Non-homeomorphic conjugate complex varieties

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- **We work over the complex number field $\mathbb{C}$.**
- **The coefficients of the (co-)homology groups are in $\mathbb{Z}$.**
- **By a lattice, we mean a finitely generated free $\mathbb{Z}$-module Λ equipped with a non-degenerate symmetric bilinear form**

$$\Lambda \times \Lambda \quad \to \quad \mathbb{Z}.$$

§1. Conjugate varieties

An affine algebraic variety $X \subset \mathbb{C}^N$ is defined by a finite number of polynomial equations:

$$X \quad : \quad f_1(x_1, \ldots, x_N) = \cdots = f_m(x_1, \ldots, x_N) = 0.$$

Let $c_{j,I} \in \mathbb{C}$ be the coefficients of the polynomial f_j:

$$f_j(x_1, \ldots, x_N) = \sum_I c_{j,I} x^I, \quad \text{where} \quad x^I = x_1^{i_1} \cdots x_N^{i_N}.$$

We then denote by

$$F_X := \mathbb{Q}(\ldots, c_{j,I}, \ldots) \quad \subset \quad \mathbb{C}$$

the minimal sub-field of $\mathbb{C}$ containing all the coefficients of the defining equations of X.

There are many other embeddings

$$\sigma \quad : \quad F_X \quad \hookrightarrow \quad \mathbb{C}$$

of the field F_X into $\mathbb{C}$.

Example.
(1) If $F_X = \mathbb{Q}(\sqrt{2}, t)$, where $t \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, then the set of embeddings $F_X \hookrightarrow \mathbb{C}$ is equal to

$$\{\sqrt{2}, -\sqrt{2}\} \times \{ \text{ transcendental complex numbers } \}.$$

(2) If all $c_{j,I}$ are algebraic over $\mathbb{Q}$, then the set of embeddings is finite, and the Galois group of the Galois closure of the algebraic extension $F_X/\mathbb{Q}$ acts on the set transitively.

For an embedding $\sigma : F_X \hookrightarrow \mathbb{C}$, we put

$$f_j^\sigma(x_1, \dots, x_N) := \sum_I c_{j,I}^\sigma x^I,$$

and denote by $X^\sigma \subset \mathbb{C}^N$ the affine algebraic variety defined by

$$f_1^\sigma = \cdots = f_m^\sigma = 0.$$

We can define X^σ for a projective or quasi-projective variety $X \subset \mathbb{P}^N$ in the same way.

Definition.
We say that two algebraic varieties X and Y are said to be *conjugate* if there exists an embedding $\sigma : F_X \hookrightarrow \mathbb{C}$ such that Y is isomorphic to X^σ.

In the language of schemes, two varieties X and Y over $\operatorname{Spec}\mathbb{C}$ are conjugate if there exists a diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\sigma^*} & \operatorname{Spec}\mathbb{C}. \end{array}$$

of the fiber product for some morphism $\sigma^* : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$.

It is obvious that being conjugate is an equivalence relation.

$$\boldsymbol{y^2 = x^3 + 6\sqrt{2}x + \sqrt{2}.}$$

$$\boldsymbol{y^2 = x^3 - 6\sqrt{2}x - \sqrt{2}.}$$

§2. Topology of conjugate varieties

Conjugate algebraic varieties cannot be distinguished by any algebraic methods. In particular, they are homeomorphic in Zariski topology.

How about their complex topology?

The following is due to Serre, Grothendieck, Artin, ...

Theorem.
Let X and Y be conjugate non-singular projective varieties.
(1) They have the same betti numbers:

$$B_i(X) = B_i(Y) \quad \text{for} \quad i = 0, \dots, 2 \dim X.$$

(2) The profinite completions of their fundamental groups are isomorphic: $\pi_1^\wedge(X) \cong \pi_1^\wedge(Y)$.

The following example is due to Serre (1964).

