Non-homeomorphic conjugate complex varieties

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- **We work over the complex number field $\mathbb{C}$.**
- **The coefficients of the (co-)homology groups are in $\mathbb{Z}$.**
- **By a lattice, we mean a finitely generated free $\mathbb{Z}$-module Λ equipped with a non-degenerate symmetric bilinear form**

$$\Lambda \times \Lambda \quad \to \quad \mathbb{Z}.$$

- **A lattice Λ is said to be *even* if $(v, v) \in 2\mathbb{Z}$ for any $v \in \Lambda$.**

§1. Conjugate varieties

An affine algebraic variety $X \subset \mathbb{C}^N$ is defined by a finite number of polynomial equations:

$$X \quad : \quad f_1(x_1, \dots, x_N) = \cdots = f_m(x_1, \dots, x_N) = 0.$$

Let $c_{j,I} \in \mathbb{C}$ be the coefficients of the polynomial f_j:

$$f_j(x_1, \dots, x_N) = \sum_I c_{j,I} x^I, \quad \text{where} \quad x^I = x_1^{i_1} \cdots x_N^{i_N}.$$

We then denote by

$$F_X := \mathbb{Q}(\dots, c_{j,I}, \dots) \quad \subset \quad \mathbb{C}$$

the minimal sub-field of $\mathbb{C}$ containing all the coefficients of the defining equations of X.

There are many other embeddings

$$\sigma \quad : \quad F_X \quad \hookrightarrow \quad \mathbb{C}$$

of the field F_X into $\mathbb{C}$.

Example.
(1) If $F_X = \mathbb{Q}(\sqrt{2}, t)$, where $t \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, then the set of embeddings $F_X \hookrightarrow \mathbb{C}$ is equal to

$$\{\sqrt{2}, -\sqrt{2}\} \times \{ \text{ transcendental complex numbers } \}.$$

(2) If all $c_{j,I}$ are algebraic over $\mathbb{Q}$, then the set of embeddings is finite, and the Galois group of the Galois closure of the algebraic extension $F_X/\mathbb{Q}$ acts on the set transitively.

For an embedding $\sigma : F_X \hookrightarrow \mathbb{C}$, we put

$$f_j^\sigma(x_1, \ldots, x_N) := \sum_I c_{j,I}^\sigma x^I,$$

and denote by $X^\sigma \subset \mathbb{C}^N$ the affine algebraic variety defined by

$$f_1^\sigma = \cdots = f_m^\sigma = 0.$$

We can define X^σ for a *projective* or *quasi-projective* variety $X \subset \mathbb{P}^N$ in the same way.

(Replace "polynomials" by "homogeneous polynomials".)

Definition.
We say that two algebraic varieties X and Y are said to be *conjugate* if there exists an embedding $\sigma : F_X \hookrightarrow \mathbb{C}$ such that Y is isomorphic (over $\mathbb{C}$) to X^σ.

In the language of schemes, two varieties X and Y over $\operatorname{Spec}\mathbb{C}$ are conjugate if there exists a diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\sigma^*} & \operatorname{Spec}\mathbb{C}. \end{array}$$

of the *fiber product* for some morphism $\sigma^* : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$.

It is obvious that being conjugate is an equivalence relation.

§2. Topology of conjugate varieties

Conjugate varieties cannot be distinguished by any algebraic methods.

In particular, they are homeomorphic in *Zariski* topology.

How about in the complex topology?

Example (Serre (1964)).
There exist conjugate non-singular projective varieties X and X^σ such that their fundamental groups are *not* isomorphic:

$$\pi_1(X) \not\cong \pi_1(X^\sigma).$$

In particular, they are not homotopically equivalent.

Other examples of non-homeomorphic conjugate varieties.

