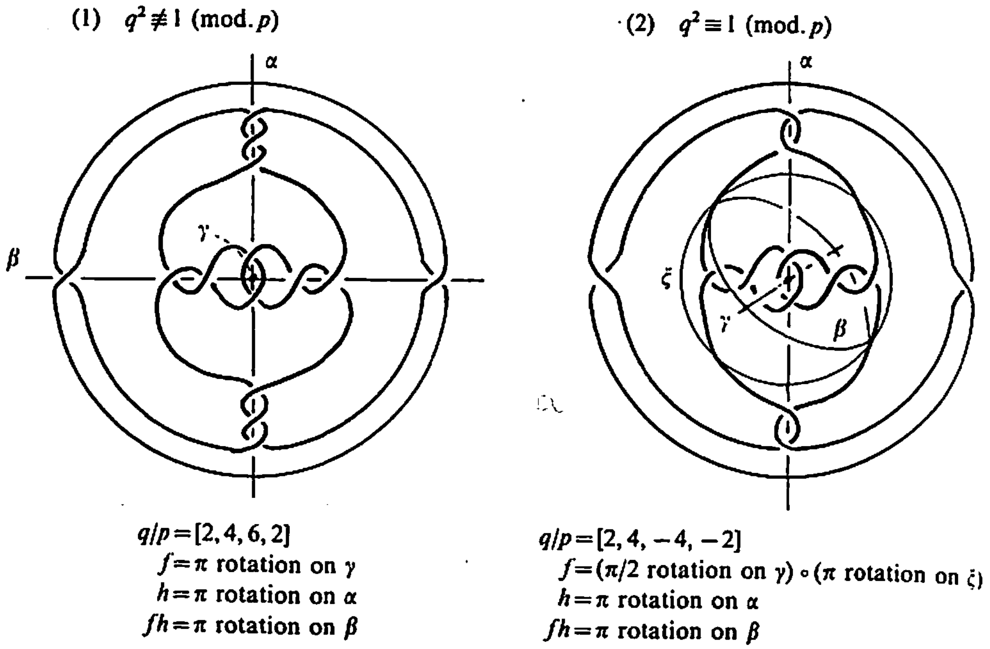


Fig. 3.2

All prime knots with 10 crossings or less are simple (see [30]), and their invertibility problems are completely solved by Hartley [13] (cf. [2, 4, 16, 30, 31]). And the prime knots with less than 10 crossings which have period 2 are completely identified by [2, 20, 27] for cyclic period and by [12] for free period. Recently, the author determined the symmetry group $\mathrm{Sym}(S^3, K) = \pi_0 \mathrm{Diff}(S^3, K)$ of every prime knot with less than 10 crossings by using the results of Thurston [41]. In the appendix, we present a table of the symmetry groups and the η-polynomials of all prime knots with less than 10 crossings. Although the amphicheirality problem on knots with 10 crossings or less is already solved by Perko [28, 29], it might be of some slight interest to remark that the problem for invertible knots with less than 10 crossings can also be solved by using the η-polynomials except 8_{20} and 9_{40} (cf. Proposition 3.4).

4. Equivariant cobordism

We say that two strongly invertible knots (K_0, h_0) and (K_1, h_1) are *equivariantly cobordant* if there is a smooth submanifold A and a smooth involution h on $(S^3 \times I, A)$, which satisfy the following conditions:

(1) A is homeomorphic to an annulus.
(2) $A \cap S^3 \times i = K_i \times i$ for each i ($i=0, 1$).
(3) The restriction of h to $(S^3 \times i, A \cap S^3 \times i)$ is equivalent to h_i for each i ($i=0, 1$).

When we consider equivariant cobordism of directed strongly invertible knots, we further require that the directions of (K_0, h_0) and (K_1, h_1) are "coherent". Let $\mathscr{S}$ be the set of all equivariant cobordism classes of directed strongly invertible knots. Then the equivariant connected sum operation naturally induces a sum operation on $\mathscr{S}$, and with respect to which $\mathscr{S}$ forms a group. The identity element of this group is represented by the trivial strongly invertible knot, and the inverse of a class $\{(K, h)\}$ is the class $\{(K, h)^*\}$, where $(K, h)^*$ is the directed strongly invertible knot obtained from (K, h) by reversing the orientations of S^3 and Fix(h).

Example 4.1. The strongly invertible knot obtained from the pretzel knot of type $(p, -p, r)$ as illustrated in Fig. 4.1 is equivariantly slice. Here p is an odd integer and r is an even integer.

