Supersingular K3 surfaces (in characteristic 2)

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We work over an algebraically closed field k of characteristic $p > 0$.

Definition

A non-singular projective surface X is called a $K3$ surface if

- $K_X \cong \mathcal{O}_X$, **and**
- $h^1(X, \mathcal{O}_X) = 0$.

Example: $K3$ surfaces as double covers of $\mathbb{P}^2$.

Let $C(x_0, x_1, x_2)$ and $F(x_0, x_1, x_2)$ be homogeneous polynomials of degree 3 and 6, respectively.

Let Y be a double cover of $\mathbb{P}^2$ defined by

$$w^2 + w \cdot C(x_0, x_1, x_2) + F(x_0, x_1, x_2) = 0,$$

and let $X \to Y$ be the minimal resolution of Y.

Then X is a $K3$ surface if and only if Y has only rational double points as its singularities; that is, if and only only if the exceptional divisor of $X \to Y$ is an ADE-configuration of smooth rational curves of self-intersection -2.

A_n

D_n

E_6, E_7, E_8

Conversely, let X be a $K3$ surface.

Let $\mathcal{L}$ be a line bundle of X with $\mathcal{L}^2 = 2$.

If the complete linear system $|\mathcal{L}|$ has no fixed components, then $\dim|\mathcal{L}| = 2$, and the morphism

$$\Phi_{|\mathcal{L}|} : X \longrightarrow \mathbb{P}^2$$

defined by $|\mathcal{L}|$ is factored as

$$X \ \longrightarrow \ Y \ \longrightarrow \ \mathbb{P}^2$$

where $Y \longrightarrow \mathbb{P}^2$ is a double cover defined by

$$w^2 + w \cdot C(x_0, x_1, x_2) + F(x_0, x_1, x_2) = 0,$$

with $\deg C = 3$ and $\deg F = 6$, and Y has only rational double points as its singularities.

Remarks

(1) Suppose that k is *not* of characteristic 2. Then we can assume that

$$C(x_0, x_2, x_2) = 0$$

by the coordinate change

$$w \mapsto w + C/2.$$

Let $B \subset \mathbb{P}^2$ be the branch curve of $Y \to \mathbb{P}^2$;

$$B = \{F(x_0, x_1, x_2) = 0\} \quad \subset \quad \mathbb{P}^2.$$

Then X is a $K3$ surface if and only if the plane curve B has only rational double points as its singularities.

(2) In characteristic 2,

$$\boxed{C(x_0, x_2, x_2) = 0} \quad \Longleftrightarrow$$

$$\boxed{X \text{ is purely inseparable over } \mathbb{P}^2} \ .$$

Let X be a $K3$ surface.

Let D be a divisor on X.
We say that D is *numerically equivalent to* 0 (denoted by $D \equiv 0$) if

$$CD = 0 \qquad \text{for any curve } C \subset X$$

holds. (Here CD is the intersection number.)

The *Néron-Severi lattice* $NS(X)$ of X is the free abelian group

$$\{\text{divisors on } X\}/\equiv \ ,$$

equipped with the non-degenerate pairing

$$[D][D'] := DD'.$$

The rank of $NS(X)$ is called the *Picard number* of X:

$$\rho(X) := \operatorname{rank} NS(X).$$

Corollary of Hodge Index Theorem

The signature of the quadratic form on $NS(X) \otimes \mathbb{R}$ defined by the intersection pairing is $(1, \rho(X) - 1)$.

In characteristic 0, we have

$$NS(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z}).$$

In particular, $\rho(X) \leq h^{1,1}(X) = 20.$

In positive characteristics, the possible values of $\rho(X)$ are $1, \ldots, 20$ and 22.

A $K3$ surface X is *supersingular* (in the sense of Shioda) if $\rho(X) = 22$ holds.

Example

Suppose that k is of characteristic 2.

Let $G(x_0, x_1, x_2)$ be a general homogeneous polynomial of degree 6. Then the purely inseparable double cover $Y_G \to \mathbb{P}^2$ defined by

$$w^2 = G(x_0, x_1, x_2)$$

has 21 rational double points of type A_1.