- **Abelson: Topologically distinct conjugate varieties with finite fundamental group.**
  Topology 13 (1974).
- **Artal Bartolo, Carmona Ruber, Cogolludo Agustín: Effective invariants of braid monodromy.**
  Trans. Amer. Math. Soc. 359 (2007).
- **S.-: On arithmetic Zariski pairs in degree 6.**
  arXiv:math/0611596, to appear in Adv. Geom.
- **S.-: Non-homeomorphic conjugate complex varieties.**
  arXiv:math/0701115
- **Easton, Vakil: Absolute Galois acts faithfully on the components of the moduli space of surfaces: A Belyi-type theorem in higher dimension.**
  arXiv:0704.3231
- **Bauer, Catanese, Grunewald: The absolute Galois group acts faithfully on the connected components of the moduli space of surfaces of general type.**
  arXiv:0706.1466
- **F. Charles: Conjugate varieties with distinct real cohomology algebras.**
  arXiv:0706.3674

Main result.

We introduce a new topological invariant

$$(B_U, \beta_U)$$

of *open* algebraic varieties U, which allows us to distinguish conjugate varieties topologically in some cases.

Combining this topological invariant with the arithmetic theory of abelian surfaces and $K3$ surfaces, we obtain examples of non-homeomorphic conjugate varieties.

Our examples are as follows:

- Zariski open subsets of abelian surfaces.
- Zariski open subsets of $K3$ surfaces.
- Arithmetic Zariski pairs in degree 6.

§3. Arithmetic Zariski pairs

Definition.
A pair $[C, C']$ of complex projective plane curves is said to be a *Zariski pair* if the following hold:

(i) There exist tubular neighborhoods $\mathcal{T} \subset \mathbb{P}^2$ of C and $\mathcal{T}' \subset \mathbb{P}^2$ of C' such that $(\mathcal{T}, C)$ and $(\mathcal{T}', C')$ are diffeomorphic.

(ii) $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are *not* homeomorphic.

Example.
The first example of Zariski pair was discovered by Zariski in 1930's, and studied by Oka. They presented a Zariski pair $[C, C']$ of plane curves of degree 6, each of which has six ordinary cusps as its only singularities. The fact $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are *not* homeomorphic follows from

$$\pi_1(\mathbb{P}^2 \setminus C) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) \quad \text{and} \quad \pi_1(\mathbb{P}^2 \setminus C') \cong \mathbb{Z}/6\mathbb{Z}.$$

Definition.

A Zariski pair $[C, C']$ is said to be an *arithmetic Zariski pair* if the following hold.

Suppose that $C = \{\Phi = 0\}$. Then there exists an embedding $\sigma : F_C \hookrightarrow \mathbb{C}$ such that C' is isomorphic (as a plane curve) to

$$C^\sigma := \{\Phi^\sigma = 0\} \quad \subset \quad \mathbb{P}^2.$$

Remark.

The Zariski pair of Zariski and Oka is not an arithmetic Zariski pair, because the pro-finite completion of

$$\pi_1(\mathbb{P}^2 \setminus C) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) \quad \text{and} \quad \pi_1(\mathbb{P}^2 \setminus C') \cong \mathbb{Z}/6\mathbb{Z}$$

are not isomorphic; there exists a surjective homomorphism from $\pi_1(\mathbb{P}^2 \setminus C)$ to the symmetric group S_3 on three letters, while there are no such homomorphism from $\pi_1(\mathbb{P}^2 \setminus C')$.

Remark.

The first example of an arithmetic Zariski pair was discovered by Artal, Carmona, Cogolludo (2007) in degree 12.

They used the invariant of *braid monodromies* in order to distinguish $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ topologically.

Example (Artal, Carmona, Cogolludo (2002)).

