K3 surfaces and lattice Theory

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Abstract

In this talk, we explain how to use the lattice theory and computer in the study of $K3$ surfaces.

1. Introduction

We work over $\mathbb{C}$.

Definition 1.1. A smooth projective surface X is called a $K3$ *surface* if there exists a nowhere vanishing holomorphic 2-form ω_X on X and $\pi_1(X) = 1$.

$K3$ surfaces are an important and interesting object, not only in algebraic geometry but also in many other branches of mathematics including theoretical physics. We consider the following geometric problems on $K3$ surfaces:

- enumerate elliptic fibrations on a given $K3$ surface,
- enumerate elliptic $K3$ surfaces up to certain equivalence relation (e.g., by the type of singular fibers, ...),
- enumerate projective models of a fixed degree (e.g., sextic double planes, quartic surfaces, ...) of a given $K3$ surface,
- enumerate projective models of a fixed degree of $K3$ surfaces up to certain equivalence relation,
- determine the automorphism group of a given $K3$ surface,
- ....

There are many works on these problems. Thanks to the Torelli-type theorem due to Piatetski-Shapiro and Shafarevich [15], some of these problems are reduced to computational problems in lattice theory, and the latter can often be solved by means of *computer*. It is important to clarify to what extent the geometric problems on $K3$ surfaces are solved by this method.

In this talk, we explain how to use lattice theory and computer in the study of $K3$ surfaces. In particular, we present some elementary but useful algorithms about lattices. We then demonstrate this method on the problems of constructing Zariski pairs of projective plane curves (that is, a study of embedding topology of plane curves), and of determining the automorphism group of a given $K3$ surface.

The methods can be applied to the supersingular $K3$ surfaces in positive characteristics (see [8, 10, 23], for example). For simplicity, however, we restrict ourselves to complex algebraic $K3$ surfaces.

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2. Lattice theory

The application of the lattice theory to the study of $K3$ surfaces started with Nikulin [11]. A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}.$$

For a lattice L, we denote by $\mathrm{O}(L)$ the orthogonal group of L, that is, the group of automorphisms of L. A lattice L is canonically embedded into its *dual lattice*

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z})$$

as a submodule of finite index. The finite abelian group

$$D_L := L^\vee / L$$

is called the *discriminant group* of L. We say that L is *unimodular* if $D_L = 0$. A lattice L is *even* if $v^2 \in 2\mathbb{Z}$ for any $v \in L$. Suppose that L is even. The $\mathbb{Z}$-valued symmetric bilinear form on L extends to a $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$, and it defines a finite quadratic form

$$q_L\colon D_L \to \mathbb{Q}/2\mathbb{Z},\ \ \bar{x} \mapsto x^2 \bmod 2\mathbb{Z},$$

which is called the *discriminant form* of L. A submodule M of $L^\vee$ containing L is said to be an *overlattice* of L if the $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$ takes values in $\mathbb{Z}$ on M. There exists a canonical bijection between the set of even overlattices of L and the set of isotropic subgroups of (D_L, q_L). The *signature* $\mathrm{sgn}(L)$ of a lattice L is the signature of the real quadratic space $L \otimes \mathbb{R}$. We say that a lattice L of rank n is *negative-definite* (resp. *hyperbolic*) if the signature of L is $(0, n)$ (resp. $(1, n-1)$).

Theorem 2.1. *Suppose that a pair of non-negative integers (s_+, s_-) and a finite quadratic form (D, q) are given. Then we can determine by an effective method whether there exists an even lattice L such that* $\mathrm{sgn}(L) = (s_+, s_-)$ *and $(D_L, q_L) \cong (D, q)$.*

See [11] or [6, Chapter 15] for the proof and the concrete description of the method.

