数論的 Zariski pair について

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- By a lattice, we mean a finitely generated free $\mathbb{Z}$-module Λ equipped with a non-degenerate symmetric bilinear form

$$\Lambda \times \Lambda \quad \to \quad \mathbb{Z}.$$

- A lattice Λ is said to be *even* if $(v, v) \in 2\mathbb{Z}$ for any $v \in \Lambda$.

§1. Conjugate varieties

A complex affine algebraic variety $X \subset \mathbb{C}^N$ is defined by a finite number of polynomial equations:

$$X \quad : \quad f_1(x_1, \ldots, x_N) = \cdots = f_m(x_1, \ldots, x_N) = 0.$$

Let $c_{j,I} \in \mathbb{C}$ be the coefficients of the polynomial f_j:

$$f_j(x_1, \ldots, x_N) = \sum_I c_{j,I} x^I, \quad \text{where} \quad x^I = x_1^{i_1} \cdots x_N^{i_N}.$$

We then denote by

$$F_X := \mathbb{Q}(\ldots, c_{j,I}, \ldots) \quad \subset \quad \mathbb{C}$$

the minimal sub-field of $\mathbb{C}$ containing all the coefficients of the defining equations of X.

There are many embeddings

$$\sigma \quad : \quad F_X \quad \hookrightarrow \quad \mathbb{C}$$

of the field F_X into $\mathbb{C}$.

Example.

(1) If $F_X = \mathbb{Q}(\sqrt{2}, t)$, where $t \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, then the set of embeddings $F_X \hookrightarrow \mathbb{C}$ is equal to

$$\{\sqrt{2}, -\sqrt{2}\} \times \{ \text{ transcendental complex numbers } \}.$$

(2) If all $c_{j,I}$ are algebraic over $\mathbb{Q}$, then the set of embeddings is finite, and the Galois group of the Galois closure of the algebraic extension $F_X/\mathbb{Q}$ acts on the set transitively.

For an embedding $\sigma : F_X \hookrightarrow \mathbb{C}$, we put

$$f_j^\sigma(x_1, \ldots, x_N) := \sum_I c_{j,I}^\sigma x^I,$$

and denote by $X^\sigma \subset \mathbb{C}^N$ the affine algebraic variety defined by

$$f_1^\sigma = \cdots = f_m^\sigma = 0.$$

We can define X^σ for a *projective* or *quasi-projective* variety $X \subset \mathbb{P}^N$ in the same way.

(Replace "polynomials" by "homogeneous polynomials".)

Definition.
We say that two algebraic varieties X and Y are said to be *conjugate* if there exists an embedding $\sigma : F_X \hookrightarrow \mathbb{C}$ such that Y is isomorphic (over $\mathbb{C}$) to X^σ.

In the language of schemes, two varieties X and Y over $\operatorname{Spec}\mathbb{C}$ are conjugate if there exists a diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\sigma^*} & \operatorname{Spec}\mathbb{C}. \end{array}$$

of the *fiber product* for some morphism $\sigma^* : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$.

It is obvious that being conjugate is an equivalence relation.

Conjugate varieties can never be distinguished by any algebraic methods.

Example.
Elliptic curves

$$\begin{aligned} E_1 &: y^2 = x^3 + \sqrt{2}x + \sqrt{3} \quad \text{and} \\ E_2 &: y^2 = x^3 - \sqrt{2}x + \sqrt{3} \end{aligned}$$

are conjugate. Their j-invariants

$$\begin{aligned} j(E_1) &= -\frac{221184}{6433} + \frac{1119744}{6433}\sqrt{2} = 211.778\ldots \quad \text{and} \\ j(E_2) &= -\frac{221184}{6433} - \frac{1119744}{6433}\sqrt{2} = -280.544\ldots \end{aligned}$$

are different. Hence they can be distinguished analytically. But they cannot be distinguished algebraically.

Conjugate varieties are homeomorphic in *Zariski* topology.
How about in the complex topology?

Example.
The betti numbers of a smooth projective complex variety X are "algebraic", that is,

$$b_i(X) = b_i(X^\sigma) \quad \text{for any } \sigma : F_X \hookrightarrow \mathbb{C},$$

in virtue of the theory of étale cohomology groups.

Example (Serre (1964)).
There exist conjugate non-singular complex projective varieties X and X^σ such that their fundamental groups are *not* isomorphic:

$$\pi_1(X) \not\cong \pi_1(X^\sigma).$$

In particular, they are not homotopically equivalent.

Grothendieck's dessins d'enfant (1984).
Let $f : C \longrightarrow \mathbb{P}^1$ be a finite covering defined over $\overline{\mathbb{Q}}$ branching only at the three points $0, 1, \infty \in \mathbb{P}^1$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consider the conjugate covering

$$f^\sigma : C^\sigma \longrightarrow \mathbb{P}^1.$$

Then f and f^σ have different topology in general.

Belyi's theorem asserts that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of topological types of the covering of $\mathbb{P}^1$ branching only at $0, 1, \infty$ is faithful.

Other examples of non-homeomorphic conjugate varieties.

