

A NOTE ON QUEBBEMANN'S EXTREMAL LATTICES OF RANK 64

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ABSTRACT. By constructing explicit examples, we show that the method of Quebbemann yields many isomorphism classes of extremal lattices of rank 64. Many of these examples have no non-trivial automorphisms.

1. INTRODUCTION

By a *lattice*, we mean an integral positive-definite lattice. Let L be an even unimodular lattice with the bilinear form $\langle \cdot, \cdot \rangle_L: L \times L \rightarrow \mathbb{Z}$. We put

$$\min(L) := \min\{\langle x, x \rangle_L \mid x \in L, x \neq 0\}.$$

It is well-known that the rank n of L is divisible by 8, and that $\min(L)$ satisfies

$$(1.1) \quad \min(L) \leq 2 + 2 \left\lfloor \frac{n}{24} \right\rfloor.$$

We say that an even unimodular lattice L of rank n is *extremal* if the equality holds in (1.1). Extremal lattices are an important research subject, because they give rise to sphere packings of high density.

Not so many *explicit* examples of extremal lattices are known. Moreover, since the construction of these examples involves very special algebraic objects, each of the known examples has a large automorphism group. For example, the automorphism group $O(\Lambda)$ of the Leech lattice Λ is of order $2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

On the other hand, it was shown in [7] that the number of isomorphism classes of extremal lattices of rank 40 is $> 8.45 \times 10^{51}$. Since this bound was proved by means of a mass formula, we do not obtain explicit examples of extremal lattices of rank 40 from this result.

We consider extremal lattices of rank 64. Quebbemann [9] gave a method to construct extremal lattices Q of rank 64 from certain ternary codes B . We call an extremal lattice of rank 64 a *Quebbemann lattice* if it is obtained by (a generalization of) this method. See Section 2 for the precise definition. A remarkable property of Quebbemann's construction is that the condition on the ternary code B required in order for the lattice Q to be extremal is an *open* condition. Therefore we expect that, by generating sufficiently general ternary codes B , we obtain many extremal lattices of rank 64. Another nice feature of this construction is that we can calculate the set $\text{Min}(Q)$ of non-zero minimal-norm vectors of a Quebbemann lattice Q (that is, the set of vectors v with $\langle v, v \rangle_Q = 6$). It turns out that, by means of $\text{Min}(Q)$, it is possible to compute the automorphism group of Q , and to compare the isomorphism class of Q with isomorphism classes of other Quebbemann lattices.

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The purpose of this note is to generalize Quebbemann's construction slightly, and to show that this method indeed yields many mutually non-isomorphic extremal lattices of rank 64, by choosing the code B (pseudo-)randomly, and that their automorphism groups are often very small. Our main result below is proved by producing Quebbemann lattices Q explicitly.

Theorem 1.1. *Quebbemann's method yields*

- (1) *at least 300 isomorphism classes of extremal lattices Q of rank 64 such that $O(Q) = \{\pm 1\}$, and*
- (2) *at least 100 isomorphism classes of extremal lattices Q of rank 64 such that $O(Q) \cong \{\pm 1\} \times \mathbb{Z}/8\mathbb{Z}$.*

See Section 2 for the method to produce Quebbemann lattices, Section 3.1 for a method to enumerate minimal-norm vectors, Section 3.2 for a method to distinguish isomorphism classes, and Section 3.3 for the computation of the automorphism groups. We exhibit a few examples in detail in Section 4. The computation data of a part of the lattices in Theorem 1.1 is available from the author's web-page [11]. (The whole data is too large to be put on a website.) These data are written in the Record format of GAP [12].

In Chapter 8.3 (d) of [2], Conway and Sloane constructed a Quebbemann lattice that is different from the one given in Quebbemann's original paper [9], and suggested that there exist several isomorphism classes of Quebbemann lattices. In [8], Quebbemann showed by means of a mass formula that there exist at least two isomorphism classes. The ease with which we can make non-isomorphic Quebbemann lattices suggests that the number of isomorphism classes is very huge.

Unimodular lattices with no non-trivial automorphisms have been studied by many authors since the work of Bannai [1]. In [5], an even unimodular lattice of rank 64 without non-trivial automorphisms is constructed. This lattice is, however, not extremal.

In [6], Nebe constructed an extremal lattice of rank 64 by a different method. The order of the automorphism group is at least 587520. In [3] and [4], the existence of extremal Type II \mathbb{Z}_{2k} -codes of length 64 was shown. The isomorphism classes and the automorphism groups of the associated extremal lattices are, however, not clear. In [10], another extremal lattice of rank 64 was constructed by means of a generalized quadratic residue code. Its automorphism group is of order 119040.

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Conventions. Elements of a vector space or a lattice are written as *row* vectors. For a lattice L or a quadratic space L , we denote by $\langle \cdot, \cdot \rangle_L$ the symmetric bilinear form on L . The automorphism group $O(L)$ of $(L, \langle \cdot, \cdot \rangle_L)$ acts on L from the *right*.

2. QUEBBEMANN LATTICE

2.1. Quebbemann's construction. We recall Quebbemann's construction [9] of extremal lattices of rank 64. See also Chapter 8.3 (d) of [2]. In fact, our construction below is slightly more general than Quebbemann's original.

Let E be the root lattice of type E_8 , that is, E is the lattice of rank 8 generated by vectors e_1, \dots, e_8 with $\langle e_i, e_i \rangle_E = 2$ that form the dual graph in Figure 2.1. It is well-known that E is unimodular. We consider the \mathbb{F}_3 -quadratic space $U := E/3E$.