Example.
There exist conjugate non-singular projective varieties X and Y such that their fundamental groups are *not* isomorphic: $\pi_1(X) \not\cong \pi_1(Y)$.

Other examples of non-homeomorphic conjugate varieties:

- **Abelson (1974).**
- **E. Artal, J. Carmona, and J.-I. Cogolludo. (2003-).**
- **Bauer, Catanese, Grunewald. (2005-).**

$$\vdots$$

Grothendieck's "dessins d'enfant".

Let $f : C \to \mathbb{P}^1$ be a finite covering of a projective line branching only at the three points $0, 1, \infty \in \mathbb{P}^1$. We have defining equations of f with coefficients in $\overline{\mathbb{Q}} \subset \mathbb{C}$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consider the conjugate covering

$$f^\sigma : C^\sigma \to \mathbb{P}^1.$$

Then f and f^σ have, in general, different topology.

§3. Main result

We introduce a new topological invariant of *open* algebraic varieties, which allows us to distinguish conjugate varieties topologically in some cases.

Combining this topological invariant with the following results, we obtain several explicit examples of non-homeomorphic conjugate varieties.

- Arithmetic theory of abelian and $K3$ surfaces due to S.- and Schütt.
- Degtyarev's theorem on the connected components of plane curves of degree 6 with only simple singularities.
- Artal, Carmona and Cogolludo's calculation of defining equations of plane curves of degree 6 with prescribed simple singularities.

Our examples consist of the following:

- Zariski open subsets of abelian surfaces.
- Zariski open subsets of $K3$ surfaces.
- Singular plane curves C of degree 6 with only simple singularities and of Milnor number 19. (In this example, the homeomorphism types of the pairs $(\mathbb{P}^2, C)$ are distinct.)

Example.

We consider the following cubic extension of $\mathbb{Q}$:

$$K := \mathbb{Q}[t]/(\varphi), \quad \text{where} \quad \varphi = 17t^3 - 18t^2 - 228t + 556.$$

The roots of $\varphi = 0$ are $\alpha, \bar{\alpha}, \beta$, where

$$\alpha = 2.590\cdots + 1.108\cdots\sqrt{-1}, \qquad \beta = -4.121\cdots .$$

There are three corresponding embeddings

$$\sigma_\alpha : K \hookrightarrow \mathbb{C}, \quad \sigma_{\bar{\alpha}} : K \hookrightarrow \mathbb{C} \quad \text{and} \quad \sigma_\beta : K \hookrightarrow \mathbb{C}.$$

There exists a homogeneous polynomial

$$\Phi(x_0, x_1, x_2) \in K[x_0, x_1, x_2]$$

of degree 6 with coefficients in K with the following properties. We consider the conjugate plane curves

$$C_\alpha = \{\Phi^{\sigma_\alpha} = 0\} \quad \text{and} \quad C_\beta = \{\Phi^{\sigma_\beta} = 0\}.$$

Then each of them has three simple singular points of type

$$A_{16} + A_2 + A_1$$

as its only singularities. In particular, there exist tubular neighborhoods $T_\alpha \subset \mathbb{P}^2$ of $C_\alpha \subset \mathbb{P}^2$ and $T_\beta \subset \mathbb{P}^2$ of $C_\beta \subset \mathbb{P}^2$ such that (T_α, C_α) is diffeomorphic to (T_β, C_β).

However, there are no homeomorphisms between the pairs $(\mathbb{P}^2, C_\alpha)$ and $(\mathbb{P}^2, C_\beta)$.

Namely, C_α and C_β form an *arithmetic Zariski pair.*

Let $X \to \mathbb{P}^2$ be the double covering of the plane branching exactly along the curve $C : \Phi = 0$, and $U \subset X$ the pull-back of $\mathbb{P}^2 \setminus C$. Then U is a variety defined over K. Consider the conjugate open varieties U^α and U^β corresponding the embeddings σ_α and σ_β. Then U^α and U^β are not homeomorphic.