Proof. We put $g(x, y) := G(x, y, 1)$. Since

$$\frac{\partial w^2}{\partial w} = 0$$

in characteristic 2, the singular points of Y_G is given by

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0.$$

The affine curves $\partial g/\partial x = 0$ and $\partial g/\partial y = 0$ intersect at 21 points transversely when G is general. (Other four intersection points are always on the line at infinity.) $\square$

Let $X_G \to Y_G$ be the minimal resolution of Y_G. Since

$$\rho(X_G) = 21 + 1 = 22,$$

the $K3$ surface X_G is supersingular.

Example (Pho D. Tai)

Suppose that k is of characteristic 5.

Let $f(x)$ be a general polynomial of degree 6, and let $B \subset \mathbb{P}^2$ be the projective completion of the affine curve defined by

$$y^5 = f(x).$$

Then $\operatorname{Sing} B$ consists of five rational double points of type A_4.

Indeed, let $\alpha_1, \dots, \alpha_5$ be the roots of $f'(x) = 0$, and let β_i be the (unique) 5-th root of $y^5 = f(\alpha_i)$. Then

$$\operatorname{Sing} B = \{(\alpha_1, \beta_1), \dots, (\alpha_5, \beta_5)\}.$$

At each singular point, B is formally isomorphic to

$$\eta^5 - \xi^2 = 0.$$

Let $Y \to \mathbb{P}^2$ be the double cover defined by

$$w^2 = y^5 - f(x),$$

and let $X \to Y$ be the minimal resolution of Y. Then

$$\rho(X) \geq 5 \times 4 + 1 = 21.$$

Hence $\rho(X) = 22$.

The *discriminant* of a lattice is the determinant of the intersection matrix.

Theorem (Artin)

Let X be a supersingular $K3$ surface in characteristic p. Then the discriminant of $NS(X)$ is of the form

$$-p^{2\sigma_X},$$

where σ_X is a positive integer ≤ 10.

The integer σ_X is called the *Artin invariant* of the supersingular $K3$ surface X.

Theorem (Artin, Shioda, Rudakov-Shafarevich)

For any pair (p, σ) of a prime integer p and a positive integer $\sigma \leq 10$, there exists a supersingular $K3$ surface X in characteristic p with Artin invariant σ.

Theorem (Rudakov-Shafarevich)

The Néron-Severi lattice of a supersingular $K3$ surface is determined uniquely (up to isomorphisms of lattices) by p and the Artin invariant.

More precisely:

Let $\Lambda_{p,\sigma}$ be the lattice of rank 22 with the following properties:
(i) even (i.e., $v^2 \in 2\mathbb{Z}$ for every $v \in \Lambda_{p,\sigma}$),
(ii) the signature is $(1, 21)$,
(iii) the cokernel of the natural embedding

$$\Lambda_{p,\sigma} \quad \hookrightarrow \quad \Lambda^{\vee}_{p,\sigma} := \mathrm{Hom}(\Lambda_{p,\sigma}, \mathbb{Z}) \quad \subset \quad \Lambda_{p,\sigma} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2\sigma}$, and
(iv) if $p = 2$, then $u^2 \in \mathbb{Z}$ for every $u \in \Lambda^{\vee}_{2,\sigma} \subset \Lambda_{2,\sigma} \otimes_{\mathbb{Z}} \mathbb{Q}$.

- **These properties determine the lattice $\Lambda_{p,\sigma}$ uniquely up to isomorphisms.**

- **If X is a supersingular $K3$ surface in characteristic p with $\sigma_X = \sigma$, then $NS(X)$ has these properties.**

Hence there exists an isomorphism

$$\phi \quad : \quad \Lambda_{p,\sigma} \quad \xrightarrow{\sim} \quad NS(X).$$

Theorem (Rudakov-Shafarevich)

Let $h \in \Lambda_{p,\sigma}$ be a vector with $h^2 = 2$ such that

$$\{ v \in \Lambda_{p,\sigma} \mid v^2 = 0, vh = 1 \} = \emptyset.$$

Then we can choose an isomorphism

$$\phi : \Lambda_{p,\sigma} \xrightarrow{\sim} NS(X)$$

in such a way that

$$\phi(h) = [\mathcal{L}],$$

where $\mathcal{L}$ is a line bundle on X whose complete linear system $|\mathcal{L}|$ has no fixed components.

Let

$$X \to Y \to \mathbb{P}^2$$

be the Stein factorization of $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$. Then the ADE-type of the singularity of Y is equal to the ADE-type of the root system

$$\{ v \in \Lambda_{p,\sigma} \mid v^2 = -2, vh = 0 \}.$$

For a supersingular $K3$ surface X, the set

$$\{\text{generically finite morphisms } X \to \mathbb{P}^2 \text{ of degree } 2\}$$

is completely determined by the characteristic p and the Artin invariant of X.