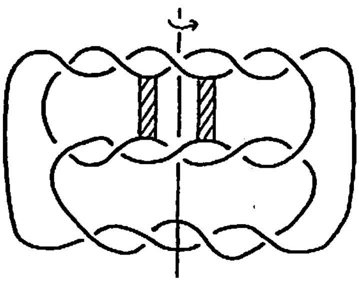


Fig. 4.1

Theorem III. *The η-polynomial gives a homomorphism from the group $\mathscr{S}$ to the additive group $\mathbf{Z}\langle t\rangle$. That is;*

(1) *If (K_0, h_0) and (K_1, h_1) are equivariantly cobordant, then*

$$\eta_{(K_0, h_0)}(t) = \eta_{(K_1, h_1)}(t) .$$

(2) *If $(K, h) = (K_1, h_1) \# (K_2, h_2)$, then*

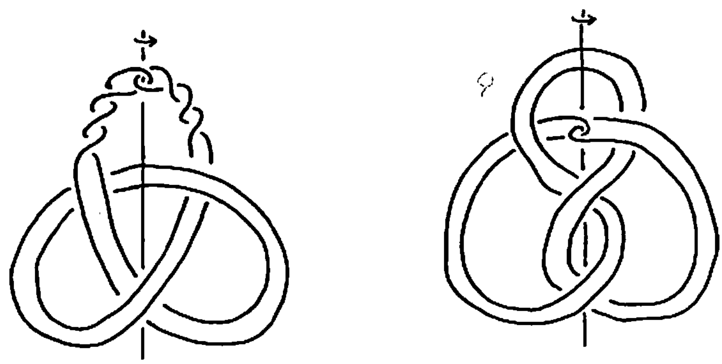
$$\eta_{(K, h)}(t) = \eta_{(K_1, h_1)}(t) + \eta_{(K_2, h_2)}(t) .$$

Proof. (1) Let $\{(S^3 \times I, A), h\}$ be an equivariant coborism between (K_0, h_0) and (K_1, h_1). By [6] p. 306, there is an h-invariant tubular neighbourhood $N(A)$ of A. We can find an annulus A' in $N(A)$ parallel to A, such that $h(A') \cap A' = \emptyset$ and $A' \cap S^3 \times i$ is a preferred longitude of K_i for each i ($i=0, 1$). On the other hand, we can see that the 4-manifold $W \equiv S^3 \times I/h$ has the homology of $S^3 \times I$ and Fix(h) is homeomorphic to an annulus, by using

[6] Chapter 3. Let p be the projection $S^3 \times I \to W$. Then $p(\mathrm{Fix}(h))$ and $p(A')$ are disjoint annuli in W and give a cobordism between the links $L(K_0, h_0)$ and $L(K_1, h_1)$. Hence, we have $\eta_{(K_0,h_0)}(t) = \eta_{(K_1,h_1)}(t)$ (see Theorem 2 of [18]).

(2) follows from the fact that the link $L(K, h)$ is obtained from $L(K_1, h_1)$ and $L(K_2, h_2)$ by a very natural fusion.

Example 4.2. The strongly invertible knots illustrated in Fig. 4.2 have trivial Alexander polynomials. But, they have nontrivial η-polynomials, and therefore they are not equivariantly slice.



$\eta(t) = [6, 0, -4, 0, 1$ $\eta(t) = [-6, 2, 4, -3, -1, 1$

Fig. 4.2

Example 4.3. There are two strongly invertible knots obtained from 6_1, and their η-polynomials are nontrivial as is seen in the appendix. Hence, they are not equivariantly slice, even though 6_1 itself is slice.

Thus the natural homomorphism from $\mathcal{S}$ to the classical knot cobordism group has the nontrivial kernel. On the other hand, by the results of Livingston [19], it is not an epimorphism.

Appendix: Table of the symmetry groups and the η-polynomials.

In the following tables, Sym^+ denotes $\pi_0 \mathrm{Diff}^+(S^3, K)$, where $\mathrm{Diff}^+(S^3, K)$ is the space of all diffeomorphisms of (S^3, K) which preserve the orientation of S^3. The symbol "A" (resp. "N") denotes that the corresponding knot is amphicheiral (resp. non-amphicheiral). $\mathrm{Sym}(S^3, K)$ can be obtained from the above data.