A sublattice L of a lattice M is said to be *primitive* if M/L is torsion free. Let M be an even *unimodular* lattice, and L a primitive sublattice of M with the orthogonal complement $L^\perp$. Then we have $(D_L, q_L) \cong (D_{L^\perp}, -q_{L^\perp})$. Conversely, if R is an even lattice such that $(D_L, q_L) \cong (D_R, -q_R)$, then there exists an even unimodular overlattice of $L \oplus R$ that contains L and R primitively.

Corollary 2.2. *Let M be an even unimodular lattice. We can determine whether a given lattice L is embedded primitively into M.*

By a *positive quadratic triple* of n-variables, we mean a triple $[Q, \lambda, c]$, where Q is a positive-definite $n \times n$ symmetric matrix with entries in $\mathbb{Q}$, λ is a column vector of length n with entries in $\mathbb{Q}$, and c is a rational number. An element of $\mathbb{R}^n$ is written as a row vector $v = [x_1, \dots, x_n]$. A positive quadratic triple $QT := [Q, \lambda, c]$ defines a quadratic function $F_{QT} : \mathbb{Q}^n \to \mathbb{Q}$ by

$$F_{QT}(v) := v\, Q\, {}^t v + 2\, v\, \lambda + c.$$

We have an algorithm to calculate the finite set

$$E(QT) := \{\ v \in \mathbb{Z}^n \ \mid\ F_{QT}(v) \le 0\ \}.$$

Let L be an even hyperbolic lattice. Then the space $\{x \in L \otimes \mathbb{R} \,|\, x^2 > 0\}$ has two connected components. Let $\mathcal{P}_L$ be one of them, and we call it a *positive cone* of L. Let $\mathrm{O}^+(L)$ denote the subgroup of $\mathrm{O}(L)$ of index 2 that preserves $\mathcal{P}_L$. For $v \in L \otimes \mathbb{R}$ with $v^2 < 0$, we put

$$(v)^\perp := \{\, x \in \mathcal{P}_L \;\mid\; \langle x, v \rangle = 0 \,\},$$

which is a real hyperplane of $\mathcal{P}_L$.

Suppose that L is a hyperbolic lattice, and that we are given vectors $h, v \in \mathcal{P}_L$. For a negative integer d, we can calculate the finite set

$$\{\, r \in L \;\mid\; \langle r, h \rangle > 0,\ \langle r, v \rangle < 0,\ \langle r, r \rangle = d \,\}.$$

By a *chamber*, we mean a closed subset

$$\{\, x \in \mathcal{P}_L \;\mid\; \langle x, v \rangle \geq 0 \ \text{ for all } \ v \in \Delta \,\}$$

of $\mathcal{P}_L$ with non-empty interior defined by a set Δ of vectors $v \in L \otimes \mathbb{R}$ with $v^2 < 0$. Let D be a chamber. A hyperplane $(v)^\perp$ of $\mathcal{P}_L$ is a *wall* of D if $(v)^\perp$ is disjoint from the interior of D and $(v)^\perp \cap D$ contains a non-empty open subset of $(v)^\perp$.

We put

$$\mathcal{R}_L := \{\, r \in L \;\mid\; r^2 = -2 \,\}.$$

Each $r \in \mathcal{R}_L$ defines a reflection $s_r : x \mapsto x + \langle x, r \rangle r$ into $(r)^\perp$, which is an element of $\mathrm{O}^+(L)$. We denote by $W(L)$ the subgroup of $\mathrm{O}^+(L)$ generated by all the reflections s_r with $r \in \mathcal{R}_L$. Then the closure in $\mathcal{P}_L$ of each connected component of

$$\mathcal{P}_L \setminus \bigcup\nolimits_{r \in \mathcal{R}_L} (r)^\perp$$

is a chamber, and is a standard fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$.

3. $K3$ surface

Example 3.1. Let A be an abelian surface, and let ι be the inversion $x \mapsto -x$ of A. Then the minimal resolution of the quotient $A/\langle \iota \rangle$ is a $K3$ surface, which is called the *Kummer surface associated with* A, and is denoted by $\mathrm{Km}(A)$.

