Automorphism groups of Enriques surfaces

(joint work with Simon Brandhorst)

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We explain an application of lattice theory to the study of geometry of Enriques/K3 surfaces.

We present a new method in the (computer-aided) calculation of automorphism groups and nef cones.

1. Goal
2. Naive method
3. Improvement
4. New results

"Vinberg" and "Conway" play important roles in this talk, as in Professor Mukai's talk on Monday.

This talk is intended to serve as an advertisement for computer-aided research of Enriques/K3 surfaces and, hopefully, of higher dimensional symplectic varieties.

Goal

For simplicity, we work over $\mathbb{C}$.

For a non-singular projective surface Z, we denote by S_Z the lattice of numerical equivalence classes of divisors on Z.

Let L_{10} be an even unimodular lattice of rank 10 with signature $(1, 9)$, which is unique up to isomorphism ($\cong U \oplus E_8$).

Suppose that Y is an Enriques surface. Then we have

$$S_Y \cong L_{10}.$$

Let $\mathcal{P}_Y \subset S_Y \otimes \mathbb{R}$ be the positive cone containing an ample class of Y. The *nef cone* of Y is defined by

$$N_Y := \{\, x \in \mathcal{P}_Y \mid \langle x, C \rangle \geq 0 \ \text{for all curves } C \text{ on } Y \,\}.$$

(More precisely, we should call it the nef-and-big cone of Y.)

We have a natural homomorphism $\mathrm{Aut}(Y) \to \mathrm{O}(S_Y, N_Y)$, where

$$\mathrm{O}(S_Y, N_Y) := \{ g \in \mathrm{O}(S_Y) \mid N_Y^g = N_Y \}.$$

We want to

- calculate a finite set of generators of $\mathrm{Aut}(Y)$ explicitly, and
- study the shape of $N_Y / \mathrm{Aut}(Y)$.

We formulate the second problem more precisely.

A lattice L is *hyperbolic* if its signature is $(1, \mathrm{rank}\, L)$. Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$, that is, $\mathcal{P}$ is one of the two connected components ofthe space of $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle > 0$. For a vector $v \in L \otimes \mathbb{Q}$ with $\langle v, v \rangle < 0$, we put

$$(v)^\perp := \{ x \in \mathcal{P} \mid \langle v, x \rangle = 0 \}.$$

A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r \rangle = -2$. A (-2)-vector $r \in L$ defines the reflection into the mirror $(r)^\perp$:

$$s_r \colon x \mapsto x + \langle x, r \rangle r.$$

The *Weyl group* $W(L)$ is defined by

$$W(L) := \langle\, s_r \mid r \text{ is a } (-2)\text{-vector} \,\rangle \quad \triangleleft \quad \mathrm{O}(L, \mathcal{P}).$$

A *standard fundamental domain* of $W(L)$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \setminus \bigcup (r)^\perp,$$

where r runs through the set of all (-2)-vectors.

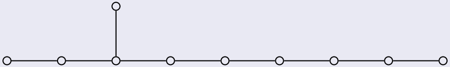
Then $W(L)$ acts on the set of standard fundamental domains simple-transitively, and we have

$$\begin{aligned} W(L) &= \langle\, s_r \mid \text{the hyperplane } (r)^\perp \text{ bounds } N \,\rangle, \\ \mathrm{O}(L, \mathcal{P}) &= W(L) \rtimes \mathrm{O}(L, N). \end{aligned}$$

Recall that $L_{10} :=$ an even unimodular hyperbolic lattice of rank 10.

Theorem (Vinberg)

A standard fumdamental domain of $W(L_{10})$ is bounded by 10 hyperplanes $(r_1)^{\perp}, \dots, (r_{10})^{\perp}$ defined by (-2)-vectors $r_1, \dots, r_{10}$ that form the dual graph below. Since this graph has no non-trivial symmetries, we have $\mathrm{O}(L_{10}, \mathcal{P}) = W(L_{10})$.