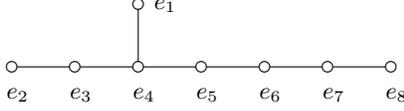


FIGURE 2.1. Dynkin diagram of type E_8

A subspace V of U is said to be *maximal isotropic* if $\dim V = 4$ and $\langle v, v \rangle_U = 0$ holds for all $v \in V$. There exists a direct sum decomposition

$$(2.1) \quad U = V \oplus W, \quad \text{where } V \text{ and } W \text{ are maximal isotropic subspaces.}$$

Let \mathcal{D} be the set of ordered pairs (V, W) of maximal isotropic subspaces of U satisfying $V \cap W = 0$. Since $\langle \cdot, \cdot \rangle_U$ is non-degenerate, we have a natural isomorphism

$$(2.2) \quad W \cong \text{Hom}(V, \mathbb{F}_3) \quad \text{for each } (V, W) \in \mathcal{D}.$$

Let S denote the orthogonal direct sum E^8 of eight copies of E . Then S is unimodular of rank 64. For $i = 1, \dots, 8$, we denote by U_i the i^{th} -factor of $S/3S = U^8$. We choose and fix an element

$$\Delta := ((V_1, W_1), \dots, (V_8, W_8))$$

of \mathcal{D}^8 . We put $T := \{1, \dots, 8\}$, and for a subset J of T , we put

$$U_J := \bigoplus_{j \in J} U_j, \quad V_J := \bigoplus_{j \in J} V_j, \quad W_J := \bigoplus_{j \in J} W_j.$$

Then we have

$$S/3S = U_T = V_T \oplus W_T.$$

We consider U_T, V_T and W_T as \mathbb{F}_3 -vector spaces. Let B be a linear subspace of V_T with $\dim B = 8$. Note that W_T can be regarded as the dual space of V_T by (2.2). We put

$$B^\perp := \{ \bar{z} \in W_T \mid \langle \bar{z}, \bar{y} \rangle_{U_T} = 0 \text{ for all } \bar{y} \in B \}.$$

Then we have $\dim B^\perp = 24$. Let $\pi: S \rightarrow S/3S = U_T$ denote the natural projection. For any elements \bar{x}, \bar{x}' of $B \oplus B^\perp \subset U_T$, we have $\langle \bar{x}, \bar{x}' \rangle_{U_T} = 0$, and hence, for any elements x, x' of $\pi^{-1}(B \oplus B^\perp)$, we have $\langle x, x' \rangle_S \equiv 0 \pmod{3}$. We denote by $Q(\Delta, B)$ the lattice whose underlying \mathbb{Z} -module is $\pi^{-1}(B \oplus B^\perp)$ and whose bilinear form $\langle \cdot, \cdot \rangle_Q$ is given by

$$\langle \cdot, \cdot \rangle_Q := \frac{1}{3} \langle \cdot, \cdot \rangle_S.$$

Then $Q := Q(\Delta, B)$ is an even lattice, and we have

$$\det Q = \left(\frac{1}{3} \right)^{64} \det S \cdot [S : Q]^2 = \left(\frac{1}{3} \right)^{64} \left(\frac{|U_T|}{|B \oplus B^\perp|} \right)^2 = 1.$$

Definition 2.1. For $J = \{i, j\} \subset T$ with $|J| = 2$, we denote by $p_J: B \rightarrow V_J$ the projection to the J -factor. We say that B satisfies *p₂-condition* if p_J is an isomorphism for all $J \subset T$ with $|J| = 2$.

Remark 2.2. If the projection $p_J: B \rightarrow V_J$ is an isomorphism, then the projection $p_{T \setminus J}: B^\perp \rightarrow W_{T \setminus J}$ to the $(T \setminus J)$ -factor is also an isomorphism. Hence, if B satisfies *p₂-condition*, then B^\perp satisfies the following *p₆-condition*: for all $J' \subset T$ with $|J'| = 6$, the projection $p_{J'}: B^\perp \rightarrow W_{J'}$ to the J' -factor is an isomorphism.

It is obvious that p_2 -condition imposes an open condition on the Grassmannian variety of 8-dimensional subspaces B of V_T .

Proposition 2.3 (Quebbemann [9]). *Let $U_T = V_T \oplus W_T$ be the decomposition of $U_T = S/3S$ associated with an element Δ of \mathcal{D}^8 , and let B be an 8-dimensional subspace of V_T . Suppose that B satisfies p_2 -condition. Then $\min(Q(\Delta, B)) = 6$ holds, that is, $Q(\Delta, B)$ is an extremal lattice of rank 64.*

Proof. We write $x \in Q(\Delta, B)$ as $x = (x_1, \dots, x_8)$, where $x_i \in E$ is the i^{th} component by the embedding $Q(\Delta, B) \hookrightarrow S = E^8$. We put

$$\bar{x} := \pi(x) = (\bar{x}_1, \dots, \bar{x}_8) \in B \oplus B^\perp \subset U_T,$$

where $\bar{x}_i \in U_i$ is $x_i \bmod 3E$. Decomposing each \bar{x}_i as $\bar{y}_i + \bar{z}_i$ with $\bar{y}_i \in V_i$ and $\bar{z}_i \in W_i$, we obtain $\bar{x} = \bar{y} + \bar{z}$, where

$$\bar{y} := (\bar{y}_1, \dots, \bar{y}_8) \in B, \quad \bar{z} := (\bar{z}_1, \dots, \bar{z}_8) \in B^\perp.$$