§4. The topological invariant

Let U be an oriented topological manifold of dimension $4n$. Let

$$\iota_U : H_{2n}(U) \times H_{2n}(U) \to \mathbb{Z}$$

be the intersection pairing.

Definition.
We put

$$J_\infty(U) := \bigcap_K \operatorname{Im}(H_{2n}(U \setminus K) \to H_{2n}(U)),$$

where K runs through the set of all compact subsets of U. We then put

$$\widetilde{B}_U := H_{2n}(U)/J_\infty(U) \quad \text{and} \quad B_U := (\widetilde{B}_U)/\text{torsion}.$$

Since any topological cycle is compact, the intersection pairing ι_U induces a symmetric bilinear form

$$\beta_U : B_U \times B_U \to \mathbb{Z}.$$

It is obvious that, if U and U' are homeomorphic, then there exists an isomorphism

$$(B_U, \beta_U) \cong (B_{U'}, \beta_{U'}),$$

and hence the isomorphism class of (B_U, β_U) is a topological invariant of U.

We study the invariant (B_U, β_U) for the space

$$U := X \setminus Y,$$

where X is a non-singular projective variety of complex dimension $2n$, and Y is a union of irreducible (possibly singular) subvarieties $Y_1 \ldots, Y_N$ of complex dimension n:

$$Y = Y_1 \cup \cdots \cup Y_N.$$

We denote by

$$\widetilde{\Sigma}_{(X,Y)} := \langle [Y_1], \ldots, [Y_N] \rangle \quad \subset \quad H_{2n}(X)$$

the submodule of $H_{2n}(X)$ generated by the homology classes $[Y_i] \in H_{2n}(X)$, and put

$$\Sigma_{(X,Y)} := (\widetilde{\Sigma}_{(X,Y)})/\text{torsion}.$$

We then put

$$\begin{aligned} \widetilde{\Lambda}_{(X,Y)} &:= \{x \in H_{2n}(X) \mid \iota_X(x, y) = 0 \text{ for any } \ y \in \widetilde{\Sigma}_{(X,Y)}\}, \\ \Lambda_{(X,Y)} &:= (\widetilde{\Lambda}_{(X,Y)})/\text{torsion}. \end{aligned}$$

Finally, we denote by

$$\begin{aligned} \sigma_{(X,Y)} &: \Sigma_{(X,Y)} \times \Sigma_{(X,Y)} \to \mathbb{Z} \quad \text{and} \\ \lambda_{(X,Y)} &: \Lambda_{(X,Y)} \times \Lambda_{(X,Y)} \to \mathbb{Z} \end{aligned}$$

the symmetric bilinear forms induced from the intersection pairing

$$\iota_X : H_{2n}(X) \times H_{2n}(X) \to \mathbb{Z}.$$

Theorem.
Let X, Y and U be as above. Suppose that $\sigma_{(X,Y)}$ is non-degenerate. Then (B_U, β_U) is isomorphic to $(\Lambda_{(X,Y)}, \lambda_{(X,Y)})$.

Sketch of the proof.

Since X is non-singular and complete, the intersection pairing ι_X on $H_{2n}(X)$/torsion is non-degenerate. Hence the assumption that $\sigma_{(X,Y)}$ is non-degenerate implies that $\lambda_{(X,Y)}$ is non-degenerate.

We consider the homomorphism

$$j_U \ : \ H_{2n}(U) \to H_{2n}(X)$$

induced by the inclusion. It is obvious that the image of j_U is contained in $\widetilde{\Lambda}_{(X,Y)}$. We first show that

$$\mathrm{Im}(j_U) = \widetilde{\Lambda}_{(X,Y)}.$$

Let $[W] \in \widetilde{\Lambda}_{(X,Y)}$ be represented by a real $2n$-dimensional topological cycle W. We can assume that $W \cap Y$ consists of a finite number of points in $Y \setminus \mathrm{Sing}(Y)$, and that the intersection of W with Y is transverse at each intersection point.