We call a standard fumdamental domain of $W(L_{10})$ a **Vinberg chamber**. The positive cone $\mathcal{P}$ of L_{10} is tessellated by Vinberg chambers, in such a way that each Vinberg chamber has 10 adjacent Vinberg chambers.

Let Y be an Enriques surface, so that

$$S_Y \cong L_{10}.$$

The nef cone N_Y is a union of Vinberg chambers, and the action of $\mathrm{Aut}(Y)$ preserves the tessellation of N_Y by Vinberg chambers. Hence $\mathrm{Aut}(Y)$ acts on the set of Vinberg chambers in N_Y.

Our goal is to **calculate a complete set of representatives of this action**.

If this task is done, then we can calculate the sets

$$\begin{aligned} \mathcal{R}(Y) &:= \text{the set of smooth rational curves on } Y\text{, and} \\ \mathcal{E}(Y) &:= \text{the set of elliptic fibrations } Y \to \mathbb{P}^1 \end{aligned}$$

modulo the action of $\mathrm{Aut}(Y)$.

Naive method

We give a general elementary algorithm.

Let (V, E) be a simple non-oriented *connected* graph, where

- V is the set of vertices and,
- E is the set of edges, which is a set of non-ordered pairs of distinct elements of V (no orientation, no multiple edges, and every edge has two distinct end-points).

The set V may be infinite, but we assume the following *local effectiveness* property:
For any $v \in V$, the set

$$\mathrm{adj}(v) := \{ v' \in V \mid \{v, v'\} \in E \}$$

is finite, and can be calculated effectively.

Suppose that a group G (possibly infinite) acts on the graph (V, E) from the right. We assume the following local effectiveness properties on G:

1. For any $v, v' \in V$, we can determine effectively whether
$$T_G(v, v') := \{ g \in G \mid v^g = v' \}$$
is empty or not, and when $T_G(v, v') \neq \emptyset$, we can calculate an element $g \in T_G(v, v')$.
2. For any $v \in V$, the stabilizer subgroup $T_G(v, v)$ of v in G is finitely generated, and a finite set of generators of $T_G(v, v)$ can be calculated effectively.

Our goal is to calculate

- a finite generating set of the group G, and
- a complete set of representatives of the orbits V/G.

Let $\sim$ denote the G-equivalence relation: $v \sim v' \iff T_G(v, v') \neq \emptyset$.
Let $V_0 \subset V$ be a non-empty finite subset with the following properties:

(A) If $v, v' \in V_0$ and $v \neq v'$, then $v \not\sim v'$.

(B) We put

$$\widetilde{V}_0 := \{ v \in V \mid v \text{ is adjacent to a vertex } v' \in V_0 \}.$$

Then, for each $v \in \widetilde{V}_0$, there is a vertex $v' \in V_0$ such that $v \sim v'$. Note that v' is unique for each $v \in \widetilde{V}_0$ by Property (A).

For each $v \in \widetilde{V}_0 - V_0$, we choose an element $h(v) \in T_G(v, v')$, where $v' \in V_0$ satisfies $v \sim v'$, and put $\mathcal{H} := \{ h(v) \mid v \in \widetilde{V}_0 - V_0 \} \subset G$.

Proposition

Let v_0 be an element of V_0. The natural mapping

$$V_0 \hookrightarrow V \twoheadrightarrow V/\!\sim\, = V/G$$

is a bijection, and the group G is generated by $T_G(v_0, v_0) \cup \mathcal{H}$.

Proof. Let $\langle\mathcal{H}\rangle \subset G$ be the subgroup generated by $\mathcal{H}$. First we show

$$(*) \qquad \forall v \in V \;\; \exists h \in \langle\mathcal{H}\rangle \;\text{ such that }\; v^h \in V_0.$$

Let a vertex $v \in V$ be fixed. A sequence

$$v_{(0)}, v_{(1)}, \dots, v_{(l)}$$

of vertices is a *path from V_0 to the orbit $v^{\langle\mathcal{H}\rangle}$* if

- $v_{(i-1)}$ and $v_{(i)}$ are adjacent for $i = 1, \dots, l$,
- the starting vertex $v_{(0)}$ is in V_0, and
- the ending vertex $v_{(l)}$ belongs to the orbit $v^{\langle\mathcal{H}\rangle}$ of v by $\langle\mathcal{H}\rangle$.

