Transcendental lattices and supersingular reduction lattices of a singular K3 surface

Keio, 2007 September

Ichiro Shimada
(Hokkaido University, Sapporo, JAPAN)

- By a lattice, we mean a finitely generated free $\mathbb{Z}$-module Λ equipped with a non-degenerate symmetric bilinear form
$$\Lambda \times \Lambda \quad \to \quad \mathbb{Z}.$$
- A lattice Λ is said to be *even* if $(v, v) \in 2\mathbb{Z}$ for any $v \in \Lambda$.
- Let Λ and Λ' be lattices. A homomorphism $\Lambda \to \Lambda'$ of $\mathbb{Z}$-modules is called an *isometry* if it preserves the symmetric bilinear forms. By definition, an isometry is injective.
- Let $\Lambda \hookrightarrow \Lambda'$ be an isometry. We denote by
$$(\Lambda \hookrightarrow \Lambda')^{\perp}$$
the *orthogonal complement* of Λ in Λ'.

§1. (Super)singular $K3$ surfaces

For a $K3$ surface X defined over a field k, we denote by $\mathrm{NS}(X)$ the Néron-Severi lattice of

$$X \otimes \bar{k},$$

where $\bar{k}$ is the algebraic closure of k; that is, $\mathrm{NS}(X)$ is the lattice of numerical equivalence classes of divisors on $X \otimes \bar{k}$ with the intersection pairing.

Definition.
A $K3$ surface X defined over a field of characteristic 0 is said to be *singular* if

$$\mathrm{rank}(\mathrm{NS}(X)) = 20.$$

A $K3$ surface X defined over a field of characteristic $p > 0$ is said to be *supersingular* if

$$\mathrm{rank}(\mathrm{NS}(X)) = 22.$$

If X is singular or supersingular, then

$$d(X) := \mathrm{disc}(\mathrm{NS}(X)).$$

is a negative integer.

Shioda and Inose showed that every singular $K3$ surface is defined over a number field.

Let X be a singular $K3$ surface defined over a number field F.

We denote by $\mathbb{Z}_F$ the integer ring of F, and by

$$\pi_F : \operatorname{Spec} \mathbb{Z}_F \to \operatorname{Spec} \mathbb{Z}$$

the natural projection. We also denote by

$$\operatorname{Emb}(F, \mathbb{C})$$

the set of embeddings of F into $\mathbb{C}$.

We consider a smooth family

$$\mathcal{X} \to U$$

of $K3$ surfaces over a non-empty Zariski open subset U of $\operatorname{Spec} \mathbb{Z}_F$ such that

$$\boxed{\text{the generic fiber } X_\eta \text{ is isomorphic to } X.}$$

For a close point $\mathfrak{p}$ of U, we denote by $X_\mathfrak{p}$ the reduction of $\mathcal{X}$ at $\mathfrak{p}$. For a prime integer p, we put

$$\mathcal{S}_p(\mathcal{X}) := \{\, \mathfrak{p} \in \pi_F^{-1}(p) \cap U \;\mid\; X_\mathfrak{p} \text{ is supersingular} \,\}.$$

We investigate the following lattices of rank 2;

- the transcendental lattice
$$T(X^\sigma) := (\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X^\sigma, \mathbb{Z}))^\perp$$
for each $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, where X^σ is the complex $K3$ surface $X \otimes_{F,\sigma} \mathbb{C}$, and
- the supersingular reduction lattice
$$L(\mathcal{X}, \mathfrak{p}) := (\mathrm{NS}(X) \hookrightarrow \mathrm{NS}(X_\mathfrak{p}))^\perp$$
for each $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$, where $\mathrm{NS}(X) \hookrightarrow \mathrm{NS}(X_\mathfrak{p})$ is the specialization isometry.

Remark.
The supersingular reduction lattices and their relation with transcendental lattices was first considered in the paper

T. Shioda: The elliptic $K3$ surfaces with a maximal singular fibre. C. R. Math. Acad. Sci. Paris 337 (2003), 461–466,

for certain elliptic $K3$ surfaces.

§2. Genera of lattices

Definition.
Two lattices

$$\lambda : \Lambda \times \Lambda \to \mathbb{Z} \quad \text{and} \quad \lambda' : \Lambda' \times \Lambda' \to \mathbb{Z}$$

are said to be *in the same genus* if

$$\lambda \otimes \mathbb{Z}_p \ : \ \Lambda \otimes \mathbb{Z}_p \times \Lambda \otimes \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{and}$$
$$\lambda' \otimes \mathbb{Z}_p \ : \ \Lambda' \otimes \mathbb{Z}_p \times \Lambda' \otimes \mathbb{Z}_p \to \mathbb{Z}_p$$

are isomorphic for any p including $p = \infty$, where $\mathbb{Z}_\infty = \mathbb{R}$.