- Abelson: Topologically distinct conjugate varieties with finite fundamental group.
  Topology 13 (1974).
- Artal Bartolo, Carmona Ruber, Cogolludo Agustín: Effective invariants of braid monodromy.
  Trans. Amer. Math. Soc. 359 (2007).
- S.-: On arithmetic Zariski pairs in degree 6.
  arXiv:math/0611596, to appear in Adv. Geom.
- S.-: Non-homeomorphic conjugate complex varieties.
  arXiv:math/0701115
- Easton, Vakil: Absolute Galois acts faithfully on the components of the moduli space of surfaces: A Belyi-type theorem in higher dimension.
  arXiv:0704.3231
- Bauer, Catanese, Grunewald: The absolute Galois group acts faithfully on the connected components of the moduli space of surfaces of general type.
  arXiv:0706.1466
- F. Charles: Conjugate varieties with distinct real cohomology algebras.
  arXiv:0706.3674

§2. Zariski pairs

Definition.
A pair $[C, C']$ of complex projective plane curves is said to be a *Zariski pair* if the following hold.

(i) There exist tubular neighborhoods $\mathcal{T} \subset \mathbb{P}^2$ of C and $\mathcal{T}' \subset \mathbb{P}^2$ of C' such that $(\mathcal{T}, C)$ and $(\mathcal{T}', C')$ are diffeomorphic.

(ii) $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are *not* homeomorphic.

Example.
The first example of a Zariski pair was discovered by Zariski in 1930's, and studied by Oka.
They presented a Zariski pair $[C, C']$ of plane curves of degree 6 with six ordinary cusps as its only singularities. The fact $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are *not* homeomorphic follows from

$$\pi_1(\mathbb{P}^2 \setminus C) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) \quad \text{and} \quad \pi_1(\mathbb{P}^2 \setminus C') \cong \mathbb{Z}/6\mathbb{Z}.$$

Hence the moduli of projective plane curves of degree 6 with 6 ordinary cusps has at least two connected components.

Remark. Degtyarev showed that there are no Zariski pairs in degree ≤ 5.

Let $[C, C']$ be the Zariski pair of 6-cuspidal sextics. Then C and C' can be deistinguished algebraically, because there is a surjective homomorphism from $\pi_1(\mathbb{P}^2 \setminus C) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ to a finite non-abelian group S_3, while there are no such homomorphisms from $\pi_1(\mathbb{P}^2 \setminus C') \cong \mathbb{Z}/6\mathbb{Z}$.

Many Zariki pairs discovered so far uses "algebraic" topological invariants in distinguishing the topology of $(\mathbb{P}^2, C)$.

Definition.
A Zariski pair $[C, C']$ is said to be an *arithmetic Zariski pair* if the following hold.
Suppose that $C = \{\Phi = 0\}$. Then there exists an embedding $\sigma : F_C \hookrightarrow \mathbb{C}$ such that C' is isomorphic (as a plane curve) to

$$C^\sigma := \{\Phi^\sigma = 0\} \quad \subset \quad \mathbb{P}^2.$$

In other words, an arithmetic Zariski pair is an *algebraically-indistinguishable* Zariski pair.

Remark.
The first example of an arithmetic Zariski pair was discovered by Artal, Carmona, Cogolludo (2007) in degree 12. They used the invariant of *braid monodromies* in order to distinguish $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ topologically.

Our aim is:

(1) to present a topological invariant of the complex plane curves that is fine enough to distinguish the conjugate curves,

(2) to present explcit examples of arithmetic Zariski pairs, and

(3) to study the topology of those examples closely, and see how the Galois action affects the topology.

§3. A topological invariant

Let V be an oriented topological manifold of real dimension 4. We put

$$H_2(V) := H_2(V, \mathbb{Z})/\text{torsion} \quad \text{and} \quad H^2(V) := H^2(V, \mathbb{Z})/\text{torsion},$$

and let

$$\iota_V \; : \; H_2(V) \times H_2(V) \; \to \; \mathbb{Z}$$

be the intersection pairing. We then put

$$J_\infty(V) \; := \; \bigcap_K \mathrm{Im}(H_2(V \setminus K) \to H_2(V)),$$

where K runs through the set of compact subsets of V, and set

$$\widetilde{B}_V := H_2(V)/J_\infty(V) \quad \text{and} \quad B_V := (\widetilde{B}_V)/\text{torsion}.$$

Since any topological cycle is compact, the intersection pairing ι_V induces a symmetric bilinear form

$$\beta_V \; : \; B_V \times B_V \; \to \; \mathbb{Z}.$$

It is obvious that the isomorphism class of (B_V, β_V) is a topological invariant of V.

For a complex smooth projective surface X, we denote by $\mathrm{NS}(X) \subset H^2(X)$ the *Néron-Severi lattice* of X; that is, the lattice generated by cohomology classes of curves on X with the intersection pairing.

Theorem.
Let X be a complex smooth projective surface, and let $C_1, \dots, C_n$ be irreducible curves on X. We put

$$V := X \setminus \bigcup C_i.$$

Suppose that the classes $[C_1], \dots, [C_n]$ span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Then (B_V, β_V) is isomorphic to the transcendental lattice

$$T(X) := (\mathrm{NS}(X) \hookrightarrow H^2(X))^{\perp}/\text{torsion}.$$

Hence $T(X)$ is a topological invariant of the open complex surface $V \subset X$.

Definition.
Two lattices

$$\lambda : \Lambda \times \Lambda \to \mathbb{Z} \quad \text{and} \quad \lambda' : \Lambda' \times \Lambda' \to \mathbb{Z}$$

are said to be *in the same genus* if

$$\lambda \otimes \mathbb{Z}_p \ : \ \Lambda \otimes \mathbb{Z}_p \times \Lambda \otimes \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{and}$$
$$\lambda' \otimes \mathbb{Z}_p \ : \ \Lambda' \otimes \mathbb{Z}_p \times \Lambda' \otimes \mathbb{Z}_p \to \mathbb{Z}_p$$

are isomorphic for any p including $p = \infty$, where $\mathbb{Z}_\infty = \mathbb{R}$.