Since (V, E) is connected, there is at least one path from V_0 to $v^{\langle\mathcal{H}\rangle}$. Suppose that we have a path from V_0 to $v^{\langle\mathcal{H}\rangle}$ of length $l > 0$. Since $v_{(1)}$ is adjacent to $v_{(0)} \in V_0$, we have $v_{(1)} \in \widetilde{V}_0$ and obtain $h_1 := h(v_{(1)}) \in \mathcal{H}$ that maps $v_{(1)}$ to a vertex in V_0.

Then

$$v_{(1)}^{h_1}, \dots, v_{(l)}^{h_1}$$

is a path from V_0 to $v^{\langle \mathcal{H} \rangle}$ of length $l-1$. Thus we obtain a path from V_0 to $v^{\langle \mathcal{H} \rangle}$ of length 0, which implies the claim $(*)$.

The injectivity of $V_0 \to V/G$ follows from Property (A) of V_0. The surjectivity follows from the claim above.

Suppose that $g \in G$. By the claim, there is an element $h \in \langle \mathcal{H} \rangle$ such that $v_0^{gh} \in V_0$. By Property (A), we have $v_0 = v_0^{gh}$ and hence $gh \in T_G(v_0, v_0)$. Therefore G is generated by the union of $\mathcal{H}$ and $T_G(v_0, v_0)$. □

We can calculate V_0 and $\mathcal{H}$ by the following procedure. This procedure terminates if and only if $|V/G| < \infty$.

Initialize $V_0 := [v_0]$, $\mathcal{H} := \{\}$, and $i := 0$.
while $i < |V_0|$ **do**
  Let v_i be the $(i+1)$st entry of the list V_0.
  Let $\mathrm{adj}(v_i)$ be the set of vertices adjacent to v_i.
  for each vertex v' in $\mathrm{adj}(v_i)$ **do**
    Set `flag` $:= \mathrm{true}$.
    for each v'' in V_0 **do**
      if $T_G(v', v'') \neq \emptyset$ **then**
        Add an element h of $T_G(v', v'')$ to $\mathcal{H}$.
        Replace `flag` by false.
        Break from the innermost for–loop.
    if `flag` $= \mathrm{true}$ **then**
      Append v' to the list V_0 as the last entry.
  Replace i by $i+1$.

Let Y be an Enriques surface. We apply the algorithm above to

$$\begin{aligned} V &= \text{the set of Vinberg chambers in } N_Y, \\ E &= \text{the usual adjacency relation of chambers}, \\ G &= \text{the image of } \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, N_Y). \end{aligned}$$

We check the local effectiveness properties.

Let $X \to Y$ be the universal covering of Y. Then X is a $K3$ surface, and we have a primitive embedding

$$S_Y(2) \hookrightarrow S_X,$$

where $S_Y(2)$ is the lattice obtained from S_Y by multiplying $\langle\ ,\ \rangle$ by 2. Let $\mathcal{P}_X \subset S_X \otimes \mathbb{R}$ be the positive cone containing an ample class and $N_X \subset \mathcal{P}_X$ the nef cone of X. We regard $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$. Then we have

$$N_Y = N_X \cap \mathcal{P}_Y.$$

Let $a \in S_Y$ be an ample class of Y. Then a is an ample class of X by $S_Y(2) \hookrightarrow S_X$. By Riemann-Roch, we have the following:

Proposition

The nef-cone N_X is equal to the standard fundamental domain of $W(S_X)$ containing a.

Hence a vector $v \in S_X \cap \mathcal{P}_X$ belongs to N_X if and only if the *set of separating* (-2)*-vectors*

$$\mathrm{Sep}_X(a, v) := \{\, r \in S_X \mid \langle r, r \rangle = -2, \ \ \langle r, a \rangle \cdot \langle r, v \rangle < 0 \,\}$$

is empty. We have an algorithm to calculate this set.