Suppose that $\langle x, x \rangle_Q \leq 4$. We show that $x = 0$. Since

$$(2.3) \quad \langle x, x \rangle_S = \sum \langle x_i, x_i \rangle_E \leq 12,$$

we see that at least two of the components x_i are zero. Since at least two of \bar{y}_i are zero, the assumption that B satisfy p_2 -condition implies $\bar{y} = 0$. Therefore we have $\bar{x} = \bar{z}$. In particular, each \bar{x}_i belongs to W_i . Since W_i is isotropic in $U = E/3E$, we have $\langle x_i, x_i \rangle_E \equiv 0 \pmod{3}$ and hence $\langle x_i, x_i \rangle_E \equiv 0 \pmod{6}$. Combining this with (2.3), we see that at most two of x_i are non-zero. Since B^\perp satisfies p_6 -condition by Remark 2.2, we see that $\bar{z} = 0$. Therefore $\bar{x} = 0$, and hence $x \in 3S$. If x were non-zero, we would have $\langle x, x \rangle_S \geq 18$, which is a contradiction. \square

Definition 2.4. An extremal lattice of rank 64 of the form $Q(\Delta, B)$, where Δ is an element of \mathcal{D}^8 and B is an 8-dimensional subspace of V_T satisfying p_2 -condition, is called a *Quebbemann lattice*.

2.2. Maximal isotropic subspaces of U . Recall that the lattice E is equipped with a basis e_1, \dots, e_8 in Figure 2.1. We write elements of E or of $U = E/3E$ as row vectors with respect to e_1, \dots, e_8 . The automorphism group $O(E)$ of E is generated by the reflections

$$x \mapsto x - \langle x, e_i \rangle_E e_i$$

with respect to the vectors e_i ($i = 1, \dots, 8$) of norm 2, and is of order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$. The natural homomorphism $O(E) \rightarrow O(U)$ to the automorphism group of the \mathbb{F}_3 -quadratic space U is injective. Let \mathcal{V} be the set of maximal isotropic subspaces V of U . We can prove the following by the standard orbit stabilizer algorithm using GAP [12].

Proposition 2.5. *The size of \mathcal{V} is 2240, and $O(E)$ acts transitively on \mathcal{V} .* \square

Let $V_0 \in \mathcal{V}$ be the maximal isotropic subspace with basis v_1, \dots, v_4 in Table 2.1. The stabilizer subgroup $\text{Stab}(V_0)$ of V_0 in $O(E)$ is of order $2^8 \cdot 3^5 \cdot 5$. Let $\mathcal{W}(V_0)$ be the set of all $W \in \mathcal{V}$ such that $V_0 \cap W = 0$. We have $|\mathcal{W}(V_0)| = 729$.

Proposition 2.6. *The action of $\text{Stab}(V_0)$ decomposes $\mathcal{W}(V_0)$ into two orbits of size 648 and 81. The orbit of size 648 contains W^{I} with basis $v_1^{*1}, \dots, v_4^{*1}$ in Table 2.1, and the orbit of size 81 contains W^{II} with basis $v_1^{*\text{II}}, \dots, v_4^{*\text{II}}$ in Table 2.1.* \square

$$\begin{aligned}
 v_1 &= (0, 0, 0, 0, 0, 0, 1, 2), & v_2 &= (0, 0, 0, 1, 2, 0, 0, 0), \\
 v_3 &= (0, 1, 0, 0, 1, 0, 0, 1), & v_4 &= (1, 0, 2, 0, 0, 0, 0, 1). \\
 \\
 v_1^{*I} &= (1, 2, 1, 0, 2, 2, 0, 2), & v_2^{*I} &= (0, 1, 2, 0, 0, 0, 0, 0), \\
 v_3^{*I} &= (2, 2, 1, 0, 1, 0, 2, 2), & v_4^{*I} &= (1, 0, 2, 0, 1, 0, 2, 1). \\
 \\
 v_1^{*II} &= (1, 2, 1, 0, 2, 2, 0, 2), & v_2^{*II} &= (1, 1, 1, 0, 0, 0, 0, 1), \\
 v_3^{*II} &= (0, 2, 0, 0, 1, 0, 2, 0), & v_4^{*II} &= (1, 2, 2, 2, 1, 0, 2, 0).
 \end{aligned}$$

TABLE 2.1. Bases of maximal isotropic subspaces V_0, W^I, W^{II}

Remark 2.7. The basis v_1^*, \dots, v_4^* of W above is dual to the basis v_1, \dots, v_4 of V_0 by the canonical pairing (2.2).