We consider the following cubic extension of $\mathbb{Q}$:

$$K := \mathbb{Q}[t]/(\varphi), \quad \text{where} \quad \varphi = 17t^3 - 18t^2 - 228t + 556.$$

The roots of $\varphi = 0$ are $\alpha, \bar{\alpha}, \beta$, where

$$\alpha = 2.590\cdots + 1.108\cdots\sqrt{-1}, \qquad \beta = -4.121\cdots .$$

There are three corresponding embeddings

$$\sigma_\alpha : K \hookrightarrow \mathbb{C}, \quad \sigma_{\bar{\alpha}} : K \hookrightarrow \mathbb{C} \quad \text{and} \quad \sigma_\beta : K \hookrightarrow \mathbb{C}.$$

There exists a homogeneous polynomial

$$\Phi(x_0, x_1, x_2) \in K[x_0, x_1, x_2]$$

of degree 6 with coefficients in K such that the plane curve

$$C = \{\Phi = 0\}$$

has three simple singular points of type

$$A_{16} + A_2 + A_1$$

as its only singularities. Consider the conjugate plane curves

$$C_\alpha = \{\Phi^{\sigma_\alpha} = 0\}, \quad C_{\bar{\alpha}} = \{\Phi^{\sigma_{\bar{\alpha}}} = 0\} \quad \text{and} \quad C_\beta = \{\Phi^{\sigma_\beta} = 0\}.$$

They show that, if C' is a plane curve possessing $A_{16} + A_2 + A_1$ as its only singularities, then C' is projectively isomorphic to C_α, $C_{\bar{\alpha}}$ or C_β.

Since simple singularities have no moduli, there are tubular neighborhoods $\mathcal{T}_\alpha \subset \mathbb{P}^2$ of $C_\alpha \subset \mathbb{P}^2$ and $\mathcal{T}_\beta \subset \mathbb{P}^2$ of $C_\beta \subset \mathbb{P}^2$ such that $(\mathcal{T}_\alpha, C_\alpha)$ is diffeomorphic to $(\mathcal{T}_\beta, C_\beta)$.

Using the new topological invariant, we can show that there are no homeomorphisms between $(\mathbb{P}^2, C_\alpha)$ and $(\mathbb{P}^2, C_\beta)$.

Let $Y_C \to \mathbb{P}^2$ be the double covering branching exactly along the curve $C : \Phi = 0$, and $U \subset Y_C$ the pull-back of $\mathbb{P}^2 \setminus C$. Then U is a variety defined over K. Consider the conjugate open varieties U_α and U_β corresponding to the embeddings σ_α and σ_β. Then the topological invariants

$$(B_{U_\alpha}, \beta_{U_\alpha}) \quad \text{and} \quad (B_{U_\beta}, \beta_{U_\beta})$$

differ.

Hence $[C_\alpha, C_\beta]$ is an arithmetic Zariski pair in degree 6.

§4. The topological invariant

Let U be an oriented topological manifold of dimension $4n$. Let

$$\iota_U : H_{2n}(U) \times H_{2n}(U) \to \mathbb{Z}$$

be the intersection pairing.

Definition.
We put

$$J_\infty(U) := \bigcap_K \operatorname{Im}(H_{2n}(U \setminus K) \to H_{2n}(U)),$$

where K runs through the set of all compact subsets of U. We then put

$$\widetilde{B}_U := H_{2n}(U)/J_\infty(U) \quad \text{and} \quad B_U := (\widetilde{B}_U)/\text{torsion}.$$

Since any topological cycle is compact, the intersection pairing ι_U induces a symmetric bilinear form

$$\beta_U : B_U \times B_U \to \mathbb{Z}.$$

It is obvious that, if U and U' are homeomorphic, then there exists an isomorphism

$$(B_U, \beta_U) \cong (B_{U'}, \beta_{U'}),$$

and hence the isomorphism class of (B_U, β_U) is a topological invariant of U.