We have the following:

Theorem (Nikulin).
Two even lattices of the same rank are in the same genus if and only if they have the same signature and their discriminant forms are isomorphic.

Definition.

Let Λ be an even lattice. Then Λ is canonically embedded into

$$\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z})$$

as a subgroup of finite index, and we have a natural symmetric bilinear form

$$\Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}$$

that extends the symmetric bilinear form on Λ. The finite abelian group

$$D_\Lambda := \Lambda^\vee / \Lambda,$$

together with the natural quadratic form

$$q_\Lambda : D_\Lambda \to \mathbb{Q}/2\mathbb{Z}$$

is called the *discriminant form of* Λ.

§3. Transcendental lattices

Let X be a singular $K3$ surface defined over a number field F. For an embedding $\sigma : F \hookrightarrow \mathbb{C}$, the transcendental lattice

$$T(X^\sigma) := (\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X^\sigma, \mathbb{Z}))^\perp$$

of the complex singular $K3$ surface

$$X^\sigma := X \otimes_{F,\sigma} \mathbb{C}$$

is an even positive-definite lattice of rank 2.

Proposition.
For $\sigma, \sigma' \in \mathrm{Emb}(F, \mathbb{C})$, the lattices $T(X^\sigma)$ and $T(X^{\sigma'})$ are in the same genus.

This follows from Nikulin's theorem. We have

$$\mathrm{NS}(X) \;\cong\; \mathrm{NS}(X^\sigma) \;\cong\; \mathrm{NS}(X^{\sigma'}).$$

Since $\mathrm{H}^2(X^\sigma, \mathbb{Z})$ is unimodular, the discriminant form of $T(X^\sigma)$ is isomorphic to (-1) times the discriminant form of $\mathrm{NS}(X^\sigma)$:

$$(D_{T(X^\sigma)}, q_{T(X^\sigma)}) \;\cong\; (D_{\mathrm{NS}(X^\sigma)}, -q_{\mathrm{NS}(X^\sigma)}).$$

The same holds for $T(X^{\sigma'})$. Hence $T(X^\sigma)$ and $T(X^{\sigma'})$ have the isomorphic discriminant forms.

For a negative integer d, we put

$$\mathcal{M}_d := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z},\ a > 0,\ c > 0, \\ b^2 - 4ac = d \end{array} \right\},$$

on which $GL_2(\mathbb{Z})$ acts by $M \mapsto {}^t g M g$, where $M \in \mathcal{M}_d$ and $g \in GL_2(\mathbb{Z})$.

We then denote by

$$\mathcal{L}_d := \mathcal{M}_d /\, GL_2(\mathbb{Z}) \quad (\text{resp.} \quad \widetilde{\mathcal{L}}_d := \mathcal{M}_d /\, SL_2(\mathbb{Z}) \ )$$

the set of isomorphism classes of even, positive-definite lattices (resp. *oriented* lattices) of rank 2 with discriminant $-d$.

Let S be a *complex* singular $K3$ surface. By the Hodge decomposition

$$T(S) \otimes \mathbb{C} = \mathrm{H}^{2,0}(S) \oplus \mathrm{H}^{0,2}(S),$$

we can define a canonical orientation on $T(S)$. We denote by

$$\widetilde{T}(S)$$

the *oriented* transcendental lattice of S, and by $[\widetilde{T}(S)] \in \widetilde{\mathcal{L}}_{d(S)}$ the isomorphism class of the oriented transcendental lattice.

Theorem (Shioda and Inose).
The map $S \mapsto [\widetilde{T}(S)]$ induces a bijection from the set of isomorphism classes of complex singular $K3$ surfaces to the set

$$\bigcup_d \widetilde{\mathcal{L}}_d$$

of isomorphism classes of even, positive-definite oriented lattices of rank 2.

We have proved the following existence theorem:

Theorem (S.- and Schütt).
Let $\mathcal{G} \subset \mathcal{L}_d$ be a genus of even positive-definite lattices of rank 2, and let

$$\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{L}}_d$$

be the pull-back of $\mathcal{G}$ by the natural projection $\widetilde{\mathcal{L}}_d \to \mathcal{L}_d$. Then there exists a singular $K3$ surface X defined over a number field F such that the set

$$\{\ [\widetilde{T}(X^\sigma)]\ \mid\ \sigma \in \mathrm{Emb}(F, \mathbb{C})\ \}\ \subset\ \widetilde{\mathcal{L}}_d$$

coincides with the oriented genus $\widetilde{\mathcal{G}}$.