Corollary 2.8. *The action of $O(E)$ decomposes \mathcal{D} into two orbits. One orbit contains (V_0, W^I) with the stabilizer subgroup G^I of order 480, and the other orbit contains (V_0, W^{II}) with the stabilizer subgroup G^{II} of order 3840. \square*

2.3. Construction of $Q(\Delta, B)$ with an automorphism of order 8. We fix a pair $(V, W) \in \mathcal{D}$, and consider the 8-tuple

$$\Delta_0 := ((V, W), \dots, (V, W)) \in \mathcal{D}^8.$$

Let G be the stabilizer subgroup of (V, W) in $O(E)$, and let γ be an element of order 8 in G . (The stabilizer subgroup G^I (resp. G^{II}) in Corollary 2.8 contains 120 elements (resp. 1360 elements) of order 8.) We define an automorphism $\tilde{\gamma}$ of $S = E^8$ by

$$x = (x_1, \dots, x_8) \mapsto x^{\tilde{\gamma}} = (x_2^{\tilde{\gamma}}, \dots, x_8^{\tilde{\gamma}}, x_1^{\tilde{\gamma}}).$$

The action of $\tilde{\gamma}$ on $S/3S = U_T$ preserves the decomposition $U_T = V_T \oplus W_T$ associated with Δ_0 above. For $v \in V_T = V^8$, we denote by $B(\gamma, v)$ the linear subspace of V_T generated by the orbit of v under the action of $\langle \tilde{\gamma} \rangle \cong \mathbb{Z}/8\mathbb{Z}$. If $B(\gamma, v)$ is of dimension 8 and satisfies p_2 -condition, then $Q(\Delta_0, B(\gamma, v))$ is a Quebbemann lattice invariant under the action of $\langle \tilde{\gamma} \rangle$ on S . In particular, the automorphism group $O(Q)$ of $Q := Q(\Delta_0, B(\gamma, v))$ contains an element

$$\tilde{\gamma}_Q := \tilde{\gamma}|_Q$$

of order 8.

3. COMPUTATIONS ON QUEBBEMANN LATTICES

We fix an 8-tuple $\Delta = ((V_1, W_1), \dots, (V_8, W_8)) \in \mathcal{D}^8$. Let B be an 8-dimensional linear subspace of $V_T = V_1 \oplus \dots \oplus V_8$ satisfying p_2 -condition, and we consider the extremal lattice $Q(\Delta, B)$ of rank 64.

3.1. Enumeration of minimal-norm vectors. In this section, we explain a method to calculate the set

$$\text{Min}(Q(\Delta, B)) := \{x \in Q(\Delta, B) \mid \langle x, x \rangle_Q = 6\} = \{x \in Q(\Delta, B) \mid \langle x, x \rangle_S = 18\}$$

a	$ N(a, E) $	$ N(a, U) $
0	1	1
2	240	240
4	2160	2160
6	6720	2240

TABLE 3.1. $N(a, E)$ and $N(a, U)$

of all minimal-norm vectors. A *norm-type* is an 8-tuple $[n_1, \dots, n_8]$ of non-negative even integers n_i such that $\sum n_i = 18$. For $x = (x_1, \dots, x_8) \in \text{Min}(Q(\Delta, B))$, we put

$$\nu(x) := [\langle x_1, x_1 \rangle_E, \dots, \langle x_8, x_8 \rangle_E],$$

and call it the *norm-type* of x . For a non-negative even integer a , we put

$$N(a, E) := \{v \in E \mid \langle v, v \rangle_E = a\},$$

and let $N(a, U) \subset U$ be the image of $N(a, E)$ by the natural projection $E \rightarrow U$. (See Table 3.1.) A minimal-norm vector $x \in \text{Min}(Q(\Delta, B))$ is said to be of *divisible type* if $x \in 3S$ holds, or equivalently, if only one of x_1, \dots, x_8 (say x_i) is non-zero and x_i is written as $3x'_i$ by some $x'_i \in N(2, E)$, or equivalently, if the norm-type $\nu(x)$ of x is obtained by a permutation of components from $[0, \dots, 0, 18]$. Since $|N(2, E)| = 240$, there exist exactly 240×8 minimal-norm vectors of divisible type.

Proposition 3.1. *Let $x \in Q(\Delta, B)$ be a minimal-norm vector that is not of divisible type. Then the norm-type $\nu(x)$ of x is obtained by a permutation of components from one of the following:*

$$(3.1) \quad \begin{array}{ll} [0, 0, 0, 0, 0, 6, 6, 6] & \text{(of type } 0^5 6^3), \\ [0, 2, 2, 2, 2, 2, 4, 4] & \text{(of type } 0^1 2^5 4^2), \\ [0, 2, 2, 2, 2, 2, 2, 6] & \text{(of type } 0^1 2^6 6^1), \\ [2, 2, 2, 2, 2, 2, 2, 4] & \text{(of type } 2^7 4^1). \end{array}$$

Proof. As in the proof of Proposition 2.3, we see that $\bar{x} := \pi(x) \in B \oplus B^\perp$ is decomposed uniquely as $\bar{y} + \bar{z}$, where $\bar{y} = (\bar{y}_1, \dots, \bar{y}_8) \in B$ and $\bar{z} = (\bar{z}_1, \dots, \bar{z}_8) \in B^\perp$. Suppose that $\bar{y} = 0$. Since x is not of divisible type, we see that $\bar{x} = \bar{z}$ is not zero. Since B^\perp satisfies p_6 -condition, at most five of $\bar{z}_1, \dots, \bar{z}_8$ are zero. Since $\bar{x}_i = \bar{z}_i \in W_i$, we have $\langle x_i, x_i \rangle_E \equiv 0 \pmod{3}$ and hence $\langle x_i, x_i \rangle_E \in \{0, 6, 12, 18\}$. Combining these, we see that $\nu(x)$ is of type $0^5 6^3$. Suppose that $\bar{y} \neq 0$. Since B satisfies p_2 -condition, at most one of $\bar{y}_1, \dots, \bar{y}_8$ is zero. Hence at most one of x_1, \dots, x_8 is zero. Therefore $\nu(x)$ is either of type $0^1 2^5 4^2$ or $0^1 2^6 6^1$ or $2^7 4^1$. \square

For $k = 1, \dots, 8$, let $e_1^{(k)}, \dots, e_8^{(k)}$ be the copy of the basis e_1, \dots, e_8 of E in the k th factor of $S = E^8$. We use the ordered set

$$(3.2) \quad e_1^{(1)}, \dots, e_8^{(1)}, e_1^{(2)}, \dots, e_8^{(2)}, \dots, e_1^{(8)}, \dots, e_8^{(8)}$$

of vectors as a basis of S and of $S/3S = U^8 = U_T$.