Let $P_{i,1}, \ldots, P_{i,k(i)}$ (resp. $Q_{i,1}, \ldots, Q_{i,l(i)}$) be the intersection points of W and Y_i with local intersection number 1 (resp. -1). Since $\iota_X([W], [Y_i]) = 0$, we have

$$k(i) = l(i).$$

Modifying W by adding the tube

$$\partial(D^{2n} \times I)$$

for each pair $(P_{i,j}, Q_{i,j})$, we obtain a topological cycle W' that is homologous to W in X and is disjoint from Y. Hence $[W] = [W']$ is represented by $W' \subset U$. Thus

$$\mathrm{Im}(j_U) = \widetilde{\Lambda}_{(X,Y)}$$

holds.

Figure

Using Mayer-Vietris sequence, we can prove

$$\mathrm{Ker}(j_U) \subseteq J_\infty(U)$$

from the assumption that $\lambda_{(X,Y)}$ is non-degenerate. By the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(j_U) & \longrightarrow & H_{2n}(U) & \xrightarrow{j_U} & \widetilde{\Lambda}_{(X,Y)} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow \tilde{v} & & \\ 0 & \longrightarrow & J_\infty(U) & \longrightarrow & H_{2n}(U) & \longrightarrow & \widetilde{B}_U & \longrightarrow & 0 \ , \end{array}$$

we obtain the isomorphism $(\Lambda_{(X,Y)}, \lambda_{(X,Y)}) \cong (B_U, \beta_U)$.

§5. Transcendental lattices

Let X be a non-singular projective variety of dimension $2n$. Then we have a natural isomorphism

$$H_{2n}(X)/\text{torsion} \cong H^{2n}(X)/\text{torsion}$$

that transforms ι_X to the cup-product $(\ ,\ )_X$. Let

$$S_X \subset H^{2n}(X)/\text{torsion}$$

be the submodule generated by the classes $[Z]$ of irreducible subvarieties Z of X with codimension n; that is, S_X is the space of *algebraic cycles* in the middle dimension. We then denote by

$$s_X : S_X \times S_X \to \mathbb{Z}$$

the restriction of $(\ ,\ )_X$ to S_X. We consider the following condition:

(N) The symmetric bilinear form s_X is non-degenerate.

Remark.
The condition (N) is satisfied for X if the Hodge conjecture

$$S_X \otimes \mathbb{Q} = H^{2n}(X, \mathbb{Q}) \cap H^{n,n}(X)$$

is true for the middle cohomology group of X. In particular, the condition (N) is satisfied if $\dim X = 2$.

Proposition.
Let X and X^σ be conjugate non-singular projective varieties. Suppose that (N) holds for both of X and X^σ. Then the map $[Z] \mapsto [Z^\sigma]$ induces an isomorphism $(S_X, s_X) \cong (S_{X^\sigma}, s_{X^\sigma})$.

Definition.
When (N) holds for X, we define the *transcendental lattice* T_X of X to be the free $\mathbb{Z}$-module

$$T_X := \{x \in H^{2n}(X)/\text{torsion} \mid (x, y)_X = 0 \text{ for any } y \in S_X\}.$$

Theorem.
Let X be a non-singular projective variety of dimension $2n$. Suppose that (N) holds for X. Let $Y_1, \ldots, Y_N$ be irreducible subvarieties of X with codimension n whose classes span $S_X \otimes \mathbb{Q}$ over $\mathbb{Q}$. We put

$$Y := \bigcup_{i=1}^{N} Y_i \quad \text{and} \quad U := X \setminus Y.$$

Then the transcendental lattice T_X of X is isomorphic to the topological invariant (B_U, β_U) of U.

Corollary.
Let X and X^σ be conjugate non-singular projective varieties of dimension $2n$. Suppose that (N) holds for both of X and X^σ. Let $Y \subset X$ and $U \subset X$ be as above. If T_{X^σ} is not isomorphic to T_X, then $U^\sigma = X^\sigma \setminus Y^\sigma$ is not homeomorphic to U.