Corollary.
Let S and S' be complex singular $K3$ surfaces. If $T(S)$ and $T(S')$ are in the same genus, then there exists an embedding $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ of the field $\mathbb{C}$ into itself such that $S \times_{\mathbb{C},\sigma} \mathbb{C}$ is isomorphic to S' as a complex surface.

Proof. Let $\widetilde{\mathcal{G}}_S \subset \widetilde{\mathcal{L}}_{d(S)}$ be the oriented genus containing $[\widetilde{T}(S)] \in \widetilde{\mathcal{L}}_{d(S)}$, and let X be the singular $K3$ surface defined over a number field F such that

$$\{ \, [\widetilde{T}(X^\sigma)] \mid \sigma \in \mathrm{Emb}(F, \mathbb{C}) \, \} = \widetilde{\mathcal{G}}_S.$$

By the theorem of Shioda-Inose, there exists $\tau \in \mathrm{Emb}(F, \mathbb{C})$ and $\tau' \in \mathrm{Emb}(F, \mathbb{C})$ such that

$$X^\tau \cong S, \quad X^{\tau'} \cong S'.$$

There exists $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ such that $\sigma \circ \tau = \tau'$. $\square$

Corollary.
Let S and S' be complex singular $K3$ surfaces. If $\mathrm{NS}(S)$ and $\mathrm{NS}(S')$ are in the same genus, then $\mathrm{NS}(S)$ and $\mathrm{NS}(S')$ are isomorphic.

Corollary.
Let S be a complex singular $K3$ surface. If S is defined over a number field L, then

$$[L:\mathbb{Q}] \geq |\widetilde{\mathcal{G}}_S|,$$

where $\widetilde{\mathcal{G}}_S \subset \widetilde{\mathcal{L}}_{d(S)}$ is the oriented genus containing $[\widetilde{T}(S)]$.

Proof. Let X be the singular $K3$ surface defined over a number field F such that $\{[\widetilde{T}(X^\sigma)] \mid \sigma \in \mathrm{Emb}(F,\mathbb{C})\} = \widetilde{\mathcal{G}}_S$. Then

$$X^{\sigma_0} \cong S \quad \text{for some} \quad \sigma_0 \in \mathrm{Emb}(F,\mathbb{C}).$$

Let Y be a $K3$ surface defined over L such that

$$Y^{\tau_0} \cong S \quad \text{for some} \quad \tau_0 \in \mathrm{Emb}(L,\mathbb{C}).$$

Then there exists a number field $M \subset \mathbb{C}$ containing both of $\sigma_0(F)$ and $\tau_0(L)$ such that

$$X \otimes M \cong Y \otimes M \qquad \text{over } M.$$

Therefore, for each $\sigma \in \mathrm{Emb}(F,\mathbb{C})$, there exists $\tau \in \mathrm{Emb}(L,\mathbb{C})$ such that $X^\sigma \cong Y^\tau$ over $\mathbb{C}$. Since there exist exactly $|\widetilde{\mathcal{G}}_S|$ isomorphism classes of complex $K3$ surfaces among X^σ, we have $|\mathrm{Emb}(L,\mathbb{C})| \geq |\widetilde{\mathcal{G}}_S|$. $\square$

§4. The set $\mathcal{S}_p(\mathcal{X})$

We fix a smooth family $\mathcal{X} \to U$ of $K3$ surfaces over an open subset $U \subset \operatorname{Spec} \mathbb{Z}_F$ such that the generic fiber X_η is singular, and investigate the set

$$\mathcal{S}_p(\mathcal{X}) := \{\, \mathfrak{p} \in \pi_F^{-1}(p) \cap U \;\mid\; X_\mathfrak{p} \text{ is supersingular } \}).$$

For a supersingular $K3$ surface Y in characteristic p, there is a positive integer $\sigma(Y) \leq 10$, which is called the *Artin invariant* of Y, such that

$$d(Y)(:= \operatorname{disc}(\mathrm{NS}(Y))) = -p^{2\sigma(Y)}.$$

Theorem.
Suppose that p does not divide $2d(X_\eta) = 2\operatorname{disc}(\mathrm{NS}(X_\eta))$. Let $\chi_p : \mathbb{F}_p^\times \to \{\pm 1\}$ be the Legendre character.

(1) If $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$, then the Artin invariant of $X_\mathfrak{p}$ is 1.

