# ENRIQUES INVOLUTIONS ON SINGULAR K3 SURFACES OF SMALL DISCRIMINANTS: COMPUTATIONAL DATA

ICHIRO SHIMADA

## 1. Introduction

This note is an explanation of the computational data obtained in the second half of the paper [2] (joint work with Davide Cesare Veniani). The data are available from the author's webpage:

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html`

The data consists of 11 files, each of which contains a `record` of `GAP` [1] that describes a data of the singular $K3$ surface $X$ with the transcendental lattice $T_X$:

(1.1)

| | | |
|---|---|---|
| `Xd3a2b1c2` | : | $T_X = [2,1,2]$ |
| `Xd4a2b0c2` | : | $T_X = [2,0,2]$ |
| `Xd7a2b1c4` | : | $T_X = [2,1,4]$ |
| `Xd8a2b0c4` | : | $T_X = [2,0,4]$ |
| `Xd12a2b0c6` | : | $T_X = [2,0,6]$ |
| `Xd12a4b2c4` | : | $T_X = [4,2,4]$ |
| `Xd15a2b1c8` | : | $T_X = [2,1,8]$ |
| `Xd16a4b0c4` | : | $T_X = [4,0,4]$ |
| `Xd20a4b2c6` | : | $T_X = [4,2,6]$ |
| `Xd24a2b0c12` | : | $T_X = [2,0,12]$ |
| `Xd36a6b0c6` | : | $T_X = [6,0,6]$ |

Each of these records is written in the `txt` file with the same name.

In the file `NKrecs.txt`, we give a list of 7 records `NKrec`, each of which describes the nef-chamber of an Enriques surface with finite automorphism group.

Every lattice $L$ is equipped with a basis, which is fixed during the whole calculation. The dual lattice $L^\vee$ is regarded as a submodule of $L \otimes \mathbb{Q}$, and hence its elements are given by a vector with rational components.

The discriminant form $q(L)$ of an even lattice $L$ is described by a record, whose components are explained in Section 3. In particular, the natural projection $L^\vee \to L^\vee/L$ and the automorphism group $\mathrm{O}(q(L))$ are given in this record, and the natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(q(L))$ is calculated by means of this record.

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## 2. Xrec

Let `Xrec` be one of the records in (1.1). Then `Xrec` has the following components:

- `Xrec.GramTX`: The Gram matrix of $T_X$.
- `Xrec.GramSX`: The Gram matrix of $S_X$.
- `Xrec.GramHX`: The Gram matrix of $H^2(X,\mathbb{Z})$.
- `Xrec.GramL`: The Gram matrix of $L_{26}$.
- `Xrec.GramR`: The Gram matrix of the orthogonal complement $R := [\iota_X]^\perp$ of the image of $\iota_X : S_X \hookrightarrow L_{26}$.
- `Xrec.m2R`: The list of $(-2)$-vectors of $R$.
- `Xrec.embTXHX`: The embedding of $T_X$ into $H^2(X,\mathbb{Z})$.
- `Xrec.embSXHX`: The embedding of $S_X$ into $H^2(X,\mathbb{Z})$.
- `Xrec.embSXL`: The embedding of $S_X$ into $L_{26}$.
- `Xrec.embRL`: The embedding of $R$ into $L_{26}$.
- `Xrec.projHXTX`: The othogonal projection $H^2(X,\mathbb{Q}) \to T_X \otimes \mathbb{Q}$.
- `Xrec.projHXSX`: The othogonal projection $H^2(X,\mathbb{Q}) \to S_X \otimes \mathbb{Q}$.
- `Xrec.projLSX`: The othogonal projection $L_{26} \otimes \mathbb{Q} \to S_X \otimes \mathbb{Q}$.
- `Xrec.projLR`:The othogonal projection $L_{26} \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$.
- `Xrec.OTX`: The elements of the group $\mathrm{O}(T_X)$.
- `Xrec.OR`: The elements of the group $\mathrm{O}(R)$.
- `Xrec.qTX`: The record of $q(T_X)$ (see Section 3).
- `Xrec.qSX`: The record of $q(S_X)$ (see Section 3).
- `Xrec.qR`: The record of $q(R)$ (see Section 3).
- `Xrec.kerOTXOqTX`: The elements of the kernel of the natural homomorphism $\mathrm{O}(T_X) \to \mathrm{O}(q(T_X))$.
- `Xrec.imageOTXOqTX`: The elements of the image of the natural homomorphism $\mathrm{O}(T_X) \to \mathrm{O}(q(T_X))$.
- `Xrec.imageOROqR`: The elements of the image of the natural homomorphism $\mathrm{O}(R) \to \mathrm{O}(q(R))$.
- `Xrec.isomqSXqTX`: The isomorphism $q(S_X) \to -q(T_X)$ induced by $H^2(X,\mathbb{Z})$.
- `Xrec.isomqTXqSX`: The inverse of `Xrec.isomqSXqTX`.
- `Xrec.isomqSXqR`: The isomorphism $q(S_X) \to -q(R)$ induced by $L_{26}$.
- `Xrec.isomqRqSX`: The inverse of `Xrec.isomqSXqTR`.
- `Xrec.OTXomega`: The elements of $\mathrm{O}(T_X, \omega)$.
- `Xrec.OqTXomega`: The elements of $\mathrm{O}(q(T_X), \omega)$.
- `Xrec.OqSXomega`: The elements of $\mathrm{O}(q(S_X), \omega)$.
- `Xrec.OqRomega`: The image of $\mathrm{O}(q(S_X), \omega)$ by the isomorphism $\mathrm{O}(q(S_X)) \to \mathrm{O}(q(R))$ induced by $L_{26}$.
- `Xrec.weyl`: The Weyl vector $\mathbf{w}$ for the Conway chamber inducing $D_0$.
- `Xrec.weylS`: The image of the Weyl vector $\mathbf{w}$ by the orthogonal projection $L_{26} \to S_X$.