We study the invariant (B_U, β_U) for an open algebraic variety

$$U := X \setminus Y,$$

where X is a non-singular projective variety of complex dimension $2n$, and Y is a union of irreducible (possibly singular) subvarieties $Y_1 \dots, Y_N$ of complex dimension n:

$$Y = Y_1 \cup \cdots \cup Y_N.$$

We denote by

$$\widetilde{\Sigma}_{(X,Y)} := \langle [Y_1], \dots, [Y_N] \rangle \quad \subset \quad H_{2n}(X)$$

the submodule of $H_{2n}(X)$ generated by the homology classes $[Y_i] \in H_{2n}(X)$, and put

$$\Sigma_{(X,Y)} := (\widetilde{\Sigma}_{(X,Y)})/\text{torsion}.$$

We then put

$$\begin{aligned} \widetilde{\Lambda}_{(X,Y)} &:= \{x \in H_{2n}(X) \mid \iota_X(x, y) = 0 \text{ for any } y \in \widetilde{\Sigma}_{(X,Y)}\}, \\ \Lambda_{(X,Y)} &:= (\widetilde{\Lambda}_{(X,Y)})/\text{torsion}. \end{aligned}$$

Finally, we denote by

$$\begin{aligned} \sigma_{(X,Y)} &: \Sigma_{(X,Y)} \times \Sigma_{(X,Y)} \to \mathbb{Z} \quad \text{and} \\ \lambda_{(X,Y)} &: \Lambda_{(X,Y)} \times \Lambda_{(X,Y)} \to \mathbb{Z} \end{aligned}$$

the symmetric bilinear forms induced from the intersection pairing

$$\iota_X : H_{2n}(X) \times H_{2n}(X) \to \mathbb{Z}.$$

Theorem.
Let X, Y and U be as above. Suppose that $\sigma_{(X,Y)}$ is non-degenerate. Then (B_U, β_U) is isomorphic to $(\Lambda_{(X,Y)}, \lambda_{(X,Y)})$.

Sketch of the proof.

We consider the homomorphism

$$j_U \ : \ H_{2n}(U) \to H_{2n}(X)$$

induced by the inclusion. It is obvious that the image of j_U is contained in $\widetilde{\Lambda}_{(X,Y)}$. We first show that

$$\mathrm{Im}(j_U) = \widetilde{\Lambda}_{(X,Y)}.$$

Let a homology class

$$[W] \in \widetilde{\Lambda}_{(X,Y)}$$

be represented by a real $2n$-dimensional topological cycle W. We can assume that $W \cap Y$ consists of a finite number of points in $Y \setminus \mathrm{Sing}(Y)$, and that the intersection of W with Y is transverse at each intersection point.

Let $P_{i,1}, \ldots, P_{i,k(i)}$ (resp. $Q_{i,1}, \ldots, Q_{i,l(i)}$) be the intersection points of W and Y_i with local intersection number 1 (resp. -1). Since $\iota_X([W],[Y_i]) = 0$, we have

$$k(i) = l(i).$$

Modifying W by adding the tube

$$\partial(D^{2n} \times I)$$

for each pair $(P_{i,j}, Q_{i,j})$, we obtain a topological cycle W' that is homologous to W in X and is disjoint from Y. Hence $[W] = [W']$ is represented by $W' \subset U$. Thus

$$\mathrm{Im}(j_U) = \widetilde{\Lambda}_{(X,Y)}$$

holds.

Figure

Since X is non-singular and complete, the intersection pairing ι_X on $H_{2n}(X)/\text{torsion}$ is non-degenerate. Hence the assumption that $\sigma_{(X,Y)}$ is non-degenerate implies that $\lambda_{(X,Y)}$ is non-degenerate.

Using Mayer-Vietris sequence, we can prove

$$\mathrm{Ker}(j_U) \subseteq J_\infty(U)$$

from the assumption that $\lambda_{(X,Y)}$ is non-degenerate.

By the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(j_U) & \longrightarrow & H_{2n}(U) & \xrightarrow{j_U} & \widetilde{\Lambda}_{(X,Y)} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow \tilde{v} & & \\
0 & \longrightarrow & J_\infty(U) & \longrightarrow & H_{2n}(U) & \longrightarrow & \widetilde{B}_U & \longrightarrow & 0 \ ,
\end{array}$$

we obtain the isomorphism $(\Lambda_{(X,Y)}, \lambda_{(X,Y)}) \cong (B_U, \beta_U)$.