(2) There exists a finite set N of prime integers containing the prime divisors of $2d(X_\eta)$ such that

$$p \notin N \quad \Rightarrow \quad \mathcal{S}_p(\mathcal{X}) = \begin{cases} \emptyset & \text{if } \chi_p(d(X_\eta)) = 1, \\ \pi_F^{-1}(p) & \text{if } \chi_p(d(X_\eta)) = -1. \end{cases}$$

§5. Supersingular reduction lattices

For simplicity, we assume that $p \neq 2$.

Theorem (Rudakov-Shafarevich).
Let p be an odd prime, and let σ be a positive integer ≤ 10. Then there exists a lattice $\Lambda_{p,\sigma}$ with the following properties, and it is unique up to isomorphism:

(i) even, rank 22,
(ii) of signature $(1, 21)$, and
(iii) the discriminant group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{2\sigma}$.

We call $\Lambda_{p,\sigma}$ the *Rudakov-Shafarevich lattice*.

Theorem (Artin-Rudakov-Shafarevich).
Let X be a supersingular $K3$ surface in odd characteristic p with the Artin invariant σ. Then $\mathrm{NS}(X)$ is isomorphic to $\Lambda_{p,\sigma}$.

Let $\mathcal{X} \to U$ be a smooth family of $K3$ surfaces over an open subset $U \subset \operatorname{Spec} \mathbb{Z}_F$ such that the generic fiber X_η is singular. Recall that the supersingular reduction lattice $L(\mathcal{X}, \mathfrak{p})$ of $\mathcal{X}$ at $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$ is defined by

$$L(\mathcal{X}, \mathfrak{p}) := (\mathrm{NS}(X_\eta) \hookrightarrow \mathrm{NS}(X_\mathfrak{p}))^\perp.$$

Suppose that

$$\boxed{p \not| \, 2d(X_\eta).}$$

Then $X_\mathfrak{p}$ is a supersingular $K3$ surface with Artin invariant 1, and hence

$$\mathrm{NS}(X_\mathfrak{p}) \cong \Lambda_{p,1}.$$

Proposition.
The image of the specialization isometry

$$\mathrm{NS}(X_\eta) \hookrightarrow \mathrm{NS}(X_\mathfrak{p})$$

is primitive, that is, the cokernel is torsion-free.

Corollary.
The supersingular reduction lattice $L(\mathcal{X}, \mathfrak{p})$ is an even, negative-definite lattice of rank 2 with discriminant $-p^2 d(X_\eta)$.

Corollary.
For $\mathfrak{p}, \mathfrak{p}' \in \mathcal{S}_p(\mathcal{X})$, the lattices $L(\mathcal{X}, \mathfrak{p})$ and $L(\mathcal{X}, \mathfrak{p}')$ are contained in the same genus.

The genus containing the lattices

$$L(\mathcal{X}, \mathfrak{p}) \qquad (\mathfrak{p} \in \mathcal{S}_p(\mathcal{X}))$$

is the genus of even, negative-definite lattices of rank 2 whose discriminant forms are isomorphic to

$$(D_{\mathrm{NS}}, -q_{\mathrm{NS}}) \oplus (D_{p,1}, q_{p,1}) \quad \cong \quad (D_T, q_T) \oplus (D_{p,1}, q_{p,1})$$

where $\mathrm{NS} = \mathrm{NS}(X_\eta)$, $T = T(X_\eta^\sigma)$ for any $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, and $(D_{p,1}, q_{p,1})$ is the discriminant form of the Rudakov-Shafarevich lattice $\Lambda_{p,1}$.

Definition.
For and $[T] \in \mathcal{L}_d$ and a prime integer $p \not| 2d$, we denote by

$$\mathcal{G}(p, T) \ \subset \ -\mathcal{L}_{p^2 d}$$

the genus consisting of even, negative-definite lattices of rank 2 whose discriminant form is isomorphic to

$$(D_T, q_T) \oplus (D_{p,1}, q_{p,1}).$$

Problem.
For a given T, does there exist a smooth family

$$\mathcal{X} \to U$$

of $K3$ surfaces over an open subset $U \subset \operatorname{Spec} \mathbb{Z}_F$ with the following properties?

(i) $(D_{\mathrm{NS}(\mathcal{X}_\eta)}, q_{\mathrm{NS}(\mathcal{X}_\eta)}) \cong (D_T, -q_T)$, and

(ii) except for a finite number of primes, if $\chi_p(d) = -1$, then the set of isomorphism classes of supersingular reduction lattices

$$L(\mathcal{X}, \mathfrak{p}) \qquad (\mathfrak{p} \in \mathcal{S}_p(\mathcal{X}) = \pi_F^{-1}(p))$$

coincides with the genus $\mathcal{G}(p, T)$.