- `Xrec.autXD0`: The elements of the group $\mathrm{aut}(X, D_0)$.
- `Xrec.D0wallorbitrecs`: The list of records `orbrec` that describe orbits of the walls of $D_0$ under the action of $\mathrm{aut}(X, D_0)$. Each `orbrec` has the following components. Let $w$ be a representative of the orbit $o$ described by `orbrec`.
  - `orbrec.definingv`: The primitive vector $v$ of $S_X^\vee$ that defines the representative $w$ of $o$.
  - `orbrec.definingvs`: The list of primitive vectors of $S_X^\vee$ that define the walls in $o$.
  - `orbrec.n`: $n = \langle v, v\rangle$.
  - `orbrec.a`: $a = \langle \mathbf{w}_S, v\rangle$.
  - `orbrec.size`: The size $|o|$ of `orbrec.definingvs`.
  - `orbrec.innout`: If $w$ is inner, then `"inner"`, whereas if $w$ is outer, then `"outer"`.

  If $w$ is inner, then `orbrec` has the following components. If $w$ is outer, then the components below are the string `"not defined"`.
  - `orbrec.type`: The type (no.) of the orbit of inner-walls given in Tables of [2].
  - `orbrec.roots`: The list of $(-2)$-vectors $r$ of $L_{26}$ such that $\langle \mathbf{w}, r\rangle = 1$ and that $(r)^\perp$ passes through $\iota_X(w) \subset \mathcal{P}_{26}$.
  - `orbrec.adetype`: The ADE-type of a fundamental root system `orbrec.roots`, which is a list of strings `"A1", "A2", ..., "E8"`.
  - `orbrec.adjweyl`: The Weyl vector $\mathbf{w}'$ corresponding to an Conway chamber $C'$ that induces the $\iota_X^*\mathcal{R}_{26}^\perp$-chamber $D'$ adjacent to $D_0$ across the wall $w$.
  - `orbrec.adjweylS`: The projection of $\mathbf{w}' \in L_{26}$ to $S_X^\vee$, which is the image of `Xrec.weylS` by an extra automorphism $g_S$ associated with $w$
  - `orbrec.gL`: An isometry $g_L \in \mathrm{O}^+(L_{26})$ that maps $C_0$ to $C'$, preserves $S_X$, and such that $g_L|S_X$ is an element of $\mathrm{aut}(X)$ that maps $D_0$ to $D'$.
  - `orbrec.gS`: An isometry $g_S \in \mathrm{aut}(X)$ that is an extra automorphism associated with $w$.
  - `orbrec.dg`: The degree $\langle \mathbf{w}^{g_S}, \mathbf{w}\rangle$.
- `Xrec.D0wallrecs`: The list of records `wallrec` that describe the walls of $D_0$. Each record `wallrec` has the following components.
  - `wallrec.no`: The number (index) of the wall, which will be used in `facerec` below.
  - `wallrec.definingv`: The primitive vector $v$ of $S_X^\vee$ that defines the wall $w$.
  - `wallrec.innout`: If $w$ is inner, then `"inner"`, whereas if $w$ is outer, then `"outer"`.