Proposition 3.2. *The ternary code $B \oplus B^\perp \subset U_T$ is generated by row vectors of a 32×64 matrix of the echelon form as in Figure 3.1, where I_8 is the identity matrix of size 8, A_i are 4×8 matrices whose row vectors form a basis of $W_i \subset U$ for $i = 3, \dots, 6$, $C_{\mu\nu}$ are some 8×8 matrices, and the blank blocks are zero matrices.*

I_8		C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}
	I_8	C_{23}	C_{24}	C_{25}	C_{26}	C_{27}	C_{28}
		A_3				C_{37}	C_{38}
			A_4				
				A_5		C_{47}	C_{48}
					A_6		

 FIGURE 3.1. Echelon form of a generator matrix of $B \oplus B^\perp$

Proof. Since the projection $p_{12}: B \rightarrow V_1 \oplus V_2$ to the (12)-factor and the projection $p_{\overline{78}}: B^\perp \rightarrow W_1 \oplus \cdots \oplus W_6$ to the (123456)-factor are both isomorphisms, the projection

$$P_{12}: B \oplus B^\perp \rightarrow U \oplus U$$

to the (12)-factor is surjective, and its kernel $\text{Ker } P_{12}$ is mapped isomorphically to $W_3 \oplus \cdots \oplus W_6$ by the projection

$$P_{3456}: \text{Ker } P_{12} \rightarrow U \oplus U \oplus U \oplus U$$

to the (3456)-factor. □

We make the symmetric group \mathfrak{S}_8 act on $S = E^8$ by

$$(x_1, \dots, x_8)^\sigma := (x_{\sigma(1)}, \dots, x_{\sigma(8)}) \quad \text{for } \sigma \in \mathfrak{S}_8.$$

For $\Delta = ((V_1, W_1), \dots, (V_8, W_8)) \in \mathcal{D}^8$, we put

$$\Delta^\sigma := ((V_{\sigma(1)}, W_{\sigma(1)}), \dots, (V_{\sigma(8)}, W_{\sigma(8)})).$$

Then we have $Q(\Delta, B)^\sigma = Q(\Delta^\sigma, B^\sigma)$ in S . If $x \in \text{Min}(Q(\Delta, B))$ is of norm-type $[n_1, \dots, n_8]$, then $x^\sigma \in \text{Min}(Q(\Delta^\sigma, B^\sigma))$ is of norm-type $[n_{\sigma(1)}, \dots, n_{\sigma(8)}]$.

Let $n = [n_1, \dots, n_8]$ be a norm-type obtained by a permutation of components from one of the norm-types in (3.1). We calculate the set $\overline{M}(n)$ of codewords $\bar{x} = \pi(x) \in B \oplus B^\perp$ corresponding $x \in \text{Min}(Q(\Delta, B))$ with $\nu(x) = n$ by the following method.

First we choose a permutation $\tau \in \mathfrak{S}_8$ such that $n^\tau = [n_{\tau(1)}, \dots, n_{\tau(8)}]$ satisfies

$$n_{\tau(1)} \leq n_{\tau(2)} \leq \dots \leq n_{\tau(8)}.$$

We then transform a generator matrix of $(B \oplus B^\perp)^\tau = B^\tau \oplus B^{\tau\perp}$ into the echelon form in Figure 3.1, and search for $\bar{x}_1, \dots, \bar{x}_8 \in U$ satisfying conditions (3.3) below by back track search; that is, if we find $(\bar{x}_1, \dots, \bar{x}_i)$ satisfying the first i conditions of (3.3), then we search for \bar{x}_{i+1} satisfying the $(i+1)^{\text{st}}$ condition of (3.3). Recall

that $N(n_i, U)$ is the image of $N(n_i, E)$ by the natural map $E \rightarrow U$.

$$\begin{aligned}
(3.3) \quad & \bar{x}_1 \in N(n_{\tau(1)}, U), \\
& \bar{x}_2 \in N(n_{\tau(2)}, U), \\
& \bar{x}_3 := \bar{x}_1 C_{13} + \bar{x}_2 C_{23} + \bar{u}_3 A_3 \in N(n_{\tau(3)}, U), \quad \text{where } \bar{u}_3 \in \mathbb{F}_3^4, \\
& \bar{x}_4 := \bar{x}_1 C_{14} + \bar{x}_2 C_{24} + \bar{u}_4 A_4 \in N(n_{\tau(4)}, U), \quad \text{where } \bar{u}_4 \in \mathbb{F}_3^4, \\
& \bar{x}_5 := \bar{x}_1 C_{15} + \bar{x}_2 C_{25} + \bar{u}_5 A_5 \in N(n_{\tau(5)}, U), \quad \text{where } \bar{u}_5 \in \mathbb{F}_3^4, \\
& \bar{x}_6 := \bar{x}_1 C_{16} + \bar{x}_2 C_{26} + \bar{u}_6 A_6 \in N(n_{\tau(6)}, U), \quad \text{where } \bar{u}_6 \in \mathbb{F}_3^4, \\
& \bar{x}_7 := \bar{x}_1 C_{17} + \bar{x}_2 C_{27} + (\bar{u}_3, \bar{u}_4) C_{37} + (\bar{u}_5, \bar{u}_6) C_{47} \in N(n_{\tau(7)}, U), \\
& \bar{x}_8 := \bar{x}_1 C_{18} + \bar{x}_2 C_{28} + (\bar{u}_3, \bar{u}_4) C_{38} + (\bar{u}_5, \bar{u}_6) C_{48} \in N(n_{\tau(8)}, U).
\end{aligned}$$