Theorem (S.-).
Yes, if

- d is odd,
- d is a fundamental discriminant, and
- T is primitive.

Definition.
A negative integer d is called a *fundamental discriminant* if it is the discriminant of an imaginary quadratic field.

Definition.
An even positive-definite lattice of rank 2 is *primitive* if it is expressed by a matrix

$$\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \quad \text{with} \quad \gcd(a, b, c) = 1.$$

Remark.
S.- proved the theorem on transcendental lattices under the assumption that

- d is a fundamental discriminant, and
- T is primitive.

Then Schütt succeeded in removing these assumptions.

§6. The theory of Shioda, Mitani and Inose

We give a sketch of the proof.

Suppose that

$$\widetilde{T} = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \quad \text{with} \quad d := b^2 - 4ac < 0$$

is given. We put

$$E' := \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z}), \qquad \text{where} \qquad \tau' = \frac{-b+\sqrt{d}}{2a}, \quad \text{and}$$

$$E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \ , \qquad \text{where} \qquad \tau = \frac{b+\sqrt{d}}{2},$$

and consider the elliptic surface

$$A := E' \times E.$$

Shioda and Mitani showed that the oriented transcendental lattice $\widetilde{T}(A)$ is isomorphic to $\widetilde{T}$. Let

$$\widetilde{A} \to A$$

be the blowing up of A at the 2-torsion points of A, and let

$$\mathrm{Km}(A) \leftarrow \widetilde{A}$$

be the quotient by the lift of the inversion of A.

Shioda and Inose showed that, on $\mathrm{Km}(A)$, there exist reduced effective divisors C and Θ such that

(i) $C = C_1 + \cdots + C_8$ and $\Theta = \Theta_1 + \cdots + \Theta_8$ are disjoint,
(ii) C is an ADE-configuration of (-2)-curves of type $\mathbb{E}_8$,
(iii) Θ is an ADE-configuration of (-2)-curves of type $8\mathbb{A}_1$,
(iv) there exists a class $[\mathcal{L}] \in \mathrm{NS}(\mathrm{Km}(A))$ such that $2[\mathcal{L}] = [\Theta]$.

Let

$$\widetilde{Y} \to \mathrm{Km}(A)$$

be the double covering branched exactly along Θ, and let

$$Y \leftarrow \widetilde{Y}$$

be the contraction of (-1)-curves on $\widetilde{Y}$ (that is, the inverse images of $\Theta_1, \ldots, \Theta_8$).

Theorem (Shioda and Inose).
The surface Y is a singular $K3$ surface, and the diagram

$$Y \longleftarrow \widetilde{Y} \longrightarrow \mathrm{Km}(A) \longleftarrow \widetilde{A} \longrightarrow A$$

induces an isomorphism

$$\widetilde{T}(Y) \cong \widetilde{T}(A) \quad (\cong \widetilde{T})$$

of the oriented transcendental lattices.

Suppose that we have a Shioda-Inose-Kummer diagram

$$\mathcal{Y} \longleftarrow \widetilde{\mathcal{Y}} \longrightarrow \mathrm{Km}(\mathcal{A}) \longleftarrow \widetilde{\mathcal{A}} \longrightarrow \mathcal{A} = \mathcal{E}' \times \mathcal{E}$$

over an open subset $U \subset \operatorname{Spec} \mathbb{Z}_F$. We denote by

$$Y_\eta \longleftarrow \widetilde{Y}_\eta \longrightarrow \mathrm{Km}(A_\eta) \longleftarrow \widetilde{A}_\eta \longrightarrow A_\eta = E'_\eta \times E_\eta$$

the generic fiber of the diagram, and by

$$Y_\mathfrak{p} \longleftarrow \widetilde{Y}_\mathfrak{p} \longrightarrow \mathrm{Km}(A_\mathfrak{p}) \longleftarrow \widetilde{A}_\mathfrak{p} \longrightarrow A_\mathfrak{p} = E'_\mathfrak{p} \times E_\mathfrak{p}$$

the fiber over a closed point $\mathfrak{p} \in U$.

Analyzing the arguments of Shioda and Inose carefully, we obtain the following theorem.

Theorem.
(1) The above diagram over η induces an isomorphism

$$\widetilde{T}(\mathrm{Y}_\eta^\sigma) \cong \widetilde{T}(A_\eta^\sigma)$$

for any $\sigma \in \mathrm{Emb}(F, \mathbb{C})$.