§5. Transcendental lattices

Let X be a non-singular projective variety of dimension $2n$. Then we have a natural isomorphism

$$H_{2n}(X)/\text{torsion} \cong H^{2n}(X)/\text{torsion}$$

that transforms ι_X to the cup-product $(\ ,\ )_X$. Let

$$S_X \subset H^{2n}(X)/\text{torsion}$$

be the submodule generated by the classes $[Z]$ of irreducible subvarieties Z of X with codimension n; that is, S_X is the space of *algebraic cycles* in the middle dimension. We then denote by

$$s_X : S_X \times S_X \to \mathbb{Z}$$

the restriction of $(\ ,\ )_X$ to S_X.

By the theory of Lefschetz decomposition and Hodge-Riemann bilinear relations, we see that s_X is non-degenerate.

Proposition.
Let X and X^σ be conjugate non-singular projective varieties. Then the map $[Z] \mapsto [Z^\sigma]$ induces an isomorphism

$$(S_X, s_X) \cong (S_{X^\sigma}, s_{X^\sigma}).$$

In other words, (S_X, s_X) is algebraic.

Definition.

We define the *transcendental lattice* T_X of X to be the free $\mathbb{Z}$-module

$$T_X := \{x \in H^{2n}(X)/\text{torsion} \mid (x,y)_X = 0 \text{ for any } y \in S_X\}.$$

Theorem.

Let X be a non-singular projective variety of dimension $2n$. Let $Y_1, \ldots, Y_N$ be irreducible subvarieties of X with codimension n whose classes $[Y_1], \ldots, [Y_N]$ span $S_X \otimes \mathbb{Q}$ over $\mathbb{Q}$. We put

$$Y := \bigcup_{i=1}^{N} Y_i \quad \text{and} \quad U := X \setminus Y.$$

Then the transcendental lattice T_X of X is isomorphic to the topological invariant (B_U, β_U) of U.

Corollary.

Let X and X^σ be conjugate non-singular projective varieties of dimension $2n$. Let $Y \subset X$ and $U \subset X$ be as above. If T_{X^σ} is not isomorphic to T_X, then $U^\sigma = X^\sigma \setminus Y^\sigma$ is not homeomorphic to U.

§6. Genus theory of lattices

Definition.
Two lattices

$$\lambda : \Lambda \times \Lambda \to \mathbb{Z} \quad \text{and} \quad \lambda' : \Lambda' \times \Lambda' \to \mathbb{Z}$$

are said to be *in the same genus* if

$$\lambda \otimes \mathbb{Z}_p \;:\; \Lambda \otimes \mathbb{Z}_p \times \Lambda \otimes \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{and}$$
$$\lambda' \otimes \mathbb{Z}_p \;:\; \Lambda' \otimes \mathbb{Z}_p \times \Lambda' \otimes \mathbb{Z}_p \to \mathbb{Z}_p$$

are isomorphic for any p including $p = \infty$, where $\mathbb{Z}_\infty = \mathbb{R}$.

Let X be a non-singular projective variety of dimension $2n$. Recall that S_X is the submodule of $H^{2n}(X)/\text{torsion}$ generated by the algebraic cycles. We consider the following condition:

(P) The submodule S_X is primitive in $H^{2n}(X)/\text{torsion}$; that is, the quotient $(H^{2n}(X)/\text{torsion})/S_X$ is torsion-free.

Remark.
The condition (P) is satisfied for X if the *integral* Hodge conjecture

$$S_X = H^{2n}(X, \mathbb{Z}) \cap H^{n,n}(X)$$

is true for X. In particular, the condition (P) is satisfied if $\dim X = 2$. There exists, however, a counter-example for (P) in higher-dimension. (Atiyah-Hirzebruch (1962).)

Theorem.
Let X and X^σ be conjugate non-singular projective varieties of dimension $2n$. Suppose that (P) holds for both of X and X^σ. Then the transcendental lattices T_X and T_{X^σ} are contained in the same genus.