If we find $\bar{x} = (\bar{x}_1, \dots, \bar{x}_8)$ satisfying all conditions in (3.3), then \bar{x} belongs to $\overline{M}(n^\tau)$ and hence

$$\bar{x}^{\tau^{-1}} = (\bar{x}_{\tau^{-1}(1)}, \dots, \bar{x}_{\tau^{-1}(8)})$$

is an element of $\overline{M}(n)$. All elements of $\overline{M}(n)$ are obtained in this way.

Using the maps $N(a, E) \rightarrow N(a, U)$, we can make from $\overline{M}(n)$ the set $M(n)$ of vectors $x \in \text{Min}(Q(\Delta, B))$ with $\nu(x) = n$. Taking the union of these sets $M(n)$ together with the set of minimal-norm vectors of divisible type, we obtain the set $\text{Min}(Q(\Delta, B))$ of all minimal-norm vectors of $Q(\Delta, B)$.

Remark 3.3. Thanks to the permutation τ , we have $|N(n_{\tau(i)}, U)| \leq |N(n_{\tau(j)}, U)|$ for $i < j$, and hence, in the back track search above, there exist few possibilities of \bar{x}_i for small indexes i . By this trick, the enumeration of $\text{Min}(Q(\Delta, B))$ becomes tractable.

Remark 3.4. We know that $|\text{Min}(L)| = 2611200$ for an extremal lattice L of rank 64 by the theory of theta series and modular forms. (See, for example, Chapter 7.7 of [2].) Hence we can confirm easily that we left no minimal-norm vectors uncounted.

3.2. Isomorphism classes. In order to distinguish isomorphism classes of two extremal lattices L and L' of rank 64, we use the *distribution of intersection patterns of minimal-norm vectors*. Let $\text{Min}(L)$ be the set of vectors $v \in L$ with $\langle v, v \rangle_L = 6$. For $v, v' \in \text{Min}(L)$, we have $\langle v, v' \rangle_L \in \{0, \pm 1, \pm 2, \pm 3, \pm 6\}$. For $k = 0, 1, 2, 3, 6$, we put

$$a_k(v) := \frac{1}{2} |\{v' \in \text{Min}(L) \mid \langle v, v' \rangle_L = k \text{ or } -k\}|.$$

We have $a_6(v) = 1$ and $\sum a_k(v) = 1305600$. The triple $a(v) := [a_1(v), a_2(v), a_3(v)]$ is called the *intersection pattern* of v . For a triple $a = [a_1, a_2, a_3]$ of non-negative integers with $a_1 + a_2 + a_3 + 1 \leq 1305600$, we put

$$\mathcal{A}_L(a) := \{v \in \text{Min}(L) \mid a(v) = a\}, \quad A_L(a) := \frac{1}{2} |\mathcal{A}_L(a)|,$$

and call the function A_L the *distribution of intersection patterns*. It is obvious that, if $A_L \neq A_{L'}$, then L and L' are not isomorphic.

Remark 3.5. In fact, the calculation of intersection patterns $a(v)$ of all elements v of $\text{Min}(Q(\Delta, B))/\{\pm 1\}$ takes most of the computation time in the proof of Theorem 1.1.

$$\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 0 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 0 \\ \hline 1 & 1 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 2 & 0 & 2 & 2 & 1 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 2 & 0 \end{array} \right]$$

TABLE 4.1. Matrix G'_0

3.3. Automorphism group. Let L be an extremal lattice of rank 64, and let Γ be a subgroup of $O(L)$. We will apply Proposition 3.7 below to $\Gamma = \{\pm 1\}$ or $\Gamma = \{\pm 1\} \times \langle \tilde{\gamma}_Q \rangle$, where $\tilde{\gamma}_Q$ is the automorphism of $Q(\Delta_0, B(\gamma, v))$ introduced in Section 2.3.

Definition 3.6. An ordered list (v_1, \dots, v_{64}) of vectors in $\text{Min}(L)$ is said to be a Γ -rigidifying basis if the following hold:

- (a) The vectors v_1, \dots, v_{64} form a basis of $L \otimes \mathbb{Q}$.
- (b) The group Γ acts on the set $\mathcal{A}_L(a(v_1))$ transitively.
- (c) Suppose that $i > 1$. Then the set

$$\{v' \in \mathcal{A}_L(a(v_i)) \mid \langle v', v_j \rangle_L = \langle v_i, v_j \rangle_L \text{ for all } j < i\}$$

consists of a single element v_i .