Theorem (S.)

(1) Every supersingular $K3$ surface in odd characteristic is birational to a double cover of $\mathbb{P}^2$ branched along a plane curve of degree 6.

(2) Every supersingular $K3$ surface in characteristic 2 is birational to a purely inseparable double cover of $\mathbb{P}^2$ that has 21 rational double points of type A_1.

Problem in odd characteristics

For each pair (p, σ), find a plane curve

$$F(x_0, x_1, x_2) = 0$$

of degree 6 such that the minimal resolution of the surface $w^2 = F(x_0, x_1, x_2)$ is a supersingular $K3$ surface in characteristic p with Artin invariant σ.

Example in characteristic 5 (Pho D. Tai)

Suppose that k is of characteristic 5, and let X be the supersingular $K3$ surface birational to

$$\{w^2 = y^5 - f(x)\} \quad \subset \quad \mathbb{A}^3,$$

where $f(x)$ is a general polynomial of degree 6. Then $NS(X)$ is isomorphic to

$$R(A_4) \oplus R(A_4) \oplus R(A_4) \oplus R(A_4) \oplus R(A_4) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

where $R(A_4)$ is the (negative-definite) root lattice given by the Cartan matrix of type A_4;

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Since $\operatorname{disc} R(A_4) = 5$, we have $\operatorname{disc} NS(X) = -5^6$, and hence

$$\sigma_X = 3.$$

Example (continued)

Conversely, suppose that X is a supersingular $K3$ surface in characteristic 5 with Artin invariant 3. Then, by the theorem of Rudakov-Shafarevich, $NS(X)$ is isomorphic to

$$R(A_4) \oplus R(A_4) \oplus R(A_4) \oplus R(A_4) \oplus R(A_4) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

It follows that X has a line bundle $\mathcal{L}$ of degree 2 ($\mathcal{L}^2 = 2$) such that

- $|\mathcal{L}|$ has no fixed components, and
- the Stein factorization of the morphism $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ defined by $|\mathcal{L}|$ is

  $$X \to Y \to \mathbb{P}^2,$$

  where Y has $5A_4$ as its singularities.

On the other hand, we can show that, if a plane curve B of degree 6 has $5A_4$ as its singularities, then B is defined by an equation of the form

$$y^5 - f(x) = 0.$$

Theorem (Pho D. Tai)

Every supersingular $K3$ surface in characteristic 5 with Artin invariant ≤ 3 is birational to a surface defined by an equation of the form

$$w^2 = y^5 - f(x).$$

Corollary

Every supersingular $K3$ surface X in characteristic 5 with Artin invariant ≤ 3 is unirational; that is, the function field $k(X)$ of X is contained in $k(\mathbb{P}^2) = k(u, v)$.

Conjecture (Artin-Shioda)

Every supersingular $K3$ surface X is unirational.

Artin-Shioda Conjecture has been verified in the following cases:

- $p = 2$,
- $p = 3$ **and** $\sigma \leq 6$, **and**
- $\sigma \leq 2$.

Suppose that k is of characteristic 2.

Then every supersingular $K3$ surface is obtained as the minimal resolution $X_G \to Y_G$ of a purely inseparable double cover

$$Y_G \quad : \quad w^2 = G(x_0, x_1, x_2)$$

of $\mathbb{P}^2$ with 21 ordinary nodes.

We put

$$\mathcal{U} := \{ \, G \mid \mathrm{Sing}(Y_G) \text{ consists of 21 ordinary nodes } \} \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)).$$

Then $\mathcal{U}$ is a Zariski open dense subset of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$.

For any $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$, we can define

$$dG \quad \in \quad \Gamma(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(6)),$$

because we are in characteristic 2 and we have $\mathcal{O}_{\mathbb{P}^2}(6) \cong \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2}$ so that the transition functions of $\mathcal{O}_{\mathbb{P}^2}(6)$ are squares:

$$g' = t^2 g \quad \Longrightarrow \quad dg' = 2g\,dt + t^2\,dg = t^2\,dg.$$

Let $Z(dG)$ be the subscheme of $\mathbb{P}^2$ defined by $dG = 0$.