Let X be a surface. Then T_X and T_{X^σ} are contained in the same genus. Let $Y_1, \dots, Y_N$ be irreducible curves of X whose classes span $S_X \otimes \mathbb{Q}$. We put

$$Y := \bigcup_{i=1}^{N} Y_i \quad \text{and} \quad U := X \setminus Y.$$

If T_X and T_{X^σ} are not isomorphic, then U and U^σ are not homeomorphic.

By the classical theory of Gauss

Disquisitiones arithmeticae,

we have a complete theory of the decomposition of the set of isomorphism classes of lattices of rank 2 (*binary lattices*) into the disjoint union of genera.

Example.
Two binary lattices

$$\begin{bmatrix} 10 & 4 \\ 4 & 22 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 & 0 \\ 0 & 34 \end{bmatrix}$$

are not isomorphic, but in the same genus.

Problem.
Can one find a surface X and $\sigma : F_X \hookrightarrow \mathbb{C}$ such that

$$T_X \cong \begin{bmatrix} 10 & 4 \\ 4 & 22 \end{bmatrix} \quad \text{and} \quad T_{X^\sigma} \cong \begin{bmatrix} 6 & 0 \\ 0 & 34 \end{bmatrix} \quad ?$$

§7. Singular $K3$ surfaces

Let X be a $K3$ surface; that is, a simply-connected surface with $K_X \cong \mathcal{O}_X$. Then $H^2(X)$ is a unimodular lattice of rank 22 with signature $(3, 19)$.

Definition.
A $K3$ surface X is said to be *singular* if the rank of the transcendental lattice

$$T(X) := T_X$$

is 2 (the possible minimum).

The transcendental lattice $T(X)$ of a singular $K3$ surface X is positive-definite. Moreover, by the Hodge decomposition

$$T(X) \otimes \mathbb{C} \cong H^{2,0}(X) \oplus H^{0,2}(X),$$

this lattice has a canonical orientation. We denote by $\widetilde{T}(X)$ the oriented transcendental lattice of X.

Definition.
We denote by

$$\mathcal{L} := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z},\ a > 0,\ c > 0, \\ 4ac - b^2 > 0 \end{array} \right\} \Big/ GL_2(\mathbb{Z})$$

the set of isomorphism classes of even positive-definite binary lattices, and by

$$\widetilde{\mathcal{L}} := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z},\ a > 0,\ c > 0, \\ 4ac - b^2 > 0 \end{array} \right\} \Big/ SL_2(\mathbb{Z})$$

the set of isomorphism classes of even positive-definite *oriented* binary lattices.

For a singular $K3$ surface X, we denote by

$$[\widetilde{T}(X)] \in \widetilde{\mathcal{L}}$$

the isomorphism class of the oriented transcendental lattice $\widetilde{T}(X)$ of X.

Theorem (Shioda and Inose).
The map $X \mapsto [\widetilde{T}(X)]$ induces a bijection from the set of isomorphism classes of complex singular $K3$ surfaces to the set of isomorphism classes of even, positive-definite oriented binary lattices.

Shioda and Inose also gave an explicit construction of a singular $K3$ surface X with a given oriented transcendental lattice.

Suppose that

$$\widetilde{T} = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \quad \text{with} \quad d := b^2 - 4ac < 0$$

is given. We put

$$E' := \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z}), \qquad \text{where} \qquad \tau' = \frac{-b+\sqrt{d}}{2a}, \quad \text{and}$$

$$E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \ , \qquad \text{where} \qquad \tau = \frac{b+\sqrt{d}}{2},$$

and consider the abelian surface

$$A := E' \times E.$$

Theorem (Shioda and Mitani).
The oriented transcendental lattice $\widetilde{T}(A)$ of the abelian surface A is isomorphic to $\widetilde{T}$.