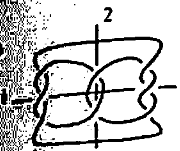
I. *2-bridge knots.* The symbol $[a_1, \cdots, a_n]$ in the third column represents the continued fraction expansion of q/p, where (p, q) is the type of the corresponding knot. The first polynomial in the last column is the η-polynomial of $I_1(a_1, \cdots, a_{n-1}; a_2/2, \cdots, a_n/2)$. If $q^2 \not\equiv 1$ (mod. p) [resp

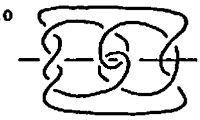
$q^2 \equiv 1$ and $q \not\equiv \pm 1$ (mod. p)], the second polynomial is the η-polynomial of $I_1(-a_n, \cdots, -a_2; -a_{n-1}/2, \cdots, -a_1/2)$ [resp. $I_2(a_1, \cdots, a_{n/2})$] (see Proposition 3.6).

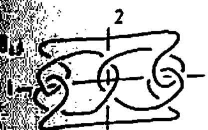
Knot	Type	Continued fraction	Sym⁺	Amphi.	η-polynomial
3_1	(3, 2)	[2, −2]	D_1	N	[2, 0, −1
4_1	(5, 2)	[2, 2]	D_2	A	[2, −1, −1, 1 [−2, 1, 1, −1
5_1	(5, 4)	[2, −2, 2, −2]	D_1	N	[2, 0, −1
5_2	(7, 2)	[4, −2]	D_2	N	[2, −1, 0, 1, −1 [4, −1, −2, 1
6_1	(9, 2)	[4, 2]	D_2	N	[−2, 1, 0, 0, 1, −1 [4, −1, −2, 1
6_2	(11, 8)	[2, −2, 2, 2,]	D_2	N	[2, 0, −1 [4, −2, −1, 2, −1
6_3	(13, 8)	[2, −2, −2, 2]	D_2	A	[0, 1, −1, −1, 1 [0, −1, 1, 1, −1
7_1	(7, 6)	[2, −2, 2, −2, 2, −2]	D_1	N	[4, 0, −2
7_2	(11, 2)	[6, −2]	D_2	N	[2, −1, 0, 0, 0, 1, −1 [6, −1, −3, 1
7_3	(13, −4)	[−4, 2, −2, 2]	D_2	N	[−4, 2, 1, −2, 1 [−2, 0, 1
7_4	(15, −4)	[−4, 4]	D_4	N	[4, −2, 0, 1, −2, 1 [−4, 0, 2
7_5	(17, 12)	[2, −2, 4, −2]	D_2	N	[4, 0, −2 [6, −2, −2, 2, −1
7_6	(19, 8)	[2, 2, 2, −2]	D_2	N	[−2, 1, 1, −1 [6, −3, −1, 2, −2, 1
7_7	(21, 8)	[2, 2, −2, −2]	D_4	N	[0, 0, 1, −1, −1, 1 [2, 0, −1
8_1	(13, 2)	[6, 2]	D_2	N	[−4, 2, 0, 0, 0, −1, 2, −1 [−6, 2, 3, −2
8_2	(17, 14)	[2, −2, 2, −2, 2, 2]	D_2	N	[0, 1, 0, −1 [14, −5, −6, 5, −1
8_3	(17, 4)	[4, 4]	D_2	A	[−4, 2, 0, −1, 2, −1 [4, −2, 0, 1, −2, 1
8_4	(19, −4)	[−4, −2, −2, 2]	D_2	N	[0, 0, 1, −1, −1, 1 [−6, 2, 1, −1, 2, −1
8_6	(23, −10)	[−2, −4, 2, −2]	D_2	N	[6, −2, −3, 2 [−2, −1, 1, 1
8_7	(23, 14)	[2, −2, −2, 2, −2, 2]	D_2	N	[−2, 2, 0, −2, 1 [0
8_8	(25, −14)	[−2, 4, 2, −2]	D_2	N	[2, 0, −1 [0, 1, −1, 0, 0, −1, 1
8_9	(25, 18)	[2, −2, 2, 2, −2, 2]	D_2	A	[0 [0
8_{11}	(27, 8)	[4, −2, 2, 2]	D_2	N	[0, −1, 1, 1, −1 [2, −1, −1, 1
8_{12}	(29, 12)	[2, 2, 2, 2]	D_2	A	[−2, 1, 1, −1 [2, −1, −1, 1
8_{13}	(29, 8)	[4, −2, −2, 2]	D_2	N	[0, 0, 0, 1, −1, −1, 1 [2, −2, 1, 1, −2, 1
8_{14}	(31, −18)	[−2, 4, −2, −2]	D_2	N	[−2, 0, 2, −1, −1, 1 [−4, 2, 1, −1, 0, −1, 1