Proposition 3.7. *If a Γ -rigidifying basis exists, then $O(L)$ coincides with Γ .*

Proof. Note that $O(L)$ preserves each subset set $\mathcal{A}_L(a)$ of $\text{Min}(L)$ for any a . In particular, we have $v^g \in \mathcal{A}_L(a(v))$ for any $g \in O(L)$ and any $v \in L$. Let g be an arbitrary element of $O(L)$. By (b), there exists an element $g' \in \Gamma$ such that $v_1^g = v_1^{g'}$. By (c), we can prove that $v_i^g = v_i^{g'}$ holds for all $i = 1, \dots, 64$ by induction on i . Then (a) implies that $g = g'$. \square

4. EXAMPLES

4.1. Examples without non-trivial automorphisms. Let V_0 be the maximal isotropic subspace of U with basis v_1, \dots, v_4 in Table 2.1. By this basis, an element of V_0 is expressed by a vector in \mathbb{F}_3^4 , and hence an element of V_0^8 is expressed by a vector in \mathbb{F}_3^{32} . Let G_0 be the 8×32 matrix with components in \mathbb{F}_3 of the form $[I_8 \mid G'_0]$, where I_8 is the identity matrix of size 8 and G'_0 is given in Table 4.1. (This matrix G'_0 was produced by choosing components pseudo-randomly.) Let B_0 be the linear subspace of V_0^8 generated by the row vectors of G_0 . Then B_0 satisfies p_2 -condition. Recall that we have given maximal isotropic subspaces W^I and W^{II} in Table 2.1. Let Q^I (resp. Q^{II}) be the Quebbemann lattice $Q(\Delta^I, B_0)$ (resp. $Q(\Delta^{II}, B_0)$), where

$$\Delta^I = ((V_0, W^I), \dots, (V_0, W^I)) \in \mathcal{D}^8, \quad \Delta^{II} = ((V_0, W^{II}), \dots, (V_0, W^{II})) \in \mathcal{D}^8.$$

Then the distributions of intersection patterns of Q^I and Q^{II} are as in Table 4.2. (The left table is of Q^I and the right is of Q^{II} .) In these tables, the intersection patterns $a = [a_1, a_2, a_3]$ are sorted by the lexicographic order. We can readily see that Q^I and Q^{II} are not isomorphic. Both of Q^I and Q^{II} have $\{\pm 1\}$ -rigidifying basis, and hence $O(Q^I)$ and $O(Q^{II})$ are equal to $\{\pm 1\}$.

no.	a_1	a_2	a_3	$A_{Q^I}(a)$	no.	a_1	a_2	a_3	$A_{Q^{II}}(a)$
1	568092	40191	612	1	1	568422	40131	602	1
2	568290	40155	606	1	2	568488	40119	600	2
3	568356	40143	604	3	3	568554	40107	598	1
4	568488	40119	600	2	4	568620	40095	596	3
5	568554	40107	598	2	5	568686	40083	594	7
6	568620	40095	596	3	6	568752	40071	592	5
7	568686	40083	594	2	7	568818	40059	590	8
8	568752	40071	592	6	8	568884	40047	588	8
9	568818	40059	590	6	9	568950	40035	586	11
10	568884	40047	588	9	10	569016	40023	584	11
			
110	579840	38055	256	5333	110	580104	38007	248	9761
111	579906	38043	254	6275	111	580170	37995	246	11289
112	579972	38031	252	7616	112	580236	37983	244	13121
113	580038	38019	250	8752	113	580302	37971	242	15330
114	580104	38007	248	10065	114	580368	37959	240	17148
115	580170	37995	246	11511	115	580434	37947	238	19598
116	580236	37983	244	13332	116	580500	37935	236	22119
117	580302	37971	242	15370	117	580566	37923	234	24532
118	580368	37959	240	17252	118	580632	37911	232	27067
119	580434	37947	238	19533	119	580698	37899	230	29774
120	580500	37935	236	22294	120	580764	37887	228	32471
			
170	583800	37335	136	17	170	584064	37287	128	2
171	583866	37323	134	9	171	584130	37275	126	1
172	583932	37311	132	6	172	584196	37263	124	2
173	583998	37299	130	5	173	584262	37251	122	1
174	584130	37275	126	3	174	584328	37239	120	1
175	584196	37263	124	2					
176	584262	37251	122	1					
		total		1305600			total		1305600

TABLE 4.2. Distributions of intersection patterns of Q^I and Q^{II}

$$\gamma := \begin{bmatrix} 2 & 1 & 2 & 4 & 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -2 & -2 & -2 & -2 & -2 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 2 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ -3 & -2 & -4 & -6 & -5 & -4 & -2 & -1 \\ 2 & 1 & 3 & 4 & 3 & 2 & 1 & 1 \end{bmatrix}, \quad \gamma' := \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 \\ -2 & -1 & -2 & -4 & -3 & -2 & -2 & -1 \\ 3 & 2 & 4 & 6 & 5 & 3 & 2 & 1 \\ -2 & -2 & -4 & -5 & -4 & -3 & -2 & -1 \\ 1 & 1 & 2 & 3 & 3 & 3 & 2 & 1 \\ -1 & -1 & -1 & -2 & -2 & -2 & -1 & 0 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 1 & 2 & 3 & 2 & 2 & 1 & 1 \end{bmatrix}$$