We then consider the Kummer surface

$$\mathrm{Km}(A).$$

Shioda and Inose showed that, on $\mathrm{Km}(A)$, there exist reduced effective divisors C and Θ such that

(1) $C = C_1 + \cdots + C_8$ and $\Theta = \Theta_1 + \cdots + \Theta_8$ are disjoint,
(2) C is an ADE-configuration of (-2)-curves of type $\mathbb{E}_8$,
(3) Θ is an ADE-configuration of (-2)-curves of type $8\mathbb{A}_1$,
(4) there exists a class $[\mathcal{L}] \in \mathrm{NS}(\mathrm{Km}(A))$ such that $2[\mathcal{L}] = [\Theta]$.

Let

$$\widetilde{Y} \to \mathrm{Km}(A)$$

be the double covering branched exactly along Θ, and let

$$Y \leftarrow \widetilde{Y}$$

be the contraction of the (-1)-curves on $\widetilde{Y}$ (that is, the inverse images of $\Theta_1, \dots, \Theta_8$).

Theorem (Shioda and Inose).
The surface Y is a singular $K3$ surface, and the diagram

$$Y \quad \longleftarrow \quad \widetilde{Y} \quad \longrightarrow \quad \mathrm{Km}(A) \quad \longleftarrow \quad \widetilde{A} \quad \longrightarrow \quad A$$

induces an isomorphism

$$\widetilde{T}(Y) \;\cong\; \widetilde{T}(A) \;(\cong\; \widetilde{T}\;)$$

of the oriented transcendental lattices.

Using this construction and the classical theory of complex multiplication in the class field theory, S.- and M. Schütt proved the following:

Theorem (S.- and M. Schütt).
Let $\mathcal{G} \subset \mathcal{L}$ be a genus of even positive-definite lattices of rank 2, and let

$$\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{L}}$$

be the pull-back of $\mathcal{G}$ by the natural projection $\widetilde{\mathcal{L}} \to \mathcal{L}$. Then there exists a singular $K3$ surface X defined over a number field F such that the set

$$\{ \ [\widetilde{T}(X^\sigma)] \ \mid \ \sigma \in \mathrm{Emb}(F, \mathbb{C}) \ \} \ \subset \ \widetilde{\mathcal{L}}$$

coincides with the oriented genus $\widetilde{\mathcal{G}}$, where $\mathrm{Emb}(F, \mathbb{C})$ denotes the set of embeddings of F into $\mathbb{C}$.

Corollary.
Let X and X' be singular $K3$ surfaces. If their transcendental lattices are in the same genus, then they are conjugate.

Corollary.

Consider two oriented lattices

$$\widetilde{T}_1 \in \widetilde{\mathcal{L}} \quad \text{and} \quad \widetilde{T}_2 \in \widetilde{\mathcal{L}}.$$

Suppose that their underlying (non-oriented) lattices are *not isomorphic* but in the same genus. Let X be a singular $K3$ surface such that $\widetilde{T}(X) \cong \widetilde{T}_1$, and let X^σ be a singular $K3$ surface conjugate to X such that $\widetilde{T}(X^\sigma) \cong \widetilde{T}_2$.

We choose a divisor D of X such that the classes of the irreducible components of D span $S_X \otimes \mathbb{Q}$. We put

$$U := X \setminus D,$$

and let $U^\sigma \subset X^\sigma$ be the Zariski open subset corresponding to U. Then U and U^σ are not homeomorphic.

§8. Arithmetic Zariski pairs of maximizing sextics

Definition.
A plane curve $C \subset \mathbb{P}^2$ of degree 6 is called a *maximizing sextic* if C has only simple singularities and the total Milnor number of C attains the possible maximum 19.

If C is a maximizing sextic, then the minimal resolution $X_C \to Y_C$ of the double covering $Y_C \to \mathbb{P}^2$ branching exactly along C is a singular $K3$ surface. We denote by $T[C]$ the transcendental lattice of X_C.