§6. Genus theory of lattices

Definition.
Two lattices

$$\lambda : \Lambda \times \Lambda \to \mathbb{Z} \quad \text{and} \quad \lambda' : \Lambda' \times \Lambda' \to \mathbb{Z}$$

are said to be *in the same genus* if

$$\lambda \otimes \mathbb{Z}_p \;:\; \Lambda \otimes \mathbb{Z}_p \times \Lambda \otimes \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{and}$$
$$\lambda' \otimes \mathbb{Z}_p \;:\; \Lambda' \otimes \mathbb{Z}_p \times \Lambda' \otimes \mathbb{Z}_p \to \mathbb{Z}_p$$

are isomorphic for any p including $p = \infty$, where $\mathbb{Z}_\infty = \mathbb{R}$.

Let X be a non-singular projective variety of dimension $2n$. Recall that S_X is the submodule of $H^{2n}(X)/\text{torsion}$ generated by algebraic cycles. We consider the following condition:

(P) The submodule S_X is primitive in $H^{2n}(X)/\text{torsion}$; that is, the quotient $(H^{2n}(X)/\text{torsion})/S_X$ is torsion-free.

Remark.
The condition (P) is satisfied for X if the *integral* Hodge conjecture

$$S_X = H^{2n}(X, \mathbb{Z}) \cap H^{n,n}(X)$$

is true for X. In particular, the condition (P) is satisfied if $\dim X = 2$. There exists, however, a counter-example for (P) in higher-dimension. (Atiyah-Hirzebruch (1962).)

Theorem.
Let X and X^σ be conjugate non-singular projective varieties of dimension $2n$. Suppose that (N) and (P) hold for both of X and X^σ. Then the transcendental lattices T_X and T_{X^σ} are contained in the same genus.

Let X be a surface. Then T_X and T_{X^σ} are contained in the same genus. Let $Y_1, \dots, Y_N$ be irreducible curves of X whose classes span $S_X \otimes \mathbb{Q}$. We put

$$Y := \bigcup_{i=1}^{N} Y_i \quad \text{and} \quad U := X \setminus Y.$$

If T_X and T_{X^σ} are not isomorphic, then U and U^σ are not homeomorphic.

Therefore we will search for lattices that are not isomorphic but in the same genus.

Gauss gave a complete description of isomorphism classes of lattices of rank 2 (binary lattices) and their decomposition into genera.

Example.
Two binary lattices

$$\begin{bmatrix} 6 & 2 \\ 2 & 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 22 \end{bmatrix}$$

are not isomorphic, but in the same genus.

§7. Singular abelian surfaces and singular $K3$ surfaces

Let A be an abelian surface; that is, a complex torus of dimension 2 that can be embedded into a projective space. Then $H^2(A)$ is a unimodular lattice of rank 6 with signature $(3,3)$.

Definition.
An abelian surface A is said to be *singular* if the rank of the transcendental lattice T_A is 2 (the possible minimum).

The transcendental lattice T_A of a singular abelian surface A is positive-definite. Moreover, by the Hodge decomposition

$$T_A \otimes \mathbb{Z} \cong H^{2,0}(A) \oplus H^{0,2}(A),$$

this lattice has a canonical orientation. We denote by $\widetilde{T}_A$ the oriented transcendental lattice of A.

Definition.
We denote by

$$\mathcal{L} := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z}, a > 0, c > 0, \\ 4ac - b^2 > 0 \end{array} \right\} / \, SL_2(\mathbb{Z})$$

the set of isomorphism classes of even positive-definite oriented binary lattices. For a singular abelian surface A, we denote by

$$[\widetilde{T}_A] \in \mathcal{L}$$

the class of the oriented transcendental lattice of A.

The following theorem is due to Shioda and Mitani (1974):

Theorem.
The map $A \mapsto [\widetilde{T}_A]$ induces a bijection from the set of isomorphism classes of singular abelian surfaces A to the set $\mathcal{L}$.