Example 3.2. A plane curve B is a *simple sextic* if B is of degree 6 and has only simple singularities. Let B be a simple sextic. We denote by $Y_B \to \mathbb{P}^2$ the double covering branched along B. Then Y_B is a normal surface with only rational double points as its singularities, and the minimal resolution X_B of Y_B is a $K3$ surface.

3.1. Lattices associated with a $K3$ surface

Suppose that X is a $K3$ surface. Then $H^2(X, \mathbb{Z})$ with the cup product is an even unimodular lattice of signature $(3, 19)$, and hence is isomorphic to

$$U^{\oplus 3} \oplus E_8^{-\oplus 2},$$

where U is the hyperbolic plane with a Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and E_8^- is the negative definite root lattice of type E_8. The *Néron-Severi lattice*

$$S_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

of cohomology classes of divisors on X is an even hyperbolic lattice of rank ≤ 20. Moreover, as a sublattice of $H^2(X, \mathbb{Z})$, S_X is primitive.

We denote by T_X the orthogonal complement of S_X in $H^2(X, \mathbb{Z})$, and call it the *transcendental lattice* of X.

For various enumeration problems, the following corollary of the surjectivity of the period map is important:

Theorem 3.3. *Let S be a primitive hyperbolic sublattice of $U^{\oplus 3} \oplus E_8^{-\oplus 2}$. Then there exists a $K3$ surface X such that $S \cong S_X$.*

Therefore, when we are given an even hyperbolic lattice S, we can determine by Corollary 2.2 whether there exists a $K3$ surface X such that $S \cong S_X$.

3.2. Singular $K3$ surfaces

Definition 3.4. A $K3$ surface X is called *singular* if $\operatorname{rank}(S_X) = 20$.

If X is a singular $K3$ surface, then its transcendental lattice $T(X) := T_X$ is an even positive-definite lattice of rank 2.

Theorem 3.5 (Shioda and Inose [24])**.** (1) *The map $X \mapsto T(X) := T_X$ is a bijection from the set of isomorphism classes of singular $K3$ surfaces to the set of isomorphism classes of oriented positive-definite even lattices of rank* 2.

(2) *Every singular $K3$ surface X is isomorphic to a double cover of* $\mathrm{Km}(E \times E')$, *where E and E' are isogenous elliptic curves with complex multiplications determined by $T(X)$.*

In particular, every singular $K3$ surface X is defined over $\overline{\mathbb{Q}}$, and a Gram matrix of S_X is easily calculated from $T(X)$.

4. Polarizations of a $K3$ surface

Let X be a $K3$ surface. We denote by $\mathcal{P}(X)$ the positive cone of $S_X \otimes \mathbb{R}$ that contains the class of an ample divisor, that is, $\mathcal{P}(X)$ contains the class of a hyperplane section of X. We then put

$$N(X) := \{ x \in \mathcal{P}(X) \mid \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } X \}.$$

By Riemann-Roch theorem on X, we have the following. See [16], for example.

Proposition 4.1. *The closed subset $N(X)$ of $\mathcal{P}(X)$ is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}(X)$.*

More precisely, let h_0 be an interior point of $N(X)$. Then, for $r \in \mathcal{R}_{S_X}$ with $\langle r, h_0 \rangle > 0$, the hyperplane $(r)^\perp$ of $\mathcal{P}(X)$ is a wall of $N(X)$ if and only if r is the class of a smooth rational curve on X.

For $v \in S_X$, we denote by $\mathcal{L}_v \to X$ the corresponding line bundle.

Definition 4.2. Let d be an even positive integer. We say that a vector $h \in S_X$ is a *polarization of degree d* if $h^2 = d$ and the complete linear system $|\mathcal{L}_h|$ is non-empty and has no fixed-components.