  - `wallrec.type`: If $w$ is inner, then the type given in Tables of [2]. If $w$ is outer, then `"not defined"`.
  - `wallrec.extraaut`: If $w$ is inner, then an extra automorphism $g \in \mathrm{aut}(X)$ associated with $w$. If $w$ is outer, then `"not defined"`.
- `Xrec.facerecs`: The list of records that describe the $N_X$-inner faces of $D_0$. An $\mathrm{aut}(X)$-equivalence class of $N_X$-inner faces of $D_0$ is described by a record `facerec` whose components are as follows. Let $f$ be a representative of the $\mathrm{aut}(X)$-equivalence class.
  - `facerec.no`: The number (index) of this $\mathrm{aut}(X)$-equivalence class, which will be used in `enrrec` below.
  - `facerec.dim`: The dimension of $f$.
  - `facerec.basis`: A basis of the vector subspace $\langle f \rangle$ of $S_X \otimes \mathbb{Q}$.
  - `facerec.passingwalls`: The list of `wallrec.no` of walls of $D_0$ containing $f$.
  - `facerec.Gammaf`: A list $\Gamma(f)$ of elements of $g \in \mathrm{aut}(X)$ such that $\{D_0^g \,|\, g \in \Gamma(f)\}$ is equal to $\mathcal{D}(f)$ and that, if $g \neq g' \in \Gamma(f)$, then $D_0^g \neq D_0^{g'}$.
  - `facerec.autXf`: The list of elements of the group $\mathrm{aut}(X, f)$,
  - `facerec.orbrecs`: The list of $\mathrm{aut}(X, D_0)$-orbits contained in this $\mathrm{aut}(X)$-equivalence class of $N_X$-inner faces of $D_0$. Let $o$ be an $\mathrm{aut}(X, D_0)$-orbit in this $\mathrm{aut}(X)$-equivalence class. Then $o$ is described by the record `orbrec` with the following components. Let $f'$ be a representative of $o$.
    - `orbrec.size`: The size of $o$.
    - `orbrec.g`: An element $g \in \mathrm{aut}(X)$ such that $f^g = f'$.
    - `orbrec.passingwalls`: The list of `wallrec.no` of walls of $D_0$ containing $f'$.
    - `orbrec.passingwalltypes`: The list of `wallrec.type` of walls containing $f'$.
- `Xrec.Yrecs`: The list of records that describe the Enriques involutions on $X$ (see Section 4).

## 3. discrec

The discriminant form $q(L)$ of an even lattice $L$ of rank $r$ with a fixed basis is described by a record `discrec`. We put

$$A(L) := L^{\vee}/L \cong \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z}$$

and fix, one and for all, a set of generators $\bar{v}_1, \dots, \bar{v}_l$ of $A(L)$, where $\bar{v}_i$ is a generator of the $i$th component $\mathbb{Z}/a_i\mathbb{Z}$. Let $b(L)\colon A(L) \times A(L) \to \mathbb{Q}/\mathbb{Z}$ be the bilinear form associated with the quadratic form $q(L)\colon A(L) \to \mathbb{Q}/2\mathbb{Z}$. The record `discrec` has the following components.

- `discrec.discg`: The list $[a_1, \dots, a_l]$.
- `discrec.discf`: The Gram matrix of $q(L)$, whose $(i,j)$-component is $q(L)(\bar{v}_i) \in \mathbb{Q}/2\mathbb{Z}$ if $i=j$ and $b(L)(\bar{v}_i, \bar{v}_j) \in \mathbb{Q}/\mathbb{Z}$ if $i \neq j$.
- `discrec.lifts`: The list of vectors $v_1, \dots, v_l \in L^\vee$, where $\bar{v}_i = v_i \mod L$.
- `discrec.proj`: The $r \times l$ matrix $M$ such that $v \mapsto vM$ gives the natural projection $L^\vee \to A(L)$, where $v \in L^\vee$ is written as a vector of length $r$ with rational components with respect to the basis of $L$.
- `discrec.Oq`: The list of elements of $\mathrm{O}(q(L))$.

Note that, by this record, we can calculate the action $q(g) \in \mathrm{O}(q(L))$ of a given isometry $g$ of $L$ on $q(L)$.

## 4. Enrrec

An Enriques involution $\varepsilon \in \mathrm{aut}(X)$ on $X$ is is described by a record `enrrec` whose components are as follows.