Since $N_Y = N_X \cap \mathcal{P}_Y$, a Vinberg chamber $D' \subset \mathcal{P}_Y$ is contained in N_Y if and only if $\mathrm{Sep}_X(a, v') = \emptyset$ for an interior point v' of D'. Hence we can determine whether $D' \in V$ or not.

Thus the local effectiveness for (V, E) holds.

For simplicity, we assume that $\operatorname{rank} S_X < 20$ and that the period ω of X is general enough so that

$$\{ g \in \mathrm{O}(T_X) \mid \omega^g \in \mathbb{C}\omega \} = \{\pm 1\},$$

where T_X is the transcendental lattice of X. (That is, X is very general in the moduli of lattice polarized $K3$ surfaces.)
For Vinberg chambers D, D' in N_Y, then there is a unique isometry $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ such that $D^g = D'$. By Torelli theorem for $K3$ surfaces, we have the following:

Proposition

An isometry $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ belongs to $G = \mathrm{Im}(\mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y))$ if and only if $\mathrm{Sep}(a, a^g) = \emptyset$ and g lifts to an isometry $\tilde{g}$ of S_X that acts as ± 1 on the discriminant group of S_X.

Hence the local effectiveness for G holds. **Thus we can apply the general algorithm, and calculate a complete set of representatives for V/G and a finite set of generators of G.**

This naive method does not work

Let Y be a *generic* Enriques surface. Since Y has no smooth rational curves, we have $N_Y = \mathcal{P}_Y$, and hence V is the set of *all* Vinberg chambers.

Theorem (Barth-Peters (1983))

The fundamental domain of the action of $\mathrm{Aut}(Y)$ *on the cone* $N_Y = \mathcal{P}_Y$ *is a union of*

$$|\mathrm{O}(L_{10} \otimes \mathbb{F}_2)| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 = 46998591897600$$

copies of Vinberg chambers.

Therefore we have $|V/G| = 46998591897600$, and hence we have to go through the `while`–loop about 47×10^{12} times.

Definition

We define the Barth-Peters number by

$$1_{\mathrm{BP}} := 46998591897600.$$

Improvement

To overcome this difficulty, we employ *Borcherds' method*; we study a lattice by embedding it in L_{26}.

Let L_{26} be an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. The standard fumdamental domain of $W(L_{26})$ was determined by Conway.

The lattice L_{26} is written as an orthogonal direct sum

$$U \oplus (\text{an even unimodular negative-definite lattice of rank } 24).$$

A vector $\mathbf{w} \in L_{26}$ is called a *Weyl vector* if $\mathbf{w}$ is written as $(1, 0, \mathbf{0})$ in a decomposition

$$L_{26} = U \oplus \Lambda,$$

where Λ is the **Leech lattice**. We fix a positive cone $\mathcal{P} \subset L_{26} \otimes \mathbb{R}$, and a Weyl vector $\mathbf{w}$ contained in the boundary $\partial \overline{\mathcal{P}}$ of $\mathcal{P}$.

A (-2)-vector $r \in L_{26}$ is a *Leech root* with respect to $\mathbf{w}$ if $\langle \mathbf{w}, r \rangle = 1$.
Under the decomposition $L_{26} = U \oplus \Lambda$ with $\mathbf{w} = (1, 0, \mathbf{0})$, Leech roots are written as

$$r_\lambda := \left(-\frac{\lambda^2}{2} - 1, 1, \lambda \right), \quad \text{where } \lambda \in \Lambda.$$

Theorem (Conway)

There is a bijection

$$\mathbf{w} \longleftrightarrow N_{\mathbf{w}}$$

between the set of Weyl vectors $\mathbf{w}$ *and the set of standard fundamental domains* $N_{\mathbf{w}}$ *of* $W(L_{26})$ *in such a way that* $N_{\mathbf{w}}$ *is bounded by* $(r_\lambda)^\perp$*, where* r_λ *are the Leech roots with respect to* $\mathbf{w}$*.*

Definition

We call a standard fundamental domain of L_{26} a *Conway chamber*.