Let h be a polarization of degree d. It is obvious that $h \in N(X)$. Since $|\mathcal{L}_h|$ is base-point free by [17], it defines a morphism Φ_h from X to a projective space of dimension $1 + d/2$. We denote by

$$X \xrightarrow{\phi_h} Y_h \xrightarrow{\psi_h} \mathbb{P}^{1+d/2}$$

the Stein factorization of Φ_h. By [3, 4], the normal surface Y_h has only rational double points as its singularities, and ϕ_h is a contraction of an ADE-configuration of smooth rational curves.

Example 4.3. In the situation of Example 3.2, we denote by

$$\rho_B : X_B \to Y_B \to \mathbb{P}^2$$

the composite of the minimal resolution and the double covering. Then the class h_B of the pull-back of a line on $\mathbb{P}^2$ by ρ_B is a polarization of degree 2, and Y_B is the projective model of (X_B, h_B).

Proposition 4.4. *The ADE-type of* $\operatorname{Sing}(Y_h)$ *is equal to the ADE-type of the root system* $\{r \in S_X \mid \langle h, r\rangle = 0, r^2 = -2\}$.

For the polarization of degree 2, we have the following:

Proposition 4.5. *Let $h \in S_X$ be a vector with $h^2 = 2$. Then h is a polarization of degree 2 if and only if $h \in N(X)$ and there exist no vectors $e \in S_X$ with $e^2 = 0$ and $\langle e, h\rangle = 1$.*

Suppose that we are given an ample class $h_0 \in S_X$. If we are given a vector $h \in S_X$ with $h^2 = 2$ and $\langle h, h_0\rangle > 0$, we can determine whether h is a polarization or not by calculating the sets

$$\{\ r \in S_X \ \mid\ \langle r, h_0\rangle > 0,\ \langle r, h\rangle < 0,\ r^2 = -2\ \},$$

and

$$\{\ e \in S_X \ \mid\ \langle e, h\rangle = 1,\ e^2 = 0\ \}.$$

Moreover, if h is a polarization of degree 2, then we can determine the ADE-type of the singularities of Y_h by Proposition 4.4.

5. Application to simple sextics

5.1. Configuration types of simple sextics

Definition 5.1. For a simple sextic B, we denote by R_B the ADE-type of $\operatorname{Sing} B$ and $\operatorname{degs} B$ the list of degrees of irreducible components of B. We say that B and B' are *of the same configuration type* and write $B \sim_{\mathrm{cfg}} B'$ if $\operatorname{degs} B = \operatorname{degs} B'$, $R_B = R_{B'}$, and their intersection patterns of irreducible components are same.

Let B be a simple sextic with the associated projective model Y_B of (X_B, h_B) as in Example 4.3. Let $\mathcal{E}_B$ be the set of exceptional curves of $X_B \to Y_B$, and let

$$\Sigma_B := \langle\, [E] \mid E \in \mathcal{E}_B \,\rangle \oplus \langle h_B\rangle \ \subset\ H^2(X_B, \mathbb{Z})$$

be the sublattice generated by the classes $[E]$ of $E \in \mathcal{E}_B$ and the polarization class h_B. Note that R_B is the ADE-type of the root system $\{[E] \mid E \in \mathcal{E}_B\}$. It is obvious that $B \sim_{\mathrm{cfg}} B'$ implies $\Sigma_B \cong \Sigma_{B'}$. We denote by

$$\overline{\Sigma}_B \ \subset\ H^2(X_B, \mathbb{Z})$$

the primitive closure of Σ_B. Then $\overline{\Sigma}_B$ must be primitively embedded into $H^2(X, \mathbb{Z})$, and satisfy $\mathcal{R}_{\overline{\Sigma}_B} = \mathcal{R}_{\Sigma_B}$.

After partial results of Urabe [26], Yang [28] classified all such $\overline{\Sigma}_B$ by computer, and made the complete list of configuration types of simple sextics. In particular, we see that the number of configuration types is 11159. See also Degtyarev [7].