(2) Except for a finite number of points $\mathfrak{p}$ of U, we have

$$Y_\mathfrak{p} \text{ is supersingular} \iff E'_\mathfrak{p} \text{ and } E_\mathfrak{p} \text{ are supersingular,}$$

and if this is the case, then the above diagram over $\mathfrak{p}$ induces

$$L(\mathcal{Y}, \mathfrak{p}) \cong (\mathrm{Hom}(E'_\eta, E_\eta) \hookrightarrow \mathrm{Hom}(E'_\mathfrak{p}, E_\mathfrak{p}))^\perp.$$

Here, for elliptic curves E_1, E_2 defined over a field k, we denote by $\mathrm{Hom}(E_1, E_2)$ the $\mathbb{Z}$-module of homomorphisms

$$\phi : E_1 \otimes \bar{k} \to E_2 \otimes \bar{k},$$

and we regard $\mathrm{Hom}(E_1, E_2)$ as a lattice by

$$(\phi, \phi) := 2 \deg \phi.$$

Thus the theorems are reduced to the statements about elliptic curves.

The lattices

$$\widetilde{T}(A_\eta^\sigma) = \widetilde{T}({E'_\eta}^\sigma \times {E_\eta}^\sigma) \qquad (\sigma \in \mathrm{Emb}(F, \mathbb{C}))$$

are calculated by the classical theory of complex multiplications in the class field theory.

The lattice

$$(\mathrm{Hom}(E'_\eta, E_\eta) \hookrightarrow \mathrm{Hom}(E'_\mathfrak{p}, E_\mathfrak{p}))^\perp$$

is calculated by Deuring's theory of endmorphism rings of supersingular elliptic curves.
We use Dorman's description of optimal embeddings of the integer ring of an imaginary quadratic fields into the Deuring order.

§7. An application to topology

We denote by $\mathrm{Emb}(\mathbb{C},\mathbb{C})$ the set of embeddings $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ of the complex number field $\mathbb{C}$ into itself.

Definition.
For a complex variety X and $\sigma \in \mathrm{Emb}(\mathbb{C},\mathbb{C})$, we define a complex variety X^σ by the following diagram of the fiber product:

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}\,\mathbb{C} & \xrightarrow{\sigma^*} & \mathrm{Spec}\,\mathbb{C}. \end{array}$$

Two complex varieties X and X' are said to be *conjugate* if there exists $\sigma \in \mathrm{Emb}(\mathbb{C},\mathbb{C})$ such that X' is isomorphic to X^σ over $\mathbb{C}$.

It is obvious from the definition that conjugate varieties are homeomorphic in Zariski topology.

Problem.
How about in the classical complex topology?

We have the following very classical:

Example (Serre (1964)).
There exist conjugate smooth projective varieties X and X^σ such that their topological fundamental groups are *not* isomorphic:

$$\pi_1(X) \ncong \pi_1(X^\sigma).$$

In particular, X and X^σ are not homotopically equivalent.

We also have the following:

Grothendieck's dessins d'enfant.
Let $f : C \longrightarrow \mathbb{P}^1$ be a finite covering defined over $\overline{\mathbb{Q}}$ branching only at the three points $0, 1, \infty \in \mathbb{P}^1$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consider the conjugate covering

$$f^\sigma : C^\sigma \longrightarrow \mathbb{P}^1.$$

Then f and f^σ are topologically distinct in general.

Belyi's theorem asserts that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of topological types of the covering of $\mathbb{P}^1$ branching only at $0, 1, \infty$ is faithful.

Other examples of non-homeomorphic conjugate varieties.

- **Abelson: Topologically distinct conjugate varieties with finite fundamental group.**
  Topology 13 (1974).
- **Artal Bartolo, Carmona Ruber, Cogolludo Agustín: Effective invariants of braid monodromy.**
  Trans. Amer. Math. Soc. 359 (2007).
- **S.-: On arithmetic Zariski pairs in degree 6.**
  arXiv:math/0611596
- **S.-: Non-homeomorphic conjugate complex varieties.**
  arXiv:math/0701115
- **Easton, Vakil: Absolute Galois acts faithfully on the components of the moduli space of surfaces: A Belyi-type theorem in higher dimension. arXiv:0704.3231**
- **Bauer, Catanese, Grunewald: The absolute Galois group acts faithfully on the connected components of the moduli space of surfaces of general type. arXiv:0706.1466**
- **F. Charles: Conjugate varieties with distinct real cohomology algebras. arXiv:0706.3674**

Using the theorem on the transcendental lattices of singular $K3$ surfaces defined over a number field, we construct examples of non-homeomorphic conjugate varieties.