Borchers method for $L_{10}(2)$

Let $L_{10}(2)$ denote the lattice obtained from L_{10} by multiplying the bilinear form $\langle\ ,\ \rangle$ by 2. We have $\mathrm{O}(L_{10}(2)) = \mathrm{O}(L_{10})$.

Theorem (S. and Brandhorst)

Up to the action of $\mathrm{O}(L_{10})$ and $\mathrm{O}(L_{26})$, there exist exactly 17 primitive embeddings of $L_{10}(2)$ into L_{26}.

$$\mathtt{12A}, \mathtt{12B}, \mathtt{20A}, \dots, \mathtt{20F}, \mathtt{40A}, \dots, \mathtt{40E}, \mathtt{96A}, \dots, \mathtt{96C}, \mathtt{infty}.$$

Recall that the positive cone $\mathcal{P}_{L_{26}}$ of L_{26} is tessellated by Conway chambers. Hence an embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ such that $\iota(\mathcal{P}_{L_{10}}) \subset \mathcal{P}_{L_{26}}$ induces a tessellation of $\mathcal{P}_{L_{10}}$ by **induced chambers**

$$\iota^{-1}(\mathcal{C}) = \mathcal{P}_{L_{10}} \cap \mathcal{C},$$

where $\mathcal{C}$ are Conway chambers such that $\iota^{-1}(\mathcal{C})$ contains a non-empty open subset of $\mathcal{P}_{L_{10}}$.

Theorem (S. and Brandhorst)

Except for the embedding of type `infty`, *the following hold.*

- *The induced chambers on $\mathcal{P}_{L_{10}}$ are isomorphic to each other under the action of $\mathrm{O}(L_{10}, \mathcal{P}_{L_{10}})$.*
- *Each induced chamber D is bounded by a finite number of walls $D \cap (r)^{\perp}$, and each wall $D \cap (r)^{\perp}$ is defined by a (-2)-vector r of L_{10}. (The name of the embedding indicates the number of walls.)*
- *Moreover, for each wall $D \cap (r)^{\perp}$, the reflection s_r maps D to the induced chamber adjacent to D across the wall $D \cap (r)^{\perp}$.*

By the second assertion, each induced chamber is tessellated by Vinberg chambers. The volume of an induced chamber is defined to be the number of Vinberg chambers contained in the induced chamber.

For the proof, we use the mass formula for positive definite lattices in a genus.

17 embeddings

No.	name	volume (by BP)	$\lvert\mathrm{aut}\rvert$	isom	NK
1	12A	1/174182400	2^2		I
2	12B	1/3870720	$2^3 \cdot 3$		II
3	20A	1/725760	$2^3 \cdot 3$		V
4	20B	1/322560	2^6		III
5	20C	1/60480	$2^3 \cdot 3 \cdot 5$	20D	VII
6	20D	1/60480	$2^3 \cdot 3 \cdot 5$	20C	VII
7	20E	1/51840	$2^3 \cdot 3 \cdot 5$		VI
8	20F	1/23040	$2^6 \cdot 5$		IV
9	40A	1/5760	$2^7 \cdot 3$		
10	40B	1/2520	$2^7 \cdot 3^2$	40C	
11	40C	1/2520	$2^7 \cdot 3^2$	40B	
12	40D	1/1440	$2^5 \cdot 3^2 \cdot 5$	40E	
13	40E	1/1440	$2^5 \cdot 3^2 \cdot 5$	40D	
14	96A	1/288	$2^{13} \cdot 3$		
15	96B	1/72	$2^{12} \cdot 3^3$	96C	
16	96C	1/72	$2^{12} \cdot 3^3$	96B	
17	infty	∞			

Rough idea

We construct a primitive embedding

$$S_X \hookrightarrow L_{26}$$

in such a way that the volume of the induced chamber of

$$S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$$

is large (for example, of type `96B` or `96C`). Instead of using Vinberg chambers of S_Y, we use the induced chambers of $S_Y(2) \hookrightarrow L_{26}$.
Then we can reduce the number of $|V/G|$, and complete the execution of the algorithm in a practical time.
(We also have to take care of automorphisms of induced chambers.)