Let
$$R = \sum a_l A_l + \sum d_m D_m + \sum e_n E_n$$
be an ADE-type such that
$$\sum a_l l + \sum d_m m + \sum e_n n = 19.$$
Using the surjectivity of the period map for $K3$ surfaces, we can determine whether there exists a maximizing sextics C such that $\mathrm{Sing}(C)$ is of type R. This task was worked out by Yang (1996).

We can also determine all possible isomorphism classes of the transcendental lattice $T[C]$.

Using computer, we obtain the following examples of arithmetic Zariski pairs of maximizing sextics.

We put

$$\boldsymbol{L[2a, b, 2c] := \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.}$$

No.	the type of Sing(C)	$T[C]$ and	$T[C']$
1	$E_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
2	$E_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
3	$E_6 + D_5 + A_6 + A_2$	$L[12,0,42]$,	$L[6,0,84]$
4	$E_6 + A_{10} + A_3$	$L[12,0,22]$,	$L[4,0,66]$
5	$E_6 + A_{10} + A_2 + A_1$	$L[18,6,24]$,	$L[6,0,66]$
6	$E_6 + A_7 + A_4 + A_2$	$L[24,0,30]$,	$L[6,0,120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$L[30,0,42]$,	$L[18,6,72]$
8	$D_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
9	$D_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
10	$D_7 + A_{12}$	$L[6,2,18]$,	$L[2,0,52]$
11	$D_7 + A_8 + A_4$	$L[18,0,20]$,	$L[2,0,180]$
12	$D_5 + A_{10} + A_4$	$L[20,0,22]$,	$L[12,4,38]$
13	$D_5 + A_6 + A_5 + A_2 + A_1$	$L[12,0,42]$,	$L[6,0,84]$
14	$D_5 + A_6 + 2A_4$	$L[20,0,70]$,	$L[10,0,140]$
15	$A_{18} + A_1$	$L[8,2,10]$,	$L[2,0,38]$
16	$A_{16} + A_3$	$L[4,0,34]$,	$L[2,0,68]$
17	$A_{16} + A_2 + A_1$	$L[10,4,22]$,	$L[6,0,34]$
18	$A_{13} + A_4 + 2A_1$	$L[8,2,18]$,	$L[2,0,70]$
19	$A_{12} + A_6 + A_1$	$L[8,2,46]$,	$L[2,0,182]$
20	$A_{12} + A_5 + 2A_1$	$L[12,6,16]$,	$L[4,2,40]$
21	$A_{12} + A_4 + A_2 + A_1$	$L[24,6,34]$,	$L[6,0,130]$
22	$A_{10} + A_9$	$L[10,0,22]$,	$L[2,0,110]$
23	$A_{10} + A_9$	$L[8,3,8]$,	$L[2,1,28]$
24	$A_{10} + A_8 + A_1$	$L[18,0,22]$,	$L[10,2,40]$
25	$A_{10} + A_7 + A_2$	$L[22,0,24]$,	$L[6,0,88]$
26	$A_{10} + A_7 + 2A_1$	$L[10,2,18]$,	$L[2,0,88]$
27	$A_{10} + A_6 + A_2 + A_1$	$L[22,0,42]$,	$L[16,2,58]$
28	$A_{10} + A_5 + A_3 + A_1$	$L[12,0,22]$,	$L[4,0,66]$
29	$A_{10} + 2A_4 + A_1$	$L[30,10,40]$,	$L[10,0,110]$
30	$A_{10} + A_4 + 2A_2 + A_1$	$L[30,0,66]$,	$L[6,0,330]$
31	$A_8 + A_6 + A_4 + A_1$	$L[22,4,58]$,	$L[18,0,70]$
32	$A_7 + A_6 + A_4 + A_2$	$L[24,0,70]$,	$L[6,0,280]$
33	$A_7 + A_6 + A_4 + 2A_1$	$L[18,4,32]$,	$L[2,0,280]$
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$L[24,0,30]$,	$L[6,0,120]$