Theorem.
Let X and X^σ be conjugate non-singular complex projective varieties of dimension 2. Suppose that $H^2(X)$ and $H^2(X^\sigma)$ are both even. Then the transcendental lattices $T(X)$ and $T(X^\sigma)$ are contained in the same genus.

This theorem follows from the theory of discriminant forms of even lattices.

Gauss gave a complete description of isomorphism classes of lattices of rank 2 (*binary lattices*) and their decomposition into genera in *Disquisitiones arithmeticae.*

§4. Singular $K3$ surfaces

Let X be a complex $K3$ surface; that is, a simply-connected surface with $K_X \cong \mathcal{O}_X$. Then $H^2(X)$ is a unimodular lattice of rank 22 with signature $(3, 19)$.

Definition.
A complex $K3$ surface X is said to be *singular* if the rank of the transcendental lattice $T(X)$ is 2 (the possible minimum).

The transcendental lattice $T(X)$ of a singular $K3$ surface X is positive-definite. Moreover, by the Hodge decomposition

$$T(X) \otimes \mathbb{C} \cong H^{2,0}(X) \oplus H^{0,2}(X),$$

this lattice has a canonical orientation. We denote by $\widetilde{T}(X)$ the oriented transcendental lattice of X.

Definition.
We put

$$\mathcal{M} := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z},\ a > 0,\ c > 0, \\ 4ac - b^2 > 0 \end{array} \right\}.$$

We then denote by

$$\mathcal{L} := \mathcal{M}/\, GL_2(\mathbb{Z})$$

the set of isomorphism classes of even positive-definite binary lattices, and by

$$\widetilde{\mathcal{L}} := \mathcal{M}/\, SL_2(\mathbb{Z})$$

the set of isomorphism classes of even positive-definite *oriented* binary lattices.

Theorem (Shioda and Inose).
The map $X \mapsto \widetilde{T}(X) \in \widetilde{\mathcal{L}}$ induces a bijection from the set of isomorphism classes of singular $K3$ surfaces to the set $\widetilde{\mathcal{L}}$.

Theorem (S.- and M. Schütt).
Let $\mathcal{G} \subset \mathcal{L}$ be a genus in $\mathcal{L}$, and let $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{L}}$ be the pull-back of $\mathcal{G}$ by the natural projection $\widetilde{\mathcal{L}} \to \mathcal{L}$. Then there exists a singular $K3$ surface X defined over a number field F such that the set

$$\{ \, [\widetilde{T}(X^\sigma)] \ | \ \sigma \in \mathrm{Emb}(F, \mathbb{C}) \, \} \ \subset \ \widetilde{\mathcal{L}}$$

coincides with the oriented genus $\widetilde{\mathcal{G}}$, where $\mathrm{Emb}(F, \mathbb{C})$ denotes the set of embeddings of F into $\mathbb{C}$.

Corollary.
Let X and X' be singular $K3$ surfaces. If their transcendental lattices are in the same genus, then they are conjugate.

Construction of examples.

Let T_1 and T_2 be even positive-definite lattices of rank 2 that are in the same genus but not isomorphic. We have a singular $K3$ surface X defined over a number field F, and embeddings $\sigma_1, \sigma_2 \in \mathrm{Emb}(F, \mathbb{C})$ such that

$$T(X^{\sigma_1}) \cong T_1 \quad \text{and} \quad T(X^{\sigma_2}) \cong T_2.$$

Let $C_1, \ldots, C_n$ be irreducible curves on X whose classes span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Enlarging F, we can assume that

$$V := X \setminus \bigcup C_i.$$

is defined over F. Then the conjugate open varieties

$$V^{\sigma_1} \quad \text{and} \quad V^{\sigma_2}$$

are not homeomorphic.

§5. Arithmetic Zariski pairs of maximizing sextics

Definition.
A complex plane curve $C \subset \mathbb{P}^2$ of degree 6 is called a *maximizing sextic* if C has only simple singularities and the total Milnor number of C attains the possible maximum 19.

If C is a maximizing sextic, then the minimal resolution $X_C \to Y_C$ of the double covering $Y_C \to \mathbb{P}^2$ branching exactly along C is a singular $K3$ surface. We denote by $T[C]$ the transcendental lattice of X_C.

Corollary.
The lattice $T[C]$ is a topological invariant of $(\mathbb{P}^2, C)$.

Using the surjectivity of the period map for complex $K3$ surfaces, we can determine whether there exists a maximizing sextics C such that $\mathrm{Sing}(C)$ is of a given ADE-type. This task was worked out by Yang (1996). We can also determine all possible isomorphism classes of the transcendental lattice $T[C]$.

Using computer, we obtain the following examples of arithmetic Zariski pairs of maximizing sextics. We put