5.2. Zariski pairs

We say that B and B' have *the same embedding topology* and write $B \sim_{\mathrm{emb}} B'$ if there exists a homeomorphism

$$\psi : (\mathbb{P}^2, B) \xrightarrow{\sim} (\mathbb{P}^2, B').$$

If $B \sim_{\mathrm{emb}} B'$, then $B \sim_{\mathrm{cfg}} B'$.

Definition 5.2. A *Zariski pair* of simple sextics is a pair $[B, B']$ of simple sextics such that $B \sim_{\mathrm{cfg}} B'$ but $B \not\sim_{\mathrm{emb}} B'$.

The notion of Zariski pairs was introduced by Artal [1], and many methods of constructing Zariski pairs are known. See the survey paper [2].

Let Θ_B denote the orthogonal complement of $\overline{\Sigma}_B$ in $H^2(X_B, \mathbb{Z})$.

Theorem 5.3. *If $B \sim_{\mathrm{emb}} B'$, then Θ_B and $\Theta_{B'}$ are isomorphic.*

Proof. In fact, Θ_B is a topological invariant of the open surface

$$U_B := \rho_B^{-1}(\mathbb{P}^2 \setminus B) \quad \subset \quad X_B,$$

because we have

$$\Theta_B \;\cong\; H^2(U_B, \mathbb{Z}) / \operatorname{Ker},$$

where $\operatorname{Ker} := \{v \in H^2(U_B) \mid \langle v, x \rangle = 0 \text{ for all } x \in H^2(U_B)\}$. Since U_B and $U_{B'}$ are homeomorphic if $B \sim_{\mathrm{emb}} B'$, Theorem 5.3 follows. □

Note that $(D_{\Theta_B}, q_{\Theta_B}) \cong (D_{\overline{\Sigma}_B}, -q_{\overline{\Sigma}_B})$. We consider the finite abelian group

$$G(B) := \overline{\Sigma}_B / \Sigma_B.$$

Corollary 5.4. *If $B \sim_{\mathrm{emb}} B'$, then we have $(D_{\overline{\Sigma}_B}, q_{\overline{\Sigma}_B}) \cong (D_{\overline{\Sigma}_{B'}}, q_{\overline{\Sigma}_{B'}})$. In particular, if $B \sim_{\mathrm{cfg}} B'$ and $|G(B)| \neq |G(B')|$, then $B \not\sim_{\mathrm{emb}} B'$.*

This corollary produces many examples of Zariski pairs of simple sextics. In fact, we can enumerate all Zariski pairs of this type. See [21].

Example 5.5. Let B be a simple sextic defined by $f^3 + g^2 = 0$, where f and g are general polynomials of degrees 2 and 3, respectively. Then $\operatorname{degs} B = [6]$, $R_B = 6A_2$. We see that $\pi_1(\mathbb{P}^2 \setminus B)$ is isomorphic to the free product $\mathbb{Z}/(2) * \mathbb{Z}/(3)$ of $\mathbb{Z}/(2)$ and $\mathbb{Z}/(3)$. Zariski [29] showed that there exists B' with $\operatorname{degs} B' = [6]$, $R_{B'} = 6A_2$ such that $\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/(2) \times \mathbb{Z}/(3)$. See also Oka [14].

For this pair, we have $G(B) \cong \mathbb{Z}/3\mathbb{Z}$ and $G(B') = 0$. The group $G(B)$ is generated by $[C] \bmod \Sigma_B$, where $[C] \in \overline{\Sigma}_B$ is the class of the conic C passing though the six cusps of B.