Let V be an oriented topological manifold of real dimension 4. We put

$$H_2(V) := H_2(V, \mathbb{Z})/\text{torsion},$$

and let

$$\iota_V \;:\; H_2(V) \times H_2(V) \;\to\; \mathbb{Z}$$

be the intersection pairing. We then put

$$J_\infty(V) \;:=\; \bigcap_K \operatorname{Im}(H_2(V \setminus K) \to H_2(V)),$$

where K runs through the set of compact subsets of V, and set

$$\widetilde{B}_V := H_2(V)/J_\infty(V) \quad \text{and} \quad B_V := (\widetilde{B}_V)/\text{torsion}.$$

Since any topological cycle is compact, the intersection pairing ι_V induces a symmetric bilinear form

$$\beta_V \;:\; B_V \times B_V \;\to\; \mathbb{Z}.$$

It is obvious that the isomorphism class of (B_V, β_V) is a topological invariant of V.

Theorem.
Let X be a complex smooth projective surface, and let $C_1, \dots, C_n$ be irreducible curves on X. We put

$$V := X \setminus \bigcup C_i.$$

Suppose that the classes $[C_1], \dots, [C_n]$ span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Then (B_V, β_V) is isomorphic to the transcendental lattice

$$T(X) := (\mathrm{NS}(X) \hookrightarrow H^2(X))^{\perp}/\text{torsion}.$$

Construction of examples.
Let T_1 and T_2 be even positive-definite lattices of rank 2 that are in the same genus but not isomorphic. We have a singular $K3$ surface X defined over a number field F, and embeddings $\sigma_1, \sigma_2 \in \mathrm{Emb}(F, \mathbb{C})$ such that

$$T(X^{\sigma_1}) \cong T_1 \quad \text{and} \quad T(X^{\sigma_2}) \cong T_2.$$

Let $C_1, \dots, C_n$ be irreducible curves on X whose classes span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Enlarging F, we can assume that

$$V := X \setminus \bigcup C_i.$$

is defined over F. Then the conjugate open varieties

$$V^{\sigma_1} \quad \text{and} \quad V^{\sigma_2}$$

are not homeomorphic.

Remark.
By the classical theory of Gauss

Disquisitiones arithmeticae,

we have a complete theory of the decomposition of the set of isomorphism classes of lattices of rank 2 (*binary lattices*) into the disjoint union of genera.

Definition.
A complex plane curve $C \subset \mathbb{P}^2$ of degree 6 is called a maximizing sextic if C has only simple singularities (double points of ADE-type) and the total Milnor number of C attains the possible maximum 19.

Remark.
If C is a maximizing sextic, the minimal resolution $X_C \to Y_C$ of the double cover $Y_C \to \mathbb{P}^2$ branching exactly along C is a singular $K3$ surface. We denote by $T[C]$ the transcendental lattice of X_C.

In the following example, we employ a calculation of Artal, Carmona and Cogolludo, and a result of Degtyarev.

We consider the following cubic extension of $\mathbb{Q}$:

$$K := \mathbb{Q}[t]/(\varphi), \quad \text{where} \quad \varphi = 17t^3 - 18t^2 - 228t + 556.$$

The roots of $\varphi = 0$ are $\alpha, \bar{\alpha}, \beta$, where

$$\alpha = 2.590\cdots + 1.108\cdots\sqrt{-1}, \qquad \beta = -4.121\cdots.$$

There are three corresponding embeddings

$$\sigma_\alpha : K \hookrightarrow \mathbb{C}, \quad \sigma_{\bar{\alpha}} : K \hookrightarrow \mathbb{C} \quad \text{and} \quad \sigma_\beta : K \hookrightarrow \mathbb{C}.$$

There exists a homogeneous polynomial

$$\Phi(x_0, x_1, x_2) \in K[x_0, x_1, x_2]$$

of degree 6 with coefficients in K such that the plane curve

$$C = \{\Phi = 0\}$$

has three simple singular points of type

$$A_{16} + A_2 + A_1$$

as its only singularities. Consider the conjugate plane curves

$$C_\alpha = \{\Phi^{\sigma_\alpha} = 0\}, \;\; C_{\bar{\alpha}} = \{\Phi^{\sigma_{\bar{\alpha}}} = 0\} \quad \text{and} \quad C_\beta = \{\Phi^{\sigma_\beta} = 0\}.$$

They show that, if C' is a plane curve possessing $A_{16} + A_2 + A_1$ as its only singularities, then C' is projectively isomorphic to C_α, $C_{\bar{\alpha}}$ or C_β.

On the other hand, by the surjectivity of the period map for complex $K3$ surfaces, we can prove that there are exactly three singular $K3$ surfaces that is a double cover of $\mathbb{P}^2$ with a sextic branch curve possessing $A_{16} + A_2 + A_1$ as its only singularities.