$$L[2a, b, 2c] := \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

No.	the type of $\mathrm{Sing}(C)$	$T[C]$ and $T[C']$	
1	$E_8 + A_{10} + A_1$	$L[6, 2, 8]$,	$L[2, 0, 22]$
2	$E_8 + A_6 + A_4 + A_1$	$L[8, 2, 18]$,	$L[2, 0, 70]$
3	$E_6 + D_5 + A_6 + A_2$	$L[12, 0, 42]$,	$L[6, 0, 84]$
4	$E_6 + A_{10} + A_3$	$L[12, 0, 22]$,	$L[4, 0, 66]$
5	$E_6 + A_{10} + A_2 + A_1$	$L[18, 6, 24]$,	$L[6, 0, 66]$
6	$E_6 + A_7 + A_4 + A_2$	$L[24, 0, 30]$,	$L[6, 0, 120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$L[30, 0, 42]$,	$L[18, 6, 72]$
8	$D_8 + A_{10} + A_1$	$L[6, 2, 8]$,	$L[2, 0, 22]$
9	$D_8 + A_6 + A_4 + A_1$	$L[8, 2, 18]$,	$L[2, 0, 70]$
10	$D_7 + A_{12}$	$L[6, 2, 18]$,	$L[2, 0, 52]$
11	$D_7 + A_8 + A_4$	$L[18, 0, 20]$,	$L[2, 0, 180]$
12	$D_5 + A_{10} + A_4$	$L[20, 0, 22]$,	$L[12, 4, 38]$
13	$D_5 + A_6 + A_5 + A_2 + A_1$	$L[12, 0, 42]$,	$L[6, 0, 84]$
14	$D_5 + A_6 + 2A_4$	$L[20, 0, 70]$,	$L[10, 0, 140]$
15	$A_{18} + A_1$	$L[8, 2, 10]$,	$L[2, 0, 38]$
16	$A_{16} + A_3$	$L[4, 0, 34]$,	$L[2, 0, 68]$
17	$A_{16} + A_2 + A_1$	$L[10, 4, 22]$,	$L[6, 0, 34]$
18	$A_{13} + A_4 + 2A_1$	$L[8, 2, 18]$,	$L[2, 0, 70]$
19	$A_{12} + A_6 + A_1$	$L[8, 2, 46]$,	$L[2, 0, 182]$
20	$A_{12} + A_5 + 2A_1$	$L[12, 6, 16]$,	$L[4, 2, 40]$
21	$A_{12} + A_4 + A_2 + A_1$	$L[24, 6, 34]$,	$L[6, 0, 130]$
22	$A_{10} + A_9$	$L[10, 0, 22]$,	$L[2, 0, 110]$
23	$A_{10} + A_9$	$L[8, 3, 8]$,	$L[2, 1, 28]$
24	$A_{10} + A_8 + A_1$	$L[18, 0, 22]$,	$L[10, 2, 40]$
25	$A_{10} + A_7 + A_2$	$L[22, 0, 24]$,	$L[6, 0, 88]$
26	$A_{10} + A_7 + 2A_1$	$L[10, 2, 18]$,	$L[2, 0, 88]$
27	$A_{10} + A_6 + A_2 + A_1$	$L[22, 0, 42]$,	$L[16, 2, 58]$
28	$A_{10} + A_5 + A_3 + A_1$	$L[12, 0, 22]$,	$L[4, 0, 66]$
29	$A_{10} + 2A_4 + A_1$	$L[30, 10, 40]$,	$L[10, 0, 110]$
30	$A_{10} + A_4 + 2A_2 + A_1$	$L[30, 0, 66]$,	$L[6, 0, 330]$
31	$A_8 + A_6 + A_4 + A_1$	$L[22, 4, 58]$,	$L[18, 0, 70]$
32	$A_7 + A_6 + A_4 + A_2$	$L[24, 0, 70]$,	$L[6, 0, 280]$
33	$A_7 + A_6 + A_4 + 2A_1$	$L[18, 4, 32]$,	$L[2, 0, 280]$
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$L[24, 0, 30]$,	$L[6, 0, 120]$

§6. Maximizing sextics of type $A_{10} + A_9$

There are 4 connected components in the moduli space of maximizing sextics of type

$$A_{10} + A_9.$$

Two of them have irreducible members, and their oriented transcendental lattices are

$$\begin{bmatrix} 10 & 0 \\ 0 & 22 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 110 \end{bmatrix}.$$

The other two have reducible members (a line and an irreducible quintic), and their oriented transcendental lattices are

$$\begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}.$$

We will consider these reducible members.

The reducible members are defined over $\mathbb{Q}(\sqrt{5})$. The defining equation is

$$C_{\pm} \quad : \quad z\cdot(G(x,y,z) \pm \sqrt{5}\cdot H(x,y,z)) = 0,$$

where

$$\begin{aligned} G(x,y,z) \;&:=\; -9\,x^4z - 14\,x^3yz + 58\,x^3z^2 - 48\,x^2y^2z - \\ &\quad -64\,x^2yz^2 + 10\,x^2z^3 + +108\,xy^3z - \\ &\quad -20\,xy^2z^2 - 44\,y^5 + 10\,y^4z, \\ H(x,y,z) \;&:=\; 5\,x^4z + 10\,x^3yz - 30\,x^3z^2 + 30\,x^2y^2z + \\ &\quad +20\,x^2yz^2 - 40\,xy^3z + 20\,y^5. \end{aligned}$$

The singular points are

$$[0:0:1]\ \ (A_{10}) \quad \text{and} \quad [1:0:0]\ \ (A_9).$$

We have two possibilities:

$$T[C_+] \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} \quad \text{and} \quad T[C_-] \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix},$$

or

$$T[C_+] \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix} \quad \text{and} \quad T[C_-] \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

Problem. Which is the case?

Remark.
This problem cannot be solved by any algebraic methods.

For simplicity, we put $X_\pm := X_{C_\pm}$. Let $D \subset X_\pm$ be the total transform of the union of the lines

$$\{z = 0\} \cup \{x = 0\},$$

on which the two singular points of $C_\pm$ locate, and let $X^0_\pm$ be the complement of D. Since the irreducible components of D span $S_{X_\pm} \otimes \mathbb{Q}$, the inclusion $X^0_\pm \hookrightarrow X_\pm$ induces a surjection

$$H_2(X^0_\pm, \mathbb{Z}) \twoheadrightarrow T(X_\pm).$$

We will describe the generators of $H_2(X^0_\pm, \mathbb{Z})$ and the intersection numbers among them.