Shioda and Mitani have also given a method of explicit construction of a singular abelian surface with a prescribed oriented transcendental lattice.

Theorem.
Let

$$M := \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$$

be a matrix representing an element of $\mathcal{L}$. We put

$$D := b^2 - 4ac \quad < \quad 0.$$

Consider the elliptic curves

$$E_1 \ := \ \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_1), \qquad \text{where} \quad \tau_1 := (b + \sqrt{D})/2, \quad \text{and}$$
$$E_2 \ := \ \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_2), \qquad \text{where} \quad \tau_2 := (-b + \sqrt{D})/(2a).$$

Then $A := E_1 \times E_2$ is a singular abelian surface such that $[\widetilde{T}_A]$ is equal to $[M]$.

Note that the elliptic curves E_1 and E_2 have complex multiplications, and hence each of them is defined over a certain number field (a class field of $\mathbb{Q}(\sqrt{D})$).

Using the classical class field theory, M. Schütt and S.- proved the following:

Theorem.
Consider two oriented lattices $\widetilde{T}_1 \in \mathcal{L}$ and $\widetilde{T}_2 \in \mathcal{L}$. Suppose that their underlying (non-oriented) lattices T_1 and T_2 are in the same genus. Then the corresponding singular abelian surfaces A_1 and A_2 are conjugate.

Combining all the results so far, we obtain the following:

Corollary.
Consider two oriented lattices $\widetilde{T}_1 \in \mathcal{L}$ and $\widetilde{T}_2 \in \mathcal{L}$. Suppose that their underlying (non-oriented) lattices are not isomorphic but in the same genus. Let A be a singular abelian surface such that $\widetilde{T}_A \cong \widetilde{T}_1$. We choose a divisor D of A such that the classes of the irreducible components of D span $S_A \otimes \mathbb{Q}$. We put

$$U := A \setminus D.$$

Let A^σ be a singular abelian surface conjugate to A such that $\widetilde{T}_{A^\sigma} \cong \widetilde{T}_2$, and let U^σ be the Zariski open subset of A^σ corresponding to U. Then U and U^σ are not homeomorphic.

Let X be a $K3$ surface; that is, a simply-connected surface with $K_X \cong \mathcal{O}_X$. Then $H^2(X)$ is a unimodular lattice of rank 22 with signature $(3, 19)$.

Definition.
A $K3$ surface X is said to be *singular* if the rank of the transcendental lattice T_X is 2 (the possible minimum).

We have the same theory for singular $K3$ surfaces as for the singular abelian surfaces by Shioda-Inose (1977), and the same theorem by S.- and Schütt.

Corollary.
If there exist two even positive-definite lattices T_1 and T_2 of rank 2 that are not isomorphic but in the same genus, then there exist non-homeomorphic conjugate varieties U_1 and U_2, where U_i is a Zariski open subset of a singular $K3$ surface X_i with the transcendental lattice isomorphic to T_i.

§8. Arithmetic Zariski pairs

We apply this corollary to the construction of examples of arithmetic Zariski pairs of maximizing sextics.

Definition.
A pair $[C, C']$ of plane curves is said to be an *arithmetic Zariski pair* if the following hold:
(i) Suppose that $C = \{\Phi = 0\}$. Then there exists an embedding $\sigma : F_C \hookrightarrow \mathbb{C}$ such that C' is isomorphic (as a plane curve) to $C^\sigma := \{\Phi^\sigma = 0\}$.
(ii) There exist tubular neighborhoods $T \subset \mathbb{P}^2$ of C and $T' \subset \mathbb{P}^2$ of C' such that (T, C) and (T', C') are diffeomorphic.
(iii) $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are *not* homeomorphic.

Definition.
A plane curve C of degree 6 is called a *maximizing sextic* if C has only simple singularities and the total Milnor number of C attains the possible maximum 19.