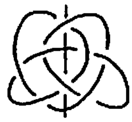
Knot	Type	Continued fraction	Sym$^+$	Amphi.	η-polynomial
9_1	(9, 8)	[2, −2, 2, −2, 2, −2, 2, −2]	D_1	N	[4, 0, −2
9_2	(15, 2)	[8, −2]	D_2	N	[2, −1, 0, 0, 0, 0, 0, 1, −1 [8, −2, −4, 2
9_3	(19, −6)	[−4, 2, −2, 2, −2, 2]	D_2	N	[−6, 3, 1, −3, 2 [−8, 2, 3, −2, 1
9_4	(21, 4)	[6, −2, 2, −2]	D_2	N	[−4, 2, 0, 0, 1, −2, 1 [2, 0, −1
9_5	(23, −4)	[−6, 4]	D_2	N	[−4, 2, 0, 0, 0, −1, 2, −1 [−6, 3, 0, −2, 3, −1
9_6	(27, 22)	[2, −2, 2, −2, 4, −2]	D_2	N	[4, −1, −1, 1, −1 [2, −3, 3, −1
9_7	(29, 20)	[2, −2, 6, −2]	D_2	N	[4, −1, −1, 1, −1 [6, −1, −3, 1
9_8	(31, 14)	[2, 4, 2, −2]	D_2	N	[−2, 0, 2, 0, −1 [4, −1, −1, 0, 0, 0, −1, 1
9_9	(31, 24)	[2, −2, 2, −4, 2, −2]	D_2	N	[4, 0, −2 [4, 0, −2
9_{10}	(33, −10)	[−4, 2, −2, 4]	D_4	N	[−6, 2, 2, −2, 1 [−4, 0, 2
9_{11}	(33, −26)	[−2, 2, −2, 2, 2, 2]	D_2	N	[−4, 1, 2, −1 [0, −1, 0, 1
9_{12}	(35, 8)	[4, 2, 2, −2]	D_2	N	[0, 0, −1, 1, 1, −1 [2, −2, 1, 1, −2, 1
9_{13}	(37, −26)	[−2, 2, −4, 4]	D_2	N	[−4, 1, 2, −1 [−6, 3, 0, −1, 2, −2, 1
9_{14}	(37, −8)	[−4, −2, 2, 2]	D_2	N	[0, 0, 0, −1, 1, 1, −1 [2, 1, −3, 0, 2, −1
9_{15}	(39, −22)	[−2, 4, 2, 2]	D_2	N	[−4, −1, 2, 1 [0, −1, 0, 1
9_{17}	(39, −14)	[−2, −2, 2, −2, 2, 2]	D_4	N	[2, −2, 0, 2, −1 [0
9_{18}	(41, 12)	[4, −2, 4, −2]	D_2	N	[2, −1, 0, 1, −1 [8, −3, −2, 2, −2, 1
9_{19}	(41, −18)	[−2, −4, 2, 2]	D_2	N	[4, −1, −2, 1 [0, 0, −1, 1, 0, 0, 1, −1
9_{20}	(41, 30)	[2, −2, 2, 2, 2, −2]	D_2	N	[4, 0, −2 [6, −2, −1, 0, −1, 2, −1
9_{21}	(43, −12)	[−4, 2, 2, 2]	D_2	N	[−4, 2, 0, −1, 1, 0, 1, −1 [−2, 1, −1, 1
9_{23}	(45, 26)	[2, −4, 4, −2]	D_4	N	[6, −2, −2, 1, 0, 1, −1 [2, 0, −1
9_{26}	(47, −34)	[−2, 2, −2, −2, 2, 2]	D_2	N	[−4, 1, 2, −1 [−2, 1, −1, 1, 1, −2, 1
9_{27}	(49, 18)	[2, 2, −2, 2, 2, −2]	D_2	N	[−2, 1, 1, −2, 1, 1, −1 [2, 1, −2, −1, 1
9_{31}	(55, −34)	[−2, 2, 2, −2, −2, 2]	D_4	N	[−2, 0, 1, 1, −1, −1, 1 [0

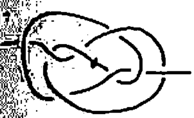
II. 3-*bridge knots*. The first row represents Sym$^+$ and amphicheirality. In case K has two inequivalent inverting involutions, the first (the second) polynomial is the η-polynomial of the inverting involution labeled 1 (resp. 2).