We put

$$f_\pm(y, z) := G(1, y, z) \pm \sqrt{5} \cdot H(1, y, z),$$

and set

$$Q_\pm := \{f_\pm(y, z) = 0\}.$$

Then $Q_\pm$ is a smooth affine quintic curve, and it intersects the line

$$L := \{z = 0\}$$

at the origin with the multiplicity 5. The open surface $X^0_\pm$ is a double covering of $\mathbb{A}^2 \setminus L$ branching along $Q_\pm$.

Let

$$\pi_\pm : X_\pm^0 \longrightarrow \mathbb{A}^2 \setminus L$$

be the double covering. We consider the projection

$$p : \mathbb{A}^2 \longrightarrow \mathbb{A}_z^1 \qquad p(y,z) := z$$

and the composite

$$q_\pm : X_\pm^0 \longrightarrow \mathbb{A}^2 \setminus L \longrightarrow \mathbb{A}_z^1 \setminus \{0\}.$$

There are four critical points of the finite covering

$$p|Q_\pm : Q_\pm \longrightarrow \mathbb{A}_z^1.$$

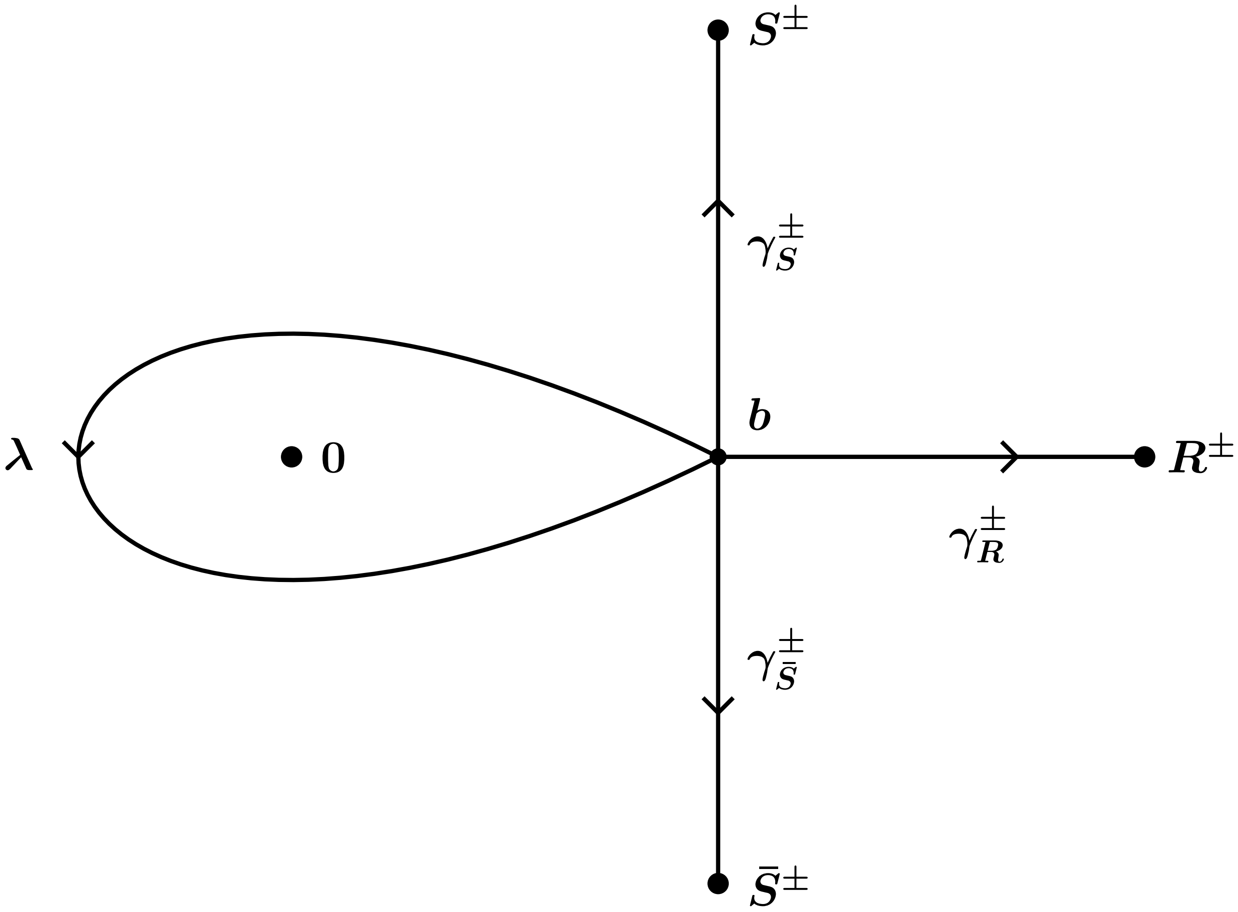
Three of them $R_\pm, S_\pm, \overline{S}_\pm$ are simple critical values, while the critical point over 0 is of multiplicity 5. Their positions are

$$R_+ = 0.42193..., \qquad S_+ = 0.23780... + 0.24431... \cdot \sqrt{-1},$$

and

$$R_- = 0.12593..., \qquad S_- = 27.542... + 45.819... \cdot \sqrt{-1}.$$

We choose a base point b on $\mathbb{A}_z^1$ as a sufficiently small positive real number (say $b = 10^{-3}$), and define the loop λ and the paths $\rho_\pm$, $\sigma_\pm$, $\bar{\sigma}_\pm$ on the z-line $\mathbb{A}_z^1$ as in the figure:



$S^{\pm}$
$\gamma_{S}^{\pm}$
b
λ
0
$R^{\pm}$
$\gamma_{R}^{\pm}$
$\gamma_{\bar{S}}^{\pm}$
$\bar{S}^{\pm}$

We put

$$\mathbb{A}^1_y := p^{-1}(b), \quad F_\pm := q_\pm^{-1}(b) = \pi_\pm^{-1}(\mathbb{A}^1_y).$$

Then the morphism

$$\pi_\pm | F_\pm : F_\pm \longrightarrow \mathbb{A}^1_y$$

is the double covering branching exactly at the five points $\mathbb{A}^1_y \cap Q_\pm$. Hence $F_\pm$ is a genus 2 curve minus one point.

We choose a system of oriented simple closed curves $a_1, \ldots, a_5$ on $F_\pm$ in such a way that their images by the double covering

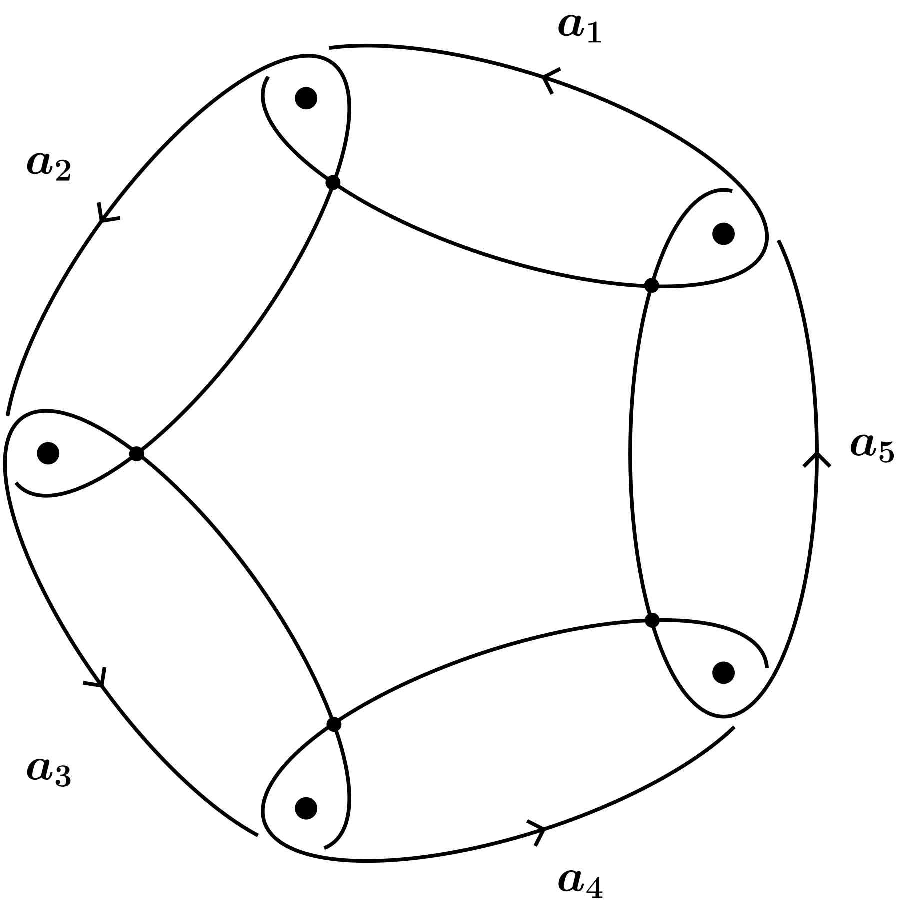
$$\pi_\pm | F_\pm : F_\pm \longrightarrow \mathbb{A}^1_y$$

are given in the figure and that the orientations are given so that

$$a_i a_{i+1} = -a_{i+1} a_i = 1$$

holds for $i = 1, \ldots, 5$, where $a_6 := a_1$. Then $H_1(F_\pm, \mathbb{Z})$ is generated by $[a_1], \ldots, [a_4]$, and we have

$$[a_5] = -[a_1] - [a_2] - [a_3] - [a_4].$$



a_1
a_2
a_3
a_4
a_5

The monodromy along the loop λ around $z = 0$ is given by

$$a_i \mapsto a_{i+1}.$$

Hence the open surface $X^0_\pm$ is homotopically equivalent to the 2-dimensional CW-complex obtained from $F_\pm$ by attaching

- **four tubes**

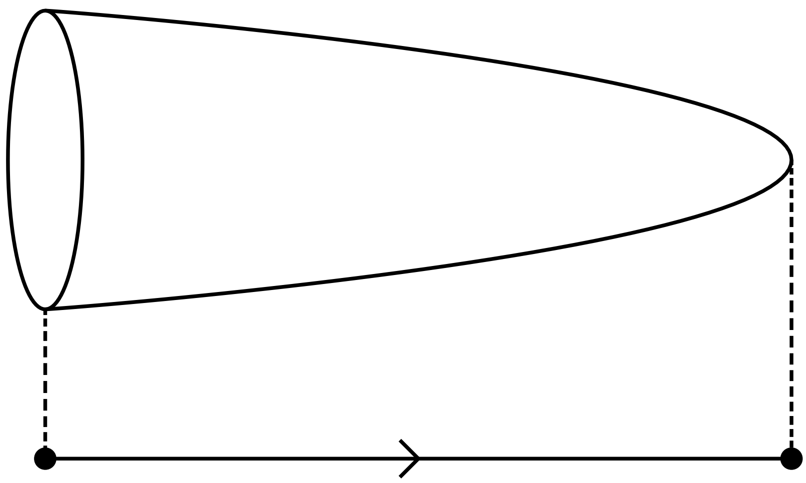
$$T_i := S^1 \times I \quad (i = 1, \ldots, 4)$$