- `enrrec.no`: The number of $\varepsilon$ in Table (Enriques involutions) of [2].
- `enrrec.fenr`: The number `facerec.no` of the $\mathrm{aut}(X)$-equivalence class of $N_X$-inner faces containing $f_\varepsilon$.
- `enrrec.invol`: The matrix representation of $\varepsilon \in \mathrm{O}^+(S_X)$.
- `enrrec.GramSY`: The Gram matrix of $S_Y$.
- `enrrec.GramP`: The Gram matrix of the orthogonal complement $P$ of the image of $\pi^* \colon S_Y(2) \hookrightarrow S_X$.
- `enrrec.embSY2SX`: The embedding $\pi^* \colon S_Y(2) \hookrightarrow S_X$.
- `enrrec.embPSX`: The embedding $P \hookrightarrow S_X$.
- `enrrec.projSXSY2`: The orthogonal projection $S_X \otimes \mathbb{Q} \to S_Y(2) \otimes \mathbb{Q}$.
- `enrrec.projSXP`: The orthogonal projection $S_X \otimes \mathbb{Q} \to P \otimes \mathbb{Q}$.
- `enrrec.m4P`: The list of $(-4)$-vectors of $P$.
- `enrrec.autXenrfenr`: The list of elements of $\mathrm{aut}(X, \varepsilon, f_\varepsilon)$.
- `enrrec.kerres`: The list of elements of the kernel of the restriction homomorphism $\mathrm{aut}(X, \varepsilon, f_\varepsilon) \to \mathrm{aut}(Y, E_0)$. The quotient of `enrrec.kerres` by $\langle \varepsilon \rangle$ is identified with the kernel of $\rho_Y \colon \mathrm{Aut}(Y) \to \mathrm{aut}(Y)$.
- `enrrec.autYE0`: The list of elements of $\mathrm{aut}(Y, E_0)$.
- `enrrec.E0NKtype`: The Nikulin-Kondo type of the walls of $E_0$. For the Enriques involution No. 24 in Table (Enriques involutions) of [2], this data is `"not defined"`.
- `enrrec.E0walls`: The list of walls of defining $(-2)$-vectors of the walls of $E_0$.
- `enrrec.E0wallrecs`: The list of records `E0wallrec`, each of which describes a wall $w$ of $E_0$. The components of the record `E0wallrec` are as follows.

- `E0wallrec.no`: The number of the wall. Except for the Enriques involution No. 24, the walls are numbered according to the corresponding Nikulin-Kondo configuration recorded in `NKrecs` (see Section 5), and hence this numbering coincides with the figures of Nikulin-Kondo configurations in [2].
- `E0wallrec.definingv`: The $(-2)$-vector $r \in \mathcal{R}_Y$ of $S_Y$ that defines the wall $w$ of $E_0$.
- `wallrec.innout`: If $w$ is inner, then `"inner"`, whereas if $w$ is outer, then `"outer"`.
- `E0wallrec.embwSX`: A basis of the minimal linear subspace of $S_X \otimes \mathbb{Q}$ containing $\pi^*(w) \subset \mathcal{P}_X$.
- `E0wallrec.passingwalls`: The list of `wallrec.no` of walls of $D_0$ passing through $\pi^*(w) \subset \mathcal{P}_X$.
- `E0wallrec.fenrw`: If $w$ is inner, then the `facerec.no` of the $\mathrm{aut}(X)$-equivalence class of $N_X$-inner faces containing $f_\varepsilon(w)$. If $w$ is outer, then `"not defined"`.
- `E0wallrec.extraaut`: If $w$ is inner, then an extra automorphism $g \in \mathrm{aut}(X, \varepsilon)$ associated with $w$. If $w$ is outer, then `"not defined"`.
- `E0wallrec.extraautY`: If $w$ is inner, then the automorphism $g|S_Y \in \mathrm{aut}(Y)$, where $g \in \mathrm{aut}(X, \varepsilon)$ is `E0wallrec.extraaut`. If $w$ is outer, then `"not defined"`.

- `enrrec.autXenrgenerators`: A generating set of $\mathrm{aut}(X, \varepsilon)$.
- `enrrec.autYgenerators`: A generating set of $\mathrm{aut}(Y)$.

## 5. NKrec

In the file `NKrecs.txt`, we give a list of 7 records `NKrec`, each of which describes the nef-chamber of an Enriques surface $Y$ with finite automorphism group. The components of `NKrec` are as follows.

- `NKrec.type`: The Nikulin-Kondo type of $Y$ (a number between 1 and 7).
- `NKrec.configmat`: The adjacency matrix of the dual graph of the smooth rational curves on $Y$.

In the figures of Nikulin-Kondo configurations in [2], the nodes are numbered according to this record.

## References

[1] The GAP Group. GAP - Groups, Algorithms, and Programming. Version 4.8.6; 2016 (http://www.gap-system.org).

[2] Ichiro Shimada and Davide Cesare Veniani. Enriques surfaces covered by singular $k3$ surfaces of small discriminants, 2018. Preprint.

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

*Email address*: ichiro-shimada@hiroshima-u.ac.jp