Remark.
If C is a maximizing sextic, the minimal resolution $X_C \to Y_C$ of the double covering $Y_C \to \mathbb{P}^2$ branching exactly along C is a singular $K3$ surface. We denote by $T[C]$ the transcendental lattice of X_C.

Remark.
If C is a maximizing sextic, then its conjugate C^σ is also a maximizing sextic and and $[C, C^\sigma]$ satisfies the condition (ii) in the definition of arithmetic Zariski pairs, because simple singularities have no moduli.

We obtain the following examples of arithmetic Zariski pairs of maximizing sextics.

We put

$$L[2a, b, 2c] := \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

No.	the type of Sing(C)	$T[C]$ and	$T[C']$
1	$E_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
2	$E_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
3	$E_6 + D_5 + A_6 + A_2$	$L[12,0,42]$,	$L[6,0,84]$
4	$E_6 + A_{10} + A_3$	$L[12,0,22]$,	$L[4,0,66]$
5	$E_6 + A_{10} + A_2 + A_1$	$L[18,6,24]$,	$L[6,0,66]$
6	$E_6 + A_7 + A_4 + A_2$	$L[24,0,30]$,	$L[6,0,120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$L[30,0,42]$,	$L[18,6,72]$
8	$D_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
9	$D_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
10	$D_7 + A_{12}$	$L[6,2,18]$,	$L[2,0,52]$
11	$D_7 + A_8 + A_4$	$L[18,0,20]$,	$L[2,0,180]$
12	$D_5 + A_{10} + A_4$	$L[20,0,22]$,	$L[12,4,38]$
13	$D_5 + A_6 + A_5 + A_2 + A_1$	$L[12,0,42]$,	$L[6,0,84]$
14	$D_5 + A_6 + 2A_4$	$L[20,0,70]$,	$L[10,0,140]$
15	$A_{18} + A_1$	$L[8,2,10]$,	$L[2,0,38]$
16	$A_{16} + A_3$	$L[4,0,34]$,	$L[2,0,68]$
17	$A_{16} + A_2 + A_1$	$L[10,4,22]$,	$L[6,0,34]$
18	$A_{13} + A_4 + 2A_1$	$L[8,2,18]$,	$L[2,0,70]$
19	$A_{12} + A_6 + A_1$	$L[8,2,46]$,	$L[2,0,182]$
20	$A_{12} + A_5 + 2A_1$	$L[12,6,16]$,	$L[4,2,40]$
21	$A_{12} + A_4 + A_2 + A_1$	$L[24,6,34]$,	$L[6,0,130]$
22	$A_{10} + A_9$	$L[10,0,22]$,	$L[2,0,110]$
23	$A_{10} + A_9$	$L[8,3,8]$,	$L[2,1,28]$
24	$A_{10} + A_8 + A_1$	$L[18,0,22]$,	$L[10,2,40]$
25	$A_{10} + A_7 + A_2$	$L[22,0,24]$,	$L[6,0,88]$
26	$A_{10} + A_7 + 2A_1$	$L[10,2,18]$,	$L[2,0,88]$
27	$A_{10} + A_6 + A_2 + A_1$	$L[22,0,42]$,	$L[16,2,58]$
28	$A_{10} + A_5 + A_3 + A_1$	$L[12,0,22]$,	$L[4,0,66]$
29	$A_{10} + 2A_4 + A_1$	$L[30,10,40]$,	$L[10,0,110]$
30	$A_{10} + A_4 + 2A_2 + A_1$	$L[30,0,66]$,	$L[6,0,330]$
31	$A_8 + A_6 + A_4 + A_1$	$L[22,4,58]$,	$L[18,0,70]$
32	$A_7 + A_6 + A_4 + A_2$	$L[24,0,70]$,	$L[6,0,280]$
33	$A_7 + A_6 + A_4 + 2A_1$	$L[18,4,32]$,	$L[2,0,280]$
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$L[24,0,30]$,	$L[6,0,120]$