Example 5.6. We have three simple sextics of degree 6

$$B_1 = C_1 + Q_1, \quad B_2 = C_2 + Q_2, \quad B_4 = C_4 + Q_4,$$

where Q_i is a quartic curve with one tacnode and C_i is a smooth conic tangent to Q_i at two points with multiplicity 4, so that we have $\operatorname{degs} B_i = [2, 4]$ and $R_{B_i} = A_3 + 2A_7$. Let $\nu \colon E_i \to Q_i$ be the normalization of Q_i. Then E_i is of genus 1 and has four special points p, q, s, t such that $\nu(p) = \nu(q)$ is the tacnode of Q_i, and $\nu(s)$ and $\nu(t)$ are the intersection points with C_i. The order of $[p+q-s-t]$ in $\operatorname{Pic}^0(E_i)$ is 1, 2 and 4 according to $i = 1, 2, 4$. We have

$$G(B_1) \cong \mathbb{Z}/2\mathbb{Z}, \quad G(B_2) \cong \mathbb{Z}/4\mathbb{Z}, \quad G(B_4) \cong \mathbb{Z}/8\mathbb{Z}.$$

Hence they are topologically distinct.

5.3. Arithmetic Zariski pairs

Definition 5.7. A Zariski pair $[B, B']$ is said to be *arithmetic* if B and B' are defined over $\overline{\mathbb{Q}}$ and conjugate by some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

A simple sextic is said to be *maximizing* if its total Milnor number is 19. If B is a maximizing simple sextic, then X_B is a singular $K3$ surface with $T(X_B) \cong \Theta_B$, and can be defined over $\overline{\mathbb{Q}}$.

Theorem 5.8 (Schütt [18] and S. [20]). *Let X and X' be singular $K3$ surfaces defined over $\overline{\mathbb{Q}}$ such that $T(X)$ and $T(X')$ have isomorphic discriminant forms. Then there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $X' \cong X^\sigma$.*

Corollary 5.9. *Let B be a maximizing sextic defined over $\overline{\mathbb{Q}}$. If the genus containing $T(X_B)$ contains more than one isomorphism class of lattices, then there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $B \not\sim_{\mathrm{emb}} B^\sigma$.*

Example 5.10. We consider the configuration type of maximizing sextics $B = L + Q$, where Q is a quintic curve with one A_{10}-singular point, and L is a line tangent to Q at one point with multiplicity 5, so that $R_B = A_9 + A_{10}$ and $\deg s B = [5, 1]$. Such maximizing sextics are projectively isomorphic to

$$z \cdot (G(x, y, z) \pm \sqrt{5} \cdot H(x, y, z)) = 0,$$

where $G(x, y, z)$ and $H(x, y, z)$ are homogenizations of

$$\begin{aligned} g(x,y) &:= -9\,x^4 - 14\,x^3y + 58\,x^3 - 48\,x^2y^2 - 64\,x^2y \\ &\quad +10\,x^2 + 108\,xy^3 - 20\,xy^2 - 44\,y^5 + 10\,y^4, \\ h(x,y) &:= 5\,x^4 + 10\,x^3y - 30\,x^3 + 30\,x^2y^2 + \\ &\quad +20\,x^2y - 40\,xy^3 + 20\,y^5. \end{aligned}$$

respectively. The genus corresponding to $(D_{\overline{\Sigma}_B}, -q_{\overline{\Sigma}_B})$ and signature $(2, 0)$ (that is, the genus containing $T(X_B)$) consists of

$$\begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \qquad \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix},$$

and they correspond to the choice of the sign of $\sqrt{5}$ in the defining equation of B. See [19] for more examples.

6. Automorphism group

We have a natural homomorphism

$$\varphi_X\colon\ \mathrm{Aut}(X) \to \mathrm{O}(S_X).$$

It is known that this homomorphism has only a finite kernel. Sterk [25] proved that $\mathrm{Aut}(X)$ is finitely generated. We put

$$\mathrm{Aut}(N(X)) := \{\ g \in \mathrm{O}^+(S_X)\ \mid\ N(X)^g = N(X)\ \}.$$

It is obvious that the image of φ_X is contained in $\mathrm{Aut}(N(X))$. We regard a non-zero holomorphic 2-form ω_X on X as a vector of $T_X \otimes \mathbb{C}$, and put

$$C_X := \{\ g \in \mathrm{O}(T_X)\ \mid\ \omega_X^g = \lambda\,\omega_X\ \text{ for some }\ \lambda \in \mathbb{C}^\times\ \}.$$