TABLE 4.3. Elements of $O(E)$ of order 8

4.2. Examples with an automorphism of order 8. Let γ be an element of $O(E)$ represented by the matrix in Table 4.3 with respect to the basis e_1, \dots, e_8 of E . Then γ is of order 8 and belongs to the stabilizer subgroup G^I of $(V_0, W^1) \in \mathcal{D}$. Let $v = (v_1, \dots, v_8) \in V_0^8$ be

$$\left(\begin{array}{c|c|c|c|c|c|c|c} 2210 & 0120 & 0201 & 1001 & 0201 & 0222 & 0122 & 0122 \end{array} \right),$$

no.	a_1	a_2	a_3	$A_Q(a)$	no.	a_1	a_2	a_3	$A_{Q'}(a)$
1	568026	40203	614	8	1	568092	40191	612	8
2	568092	40191	612	16	2	568224	40167	608	24
3	568290	40155	606	24	3	568290	40155	606	8
4	568356	40143	604	16	4	568488	40119	600	8
5	568422	40131	602	16	5	568686	40083	594	16
			
100	580104	38007	248	11240	100	580104	38007	248	10688
101	580170	37995	246	12984	101	580170	37995	246	12344
102	580236	37983	244	14840	102	580236	37983	244	14656
103	580302	37971	242	16712	103	580302	37971	242	16064
104	580368	37959	240	18800	104	580368	37959	240	19240
105	580434	37947	238	20808	105	580434	37947	238	20104
106	580500	37935	236	23184	106	580500	37935	236	22984
107	580566	37923	234	25304	107	580566	37923	234	25128
108	580632	37911	232	27416	108	580632	37911	232	28064
109	580698	37899	230	29720	109	580698	37899	230	29128
110	580764	37887	228	32472	110	580764	37887	228	32304
			
155	583734	37347	138	40	155	583734	37347	138	32
156	583800	37335	136	24	156	583800	37335	136	48
157	583866	37323	134	16	157	583866	37323	134	24
158	583932	37311	132	8	158	583932	37311	132	24
					159	583998	37299	130	16
					160	584064	37287	128	8
					161	584196	37263	124	8
			total	1305600				total	1305600

TABLE 4.4. Distributions of intersection patterns of Q and Q'

where each component v_i is written with respect to the basis of V_0 in Table 2.1. Then the subspace $B(\gamma, v)$ of V_0^8 satisfies p_2 -condition, and we obtain a Quebbemann lattice $Q := Q(\Delta^I, B(\gamma, v))$ with an automorphism $\tilde{\gamma}_Q$ of order 8, where $\Delta^I \in \mathcal{D}^8$ is given in the previous subsection. By the method of Γ -rigidifying basis, we see that $O(Q) = \{\pm 1\} \times \langle \tilde{\gamma}_Q \rangle$. The action of $O(Q)$ decomposes $\text{Min}(Q)$ into $2611200/16 = 163200$ orbits. The distribution of intersection patterns is given in Table 4.4 (left).

Let γ' be an element of $O(E)$ given in Table 4.3, which is of order 8 and belongs to the stabilizer subgroup G^{II} of $(V_0, W^{\text{II}}) \in \mathcal{D}$. Let $v' \in V_0^8$ be

$$(2220 \mid 0102 \mid 2120 \mid 2220 \mid 2202 \mid 1202 \mid 2220 \mid 2112) .$$

Then $B(\gamma', v')$ satisfies p_2 -condition, and we obtain a Quebbemann lattice $Q' := Q(\Delta^{\text{II}}, B(\gamma', v'))$. We see that $O(Q') = \{\pm 1\} \times \langle \tilde{\gamma}_{Q'} \rangle$, and its action decomposes $\text{Min}(Q')$ into 163200 orbits. The distribution of intersection patterns is given in Table 4.4 (right).

REFERENCES

[1] Etsuko Bannai. Positive definite unimodular lattices with trivial automorphism groups. *Mem. Amer. Math. Soc.*, 85(429):iv+70, 1990.

- [2] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999.
- [3] Masaaki Harada, Masaaki Kitazume, and Michio Ozeki. Ternary code construction of unimodular lattices and self-dual codes over \mathbb{Z}_6 . *J. Algebraic Combin.*, 16(2):209–223, 2002.
- [4] Masaaki Harada and Tsuyoshi Mieozaki. On the existence of extremal Type II \mathbb{Z}_{2k} -codes. *Math. Comp.*, 83(287):1427–1446, 2014.
- [5] Yoshio Mimura. A construction of unimodular lattices without non-trivial automorphisms (a survey). *Sūrikaiseikikenkyūsho Kōkyūroku*, 1476:105–110, 2006.
- [6] Gabriele Nebe. Some cyclo-quaternionic lattices. *J. Algebra*, 199(2):472–498, 1998.
- [7] Meinhard Peters. Definite unimodular 48-dimensional quadratic forms. *Bull. London Math. Soc.*, 15(1):18–20, 1983.
- [8] H.-G. Quebbemann. An application of Siegel’s formula over quaternion orders. *Mathematika*, 31(1):12–16, 1984.
- [9] H.-G. Quebbemann. A construction of integral lattices. *Mathematika*, 31(1):137–140, 1984.
- [10] Ichiro Shimada. An even extremal lattice of rank 64. *J. Number Theory*, 185:1–15, 2018.
- [11] Ichiro Shimada. A note on Quebbemann’s extremal lattices of rank 64: computation data. <http://www.math.sci.hiroshima-u.ac.jp/shimada/lattice.html>, 2021.
- [12] The GAP Group. *GAP - Groups, Algorithms, and Programming*. Version 4.11.0; 2020 (<http://www.gap-system.org>).

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