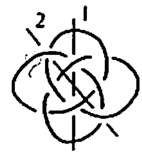
$8_{[illegible]}$

D_2 N
[−2, 0, 1
[−6, 2, 3, −2

8_{10}

D_1 N
[−2, 0, 1

8_{14}

D_2 N
[2, 0, 0, −1, −1, 1
[−4, −1, 2, 1

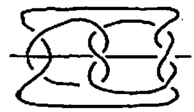
8_{16}

D_1 N
[−2, 0, 1

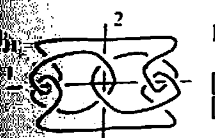
8_{17}

1 A
Noninvertible

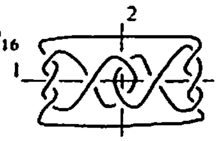
8_{18}

D_4 A
[2, −1, −1, 1
[−2, 1, 1, −1

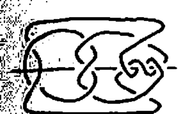
8_{19}

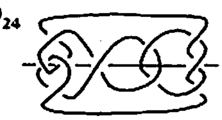
D_1 N
[−6, 1, 3, −1

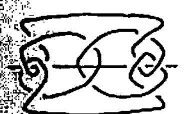
8_{20}

D_1 N
[0

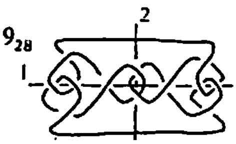
8_{21}

D_2 N
[2, 1, −2, −1, 1
[4, −1, −2, 1

9_{16}

D_2 N
[−4, 0, 2
[−8, 2, 4, −2

9_{22}

D_1 N
[−2, 0, 1

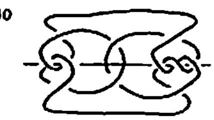
9_{24}

D_1 N
[2, −1, −1, 1

9_{25}

D_1 N
[4, −1, −2, 1

9_{28}

D_2 N
[2, 1, −2, 0, 0, −1, 1
[2, −1, −1, 1

9_{29}

D_1 N
[2, −1, −1, 1

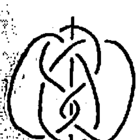
9_{30}

D_1 N
[−2, 2, 0, −1, 0, −1, 1

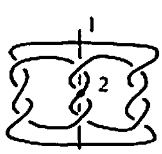
9_{32}

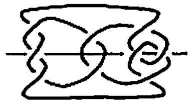
1 N
Noninvertible

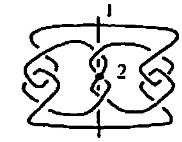
9_{33}

1 N
Noninvertible

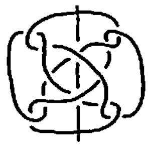
9_{34}

D_1 N
[0, 1, −1, −1, 1

9_{35}

D_6 N
[4, −3, 0, 3, −1
[6, −1, −3, 1

9_{36}

D_1 N
[−2, 0, 1

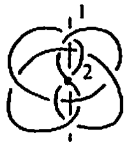
9_{37}


D_2 N
[0
[−2, 0, 1

9_{38}

D_1 N
[4, 0, −2

9_{39}

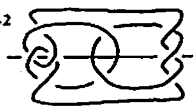
D_1 N
[−4, 0, 2

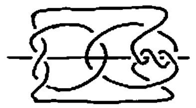
9_{40}


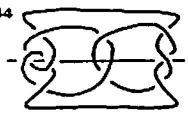
D_6 N
[−2, 0, 1
[2, 0, −1

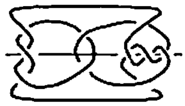
9_{41}

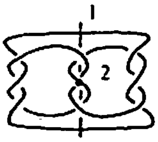
D_3 N
[0, 1, 0, −1

9_{42}

D_1 N
[2, −2, 0, 2, −1

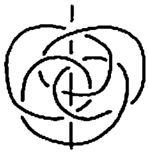
9_{43}

D_1 N
[−6, 1, 3, −1

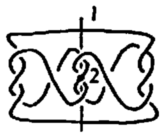
9_{44}

D_1 N
[4, −1, −2, 1

9_{45}

D_1 N
[2, 1, −2, −1, 1

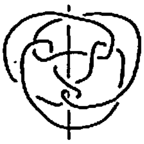
9_{46}


D_2 N
[6, −3, −2, 3, −1
[0, 1, 0, −1

9_{47}

D_3 N
[2, −1, −1, 1

9_{48}


D_6 N
[4, −2, −1, 2, −1
[2, 0, −1

9_{49}

D_3 N
[−6, 2, 2, −2, 1

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