Since $H := H^2(X, \mathbb{Z})$ is unimodular, the subgroup $H/(S_X \oplus T_X)$ of the discriminant group $D_{S_X} \oplus D_{T_X}$ of $S_X \oplus T_X$ is the graph of an isomorphism

$$\delta_{ST} \colon (D_{S_X}, q_{S_X}) \xrightarrow{\sim} (D_{T_X}, -q_{T_X}),$$

which induces an isomorphism

$$\delta_{ST*} \colon \mathrm{O}(q_{S_X}) \xrightarrow{\sim} \mathrm{O}(q_{T_X}).$$

In general, for an even lattice L, we have a natural homomorphism $\eta_L \colon \mathrm{O}(L) \to \mathrm{O}(q_L)$. Since $\mathrm{O}(q_{T_X})$ is finite, the subgroup

$$G_X := \{\, g \in \mathrm{O}^+(S_X) \mid \delta_{ST*}(\eta_{S_X}(g)) \in \eta_{T_X}(C_X) \,\}$$

of $\mathrm{O}^+(S_X)$ has finite index. As a corollary of the Torelli-type theorem due to Piatetski-Shapiro and Shafarevich [15], we have the following:

Theorem 6.1 (Piatetski-Shapiro and Shafarevich [15]). *The image of φ_X is equal to $\mathrm{Aut}(N(X)) \cap G_X$.*

6.1. Borcherds method

Therefore, the calculation of the image of φ_X is reduced to the following lattice-theoretic problem.

Problem 6.2. Suppose that the following objects are given: an even hyperbolic lattice S and a positive cone $\mathcal{P}(S)$ of $S \otimes \mathbb{R}$, a standard fundamental domain N of the action of $W(S)$ on $\mathcal{P}(S)$, and a subgroup G of $\mathrm{O}^+(S)$ with finite index. Calculate a finite set of generators of the group $\mathrm{Aut}(N) \cap G$.

Remark 6.3. The lattices for which $\mathrm{Aut}(N)$ is finite are classified by Nikulin [12, 13] and Vinberg [27]. Therefore we will be concerned with the cases where $\mathrm{Aut}(N)$ is infinite.

Let L_n be the even hyperbolic unimodular lattice of rank $n = 10, 18$ or 26. Then L_n is unique up to isomorphisms. A standard fundamental domain of the action of $W(L_n)$ on $\mathcal{P}(L_n)$ is called a *Conway chamber*. Let $\mathcal{D}$ be a Conway chamber. We say that a vector $w \in L_n$ is a *Weyl vector of* $\mathcal{D}$ if the set of walls of $\mathcal{D}$ is given by

$$\{\, (r)^\perp \mid r^2 = -2, \ \langle w, r\rangle = 1 \,\}.$$

Theorem 6.4 (Conway [5]). *A Weyl vector exists.*

In fact, Conway [5] gave an explicit description of Weyl vectors.

Example 6.5. Let U denote the hyperbolic plane and let Λ be the *negative-definite* Leech lattice. Then we have $L_{26} \cong U \oplus \Lambda$. Under this isomorphism, we denote vectors of L_{26} by (x, y, λ), where $(x, y) \in U$ and $\lambda \in \Lambda$. Then $w_0 := (1, 0, 0)$ is a Weyl vector of the Conway chamber

$$\mathcal{D}_0 := \{\, x \in \mathcal{P}(L_{26}) \mid \langle x, r_\lambda\rangle \geq 0 \ \text{ for any } \lambda \in \Lambda \,\},$$

where $r_\lambda := (-1 - \lambda^2/2, 1, \lambda) \in L_{26}$. Hence $\mathrm{Aut}(\mathcal{D}_0) \subset \mathrm{O}^+(L_{26})$ is isomorphic to the Conway group Co_∞.