  with $\partial T_i = a_{i+1} - a_i$, and

- **three thimbles**

$$\Theta(\rho_\pm), \quad \Theta(\sigma_\pm), \quad \Theta(\bar{\sigma}_\pm)$$

  corresponding to the vanishing cycles on $F_\pm$ for the simple critical values $R_\pm$, $S_\pm$ and $\overline{S}_\pm$.

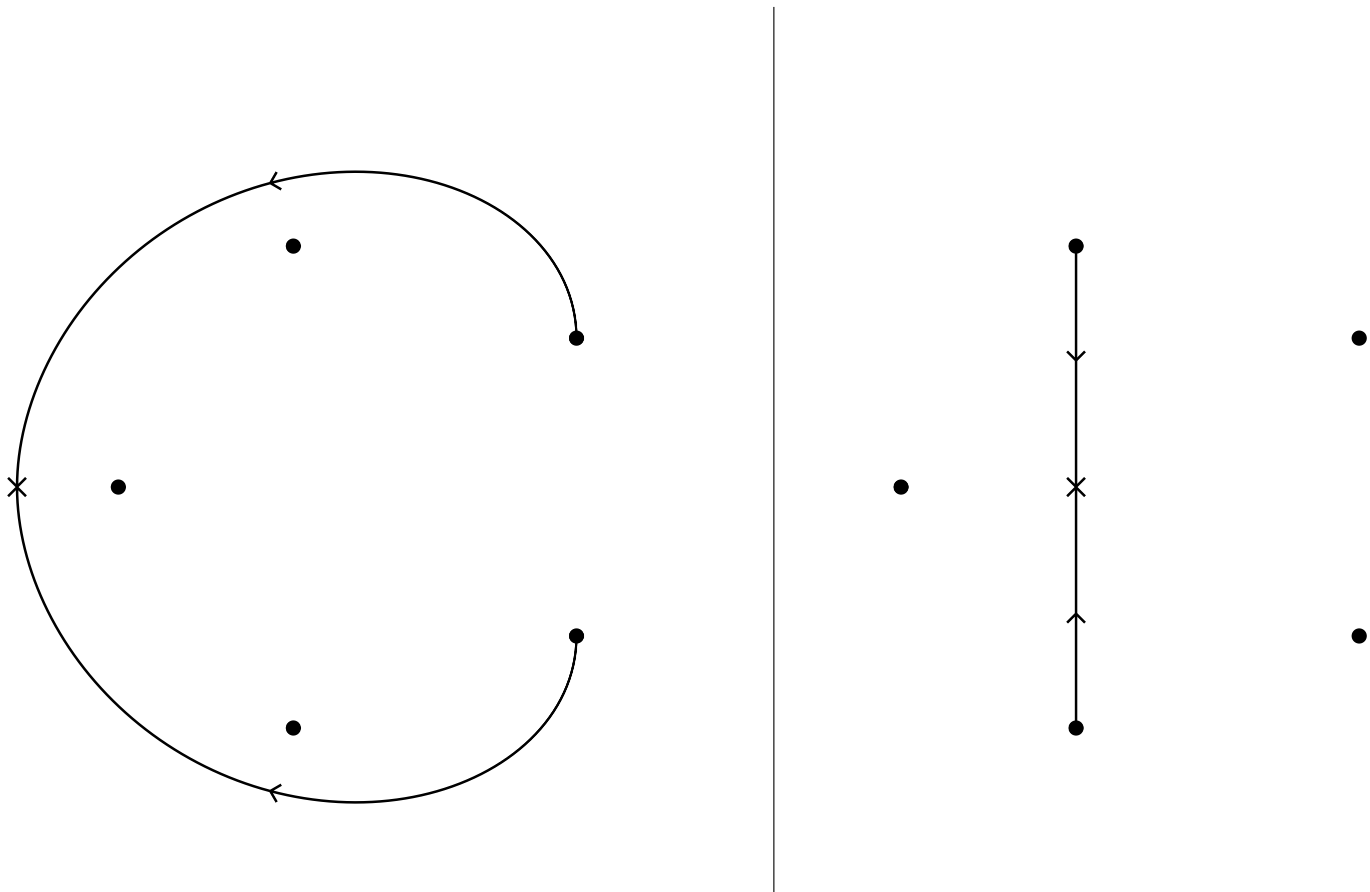


Hence the homology group $H_2(X^0_\pm, \mathbb{Z})$ is equal to the kernel of the homomorphism

$$\bigoplus_{i=1}^{4} \mathbb{Z}[T_i] \oplus \mathbb{Z}[\Theta(\rho_\pm)] \oplus \mathbb{Z}[\Theta(\sigma_\pm)] \oplus \mathbb{Z}[\Theta(\bar{\sigma}_\pm)] \longrightarrow \bigoplus_{i=1}^{4} \mathbb{Z}[a_i]$$

given by $[M] \mapsto [\partial(M)]$. Therefore the problem is reduced to the calculation of the vanishing cycles $\partial\Theta(\rho_\pm)$, $\partial\Theta(\sigma_\pm)$ and $\partial\Theta(\bar{\sigma}_\pm)$.