Their oriented transcendental lattices are

$$[10, \pm 4, 22] := \begin{bmatrix} 10 & \pm 4 \\ \pm 4 & 22 \end{bmatrix} \quad \text{and} \quad [6, 0, 34] := \begin{bmatrix} 6 & 0 \\ 0 & 34 \end{bmatrix}.$$

Therefore we have

$$T[C_\alpha] \cong [10, 4, 22] \text{ or } [10, -4, 22] \quad \text{and} \quad T[C_\beta] \cong [6, 0, 34].$$

Let $V \subset Y_C$ be the pull-back of $\mathbb{P}^2 \setminus C$ by $Y_C \to \mathbb{P}^2$, which is a smooth open surface defined over K. Then the conjugate varieties V^{σ_α} and V^{σ_β} are not homeomorphic.

By the same method, we construct examples of pairs of non-homeomorphic conjugate varieties as double covers of complements of maximizing sextics.

1	$E_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
2	$E_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
3	$E_6 + D_5 + A_6 + A_2$	$L[12,0,42]$,	$L[6,0,84]$
4	$E_6 + A_{10} + A_3$	$L[12,0,22]$,	$L[4,0,66]$
5	$E_6 + A_{10} + A_2 + A_1$	$L[18,6,24]$,	$L[6,0,66]$
6	$E_6 + A_7 + A_4 + A_2$	$L[24,0,30]$,	$L[6,0,120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$L[30,0,42]$,	$L[18,6,72]$
8	$D_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
9	$D_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
10	$D_7 + A_{12}$	$L[6,2,18]$,	$L[2,0,52]$
11	$D_7 + A_8 + A_4$	$L[18,0,20]$,	$L[2,0,180]$
12	$D_5 + A_{10} + A_4$	$L[20,0,22]$,	$L[12,4,38]$
13	$D_5 + A_6 + A_5 + A_2 + A_1$	$L[12,0,42]$,	$L[6,0,84]$
14	$D_5 + A_6 + 2A_4$	$L[20,0,70]$,	$L[10,0,140]$
15	$A_{18} + A_1$	$L[8,2,10]$,	$L[2,0,38]$
16	$A_{16} + A_3$	$L[4,0,34]$,	$L[2,0,68]$
17	$A_{16} + A_2 + A_1$	$L[10,4,22]$,	$L[6,0,34]$
18	$A_{13} + A_4 + 2A_1$	$L[8,2,18]$,	$L[2,0,70]$
19	$A_{12} + A_6 + A_1$	$L[8,2,46]$,	$L[2,0,182]$
20	$A_{12} + A_5 + 2A_1$	$L[12,6,16]$,	$L[4,2,40]$
21	$A_{12} + A_4 + A_2 + A_1$	$L[24,6,34]$,	$L[6,0,130]$
22	$A_{10} + A_9$	$L[10,0,22]$,	$L[2,0,110]$
23	$A_{10} + A_9$	$L[8,3,8]$,	$L[2,1,28]$
24	$A_{10} + A_8 + A_1$	$L[18,0,22]$,	$L[10,2,40]$
25	$A_{10} + A_7 + A_2$	$L[22,0,24]$,	$L[6,0,88]$
26	$A_{10} + A_7 + 2A_1$	$L[10,2,18]$,	$L[2,0,88]$
27	$A_{10} + A_6 + A_2 + A_1$	$L[22,0,42]$,	$L[16,2,58]$
28	$A_{10} + A_5 + A_3 + A_1$	$L[12,0,22]$,	$L[4,0,66]$
29	$A_{10} + 2A_4 + A_1$	$L[30,10,40]$,	$L[10,0,110]$
30	$A_{10} + A_4 + 2A_2 + A_1$	$L[30,0,66]$,	$L[6,0,330]$
31	$A_8 + A_6 + A_4 + A_1$	$L[22,4,58]$,	$L[18,0,70]$
32	$A_7 + A_6 + A_4 + A_2$	$L[24,0,70]$,	$L[6,0,280]$
33	$A_7 + A_6 + A_4 + 2A_1$	$L[18,4,32]$,	$L[2,0,280]$
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$L[24,0,30]$,	$L[6,0,120]$

Preprints are available from my web-site:

http://www.math.sci.hokudai.ac.jp/~shimada/preprints.html

- **Non-homeomorphic conjugate complex varieties (preprint).**
- **On arithmetic Zariski pairs in degree 6 (to appear in Adv. Geom.)**
- **Transcendental lattices and supersingular reduction lattices of a singular K3 surface (to appear in Trans. Amer. Math. Soc.)**
- **On normal K3 surfaces (to appear in Michigan Math. J.)**