We assume that S is embedded in L_n primitively, and that any element of G can be extended to an isometry of L_n. Moreover, when $n = 26$, we further assume that the orthogonal complement of S in L_{26} cannot be embedded into Λ.

A Conway chamber $\mathcal{D}$ is said to be *S-nondegenerate* if $D := \mathcal{D} \cap \mathcal{P}(S)$ contains a non-empty open subset of $\mathcal{P}(S)$. In this case, we say that D is an *induced chamber*. Since $\mathcal{P}(L_n)$ is tiled by Conway chambers, $\mathcal{P}(S)$ is tiled by induced chambers. Moreover, since $\mathcal{R}_S \subset \mathcal{R}_{L_n}$, the given standard fundamental domain N of the action of $W(S)$ on $\mathcal{P}(S)$ is a union of induced chambers. Two induced chambers D and D' are said to be *G-congruent* if there exists $g \in G$ such that $D' = D^g$.

Proposition 6.6 ([22]). (1) *The number of G-congruence classes of induced chambers is finite.* (2) *The number of walls of an induced chamber $D = \mathcal{D} \cap \mathcal{P}(S)$ is finite, and we can calculate the set of walls of D from the Weyl vector of $\mathcal{D}$.*

Hence $\mathrm{Aut}(D) \cap G = \{g \in G \mid D^g = D\}$ is finite for any induced chamber D. Moreover, for two induced chambers D and D', we can determine whether D and D' are G-congruent or not.

Borcherds method makes a complete set

$$\mathbb{D} := \{D_0, \dots, D_m\}$$

of representatives of all G-congruence classes of induced chambers contained in N. We start from an induced chamber D_0 contained in N, set $\Gamma := \{\}$ and $\mathbb{D} := [D_0]$, and proceed as follows. For an induced chamber $D_i \in \mathbb{D} = [D_0, \dots, D_k]$, we calculate the set of walls of D_i and the finite group $\mathrm{Aut}(D_i) \cap G$. We append a set of generators of $\mathrm{Aut}(D_i) \cap G$ to Γ. For each wall $(v)^\perp$ of D_i that is not a wall of N, we calculate the induced chamber D' adjacent to D_i along $(v)^\perp$, and determine whether D' is G-congruent to some $D_j \in \mathbb{D}$. If there are no such D_j, then we set $D_{k+1} := D'$ and append it to $\mathbb{D}$ as a representative of a new G-congruence class. If there exist $D_j \in \mathbb{D}$ and $g \in G$ such that $D' = D_j^g$, then we append g to Γ. We repeat this process until we reach the end of the list $\mathbb{D}$. By Proposition 6.6, this algorithm terminates. Then the group $\mathrm{Aut}(N) \cap G$ is generated by the elements in the finite set Γ.

Example 6.7 (Kondo [9]). Let C be a generic genus 2 curve, and $\mathrm{Jac}(C)$ its Jacobian. For $X = \mathrm{Km}(\mathrm{Jac}(C))$, we have $\mathrm{rank}(S_X) = 17$. The subgroup G_X is of index 32 in $\mathrm{O}^+(S_X)$. We have $\mathbb{D} = \{D_0\}$, and $|\mathrm{Aut}(D_0) \cap G_X| = 32$. The induced chamber D_0 has 316 walls, which are decomposed by the action of $\mathrm{Aut}(D_0) \cap G_X$ into 23 orbits as

$$316 = 32 \times 1 + 4 \times 15 + 32 \times 7.$$

The first orbit consists of 32 walls of $N(X)$. From the other 22 orbits, we obtain extra automorphisms. Hence the image of φ_X is generated by $\mathrm{Aut}(D_0) \cap G_X$ and 22 extra automorphisms.

Example 6.8. Let X be a $K3$ surface with $\mathrm{rank}(S_X) = 20$ and

$$T_X = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix},$$

which is unique up to isomorphisms. Then we have $|\mathbb{D}| = 1098$. The output Γ consists of 789 elements.

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