For example, for Barth-Peters generic Enriques surfaces, by using the embedding `96C`, we can complete the algorithm by going through the `while`–loop only about 72 (+ contribution from the boundary) times.

Main results

We need the notion of $(\tau, \bar{\tau})$-generic Enriques surfaces to state the main results, where τ and $\bar{\tau}$ are ADE-types of the same rank. Since we have no time, we only give examples.

Examples

- The generic Enriques surface of Barth-Peters is $(0, 0)$-generic.
- A general nodal Enriques surface is (A_1, A_1)-generic. More generally, if Y is an Enriques surface that is very general in the moduli of Enriques surfaces containing n disjoint smooth rational curves, then Y is (nA_1, nA_1)-generic.
- If Y is very general in the moduli of Enriques surfaces containing two smooth rational curves whose dual graph is ○—○, then Y is (A_2, A_2)-generic. We say that such an Enriques surface Y is *general cuspidal*.

Volume formula

We calculate the volume $\mathrm{vol}(N_Y/\operatorname{Aut}(Y))$ to be the number of orbits $|V/G|$. Recall that $1_{\mathrm{BP}} := 46998591897600$.

Theorem (S. and Brandhorst)

Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface. Then we have

$$\mathrm{vol}(N_Y/\operatorname{Aut}(Y)) = |V/G| = \frac{c_{(\tau,\bar{\tau})}}{|W(R_\tau)|} \cdot 1_{\mathrm{BP}},$$

where $W(R_\tau)$ is the Weyl group of type τ, and $c_{(\tau,\bar{\tau})} \in \{1, 2\}$ is the number of numerically trivial automorphisms of Y, that is, the size of the kernel of $\rho\colon \operatorname{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)$.

Example

- If Y is generic, then $|V/G| = 1_{\mathrm{BP}}$. This is the definition of 1_{BP}.
- If Y is general nodal, then $|V/G| = 1_{\mathrm{BP}}/2$.
  If Y is general n-nodal, then $|V/G| = 1_{\mathrm{BP}}/2^n n!$ for $n \leq 8$.
- If Y is general cuspidal, then $|V/G| = 1_{\mathrm{BP}}/6$.

There are two good things about this formula.

- We have a proof that **does not use computer**.
- We can make an explicit list of representatives of V/G, and hence we can confirm the formula by computer.

We can calculate the sets

$$\begin{aligned} \mathcal{R}(Y) &:= \text{the set of smooth rational curves on } Y, \text{ and} \\ \mathcal{E}(Y) &:= \text{the set of elliptic fibrations } Y \to \mathbb{P}^1 \end{aligned}$$

modulo the action of $\mathrm{Aut}(Y)$.

Example

- If Y is general nodal, then $|\mathcal{R}(Y)/\mathrm{Aut}(Y)| = 1$. This had been proved by Cossec-Dolgachev.
- If Y is general n-nodal with $n \le 6$, then $|\mathcal{R}(Y)/\mathrm{Aut}(Y)| = n$.
- If Y is general cuspidal, then $|\mathcal{R}(Y)/\mathrm{Aut}(Y)| = 1$.
- ...

Theorem (Barth-Peters)

Let Y be a generic Enriques surface. Then $|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 527$.

We generalize this theorem as follows:

Theorem (S. and Brandhorst)

Let Y be a general nodal Enriques surface. Then

$$|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 136 + 255.$$

In the representatives of elements of $\mathcal{E}(Y)/\operatorname{Aut}(Y)$,
136 *elliptic fibrations have no reducible fibers, and*
255 *elliptic fibrations have one non-multiple reducible fiber of type* A_1.

. . . .

Our preprints are available from:

Borcherds' method for Enriques surfaces
Simon Brandhorst, Ichiro Shimada
arXiv:1903.01087

Automorphism groups of certain Enriques surfaces
Simon Brandhorst, Ichiro Shimada
arXiv:2012.10622

Thank you very much for listening!