When z moves from b to $R_{\pm}$ along the path $\rho_{\pm}$, the branch points $p^{-1}(z) \cap Q_{\pm}$ moves as follows:



$\mathcal{Q}^{+}(z)$ $\qquad$ $\mathcal{Q}^{-}(z)$

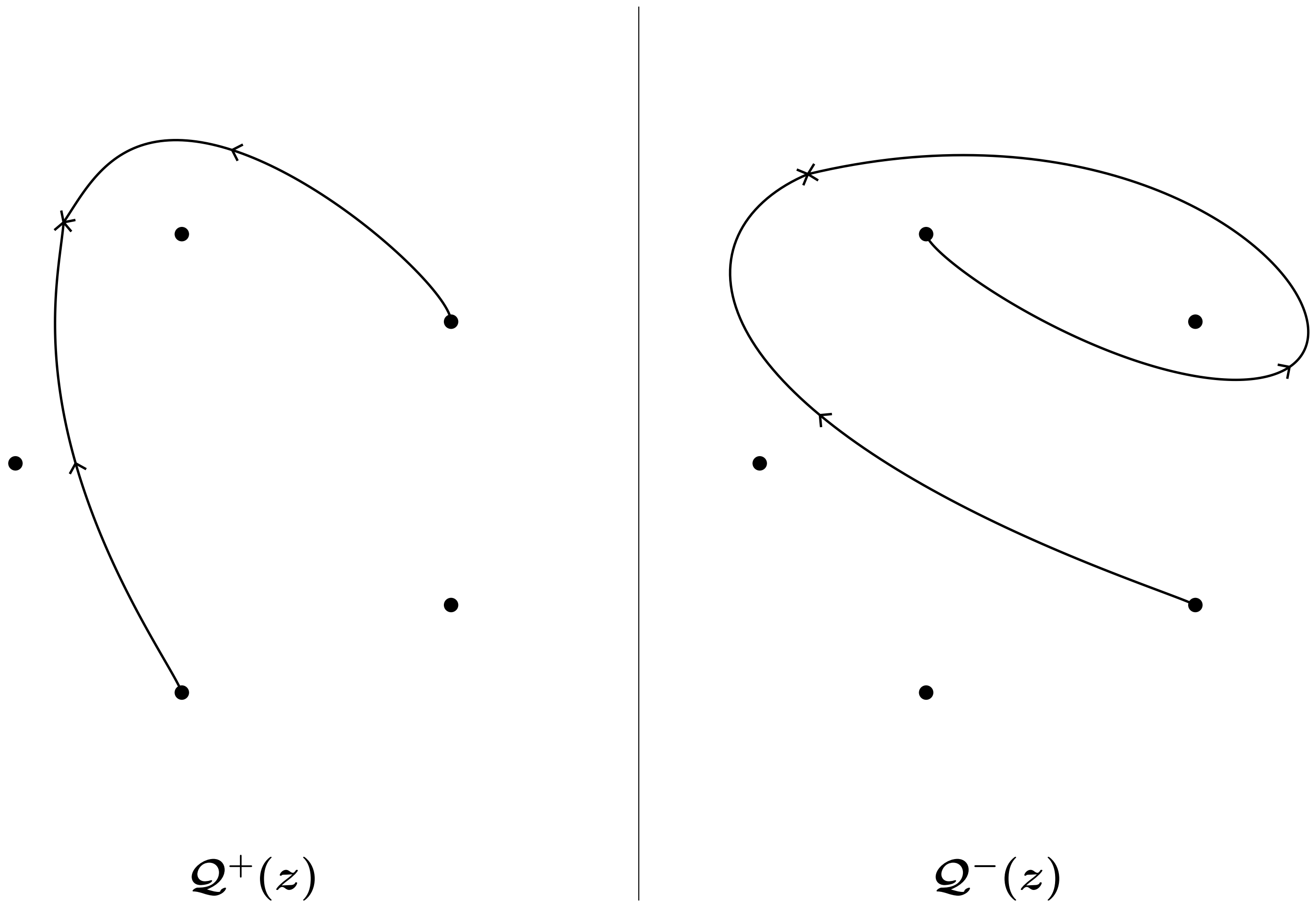
Therefore, putting an orientation on the thimble, we have

$$[\partial\Theta(\rho_{+})] = [a_1] - [a_2] + [a_3] - [a_4],$$

while

$$[\partial\Theta(\rho_{-})] = [a_2] + [a_3].$$

When z moves from b to $S_\pm$ along the path $\sigma_\pm$, the branch points $p^{-1}(z) \cap Q_\pm$ moves as follows:



Therefore, putting an orientation on the thimble, we have

$$[\partial\Theta(\sigma_+)] = [a_1] - [a_2] - [a_3],$$

while

$$[\partial\Theta(\sigma_-)] = 2[a_1] - [a_2] - [a_3] - [a_4].$$

By this calculation, we obtain the following:

Proposition.

$$\boldsymbol{T}[\boldsymbol{C}_+] \cong \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{28} \end{bmatrix}, \quad \boldsymbol{T}[\boldsymbol{C}_-] \cong \begin{bmatrix} \mathbf{8} & \mathbf{3} \\ \mathbf{3} & \mathbf{8} \end{bmatrix}.$$

Problem.

$$\boldsymbol{\pi}_1(\mathbb{P}^2 \setminus \boldsymbol{C}_+) \cong \boldsymbol{\pi}_1(\mathbb{P}^2 \setminus \boldsymbol{C}_-)?$$