Then we have

$$\mathrm{Sing}(Y_G) = \pi_G^{-1}(Z(dG)),$$

where $\pi_G : Y_G \to \mathbb{P}^2$ is the covering morphism.

Hence

$$\mathcal{U} = \{\ G \ \mid\ Z(dG) \text{ is reduced of dimension } 0\ \}.$$

(Note that $c_2(\Omega^1_{\mathbb{P}^2}(6)) = 21$.)

We put

$$\mathcal{V} := H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)).$$

Because we have $d(G + H^2) = dG$ **for** $H \in \mathcal{V}$**, the additive group** $\mathcal{V}$ **acts on the space** $\mathcal{U}$ **by**

$$(G, H) \in \mathcal{U} \times \mathcal{V} \quad \mapsto \quad G + H^2 \in \mathcal{U}.$$

Let G **and** G' **be homogeneous polynomials in** $\mathcal{U}$**.**
Then the following conditions are equivalent:
(i) Y_G **and** $Y_{G'}$ **are isomorphic over** $\mathbb{P}^2$**,**
(ii) $Z(dG) = Z(dG')$**, and**
(iii) there exist $c \in k^\times$ **and** $H \in \mathcal{V}$ **such that**
$G' = c\,G + H^2$**.**

Therefore a moduli space $\mathcal{M}$ **of polarized supersingular** $K3$ **surfaces of degree 2 is constructed by**

$$\mathcal{M} = GL(3, k)\backslash\mathcal{U}/\mathcal{V}.$$

Let

$$(\mathcal{U}/\mathcal{V})^s \ \subset \ \mathcal{U}/\mathcal{V}$$

be the locus of stable points with respect to the action of $GL(3, k)$ on the vector space $\mathcal{U}/\mathcal{V}$.

By Hilbert-Mumford criterion, we can see that

$(\mathcal{U}/\mathcal{V})^s = \{ \ G \ \mid \ Y_G$ **has only rational double points** $\}$.

If $[G] \in (\mathcal{U}/\mathcal{V})^s$, then $\mathrm{Sing}(Y_G)$ consists of rational double points of type

$$A_1, \quad D_{2m}, \quad E_7 \quad \text{or} \quad E_8.$$

We can calculate the Artin invariant of X_G from $G \in \mathcal{U}$.

Example

If $G \in \mathcal{U}$ is general, the Artin invariant of X_G is 10.

Let G_1 and G_2 be general homogeneous polynomials such that

$$\deg G_1 + \deg G_2 = 6.$$

Then

$$G = G_1 G_2$$

is a member of $\mathcal{U}$, and the Artin invariant of X_G is 9.

The proper transform of the plane curve C_1 defined by $G_1 = 0$ is non-reduced divisor on X_G. It is written as $2F_1$. The class of the curve F_1 gives an extra algebraic cycles.

Example

Let $L_1, \ldots, L_6$ be general linear forms of $\mathbb{P}^2$. Then

$$G := L_1 L_2 \ldots L_6$$

is a member of $\mathcal{U}$, and the Artin invariant of X_G is 5.

Example

We put

$$G[a] := x_0 x_1 x_2 \left(x_0^3 + x_1^3 + x_2^3\right) + a\, x_0^3 x_1^3,$$

where a is a parameter.

Then $G[a]$ is a member of $\mathcal{U}$ for any a, and the Artin invariant of $X_{G[a]}$ is

$$\begin{cases} 2 & \text{if } a \neq 0, \\ 1 & \text{if } a = 0. \end{cases}$$

Compactification of the moduli.

Rudakov, Shafarevich and Zink showed that, at least in characteristic > 3, every smooth family of of supersingular $K3$ surfaces can be extended, after base change by finite covering and birational transformation of the total space, to a *non-degenerate complete* family.

Problem

Construct explicitly a non-degenerate completion of a finite cover of the moduli space $\mathcal{M}$.

Example in characteristic 3 (S. and De Qi Zhang)

In characteristic 3, every supersingular $K3$ with Artin invariant ≤ 6 is birational to a purely inseparable triple cover Y of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$w^3 = f(x_0, x_1; y_0, y_1),$$

where f is a bi-homogeneous polynomial of degree $(3, 3)$, and Y has 10 rational double points of type A_2 as its only singularities.

The minimal resolution of the surface

$$w^3 = (x_0^3 - x_0^2 x_1)(y_0^3 - y_0^2 y_1)$$

is a supersingular $K3$ surface with Artin invariant 1.