Conditions for Robustness to Nonnormality of Test Statistics in a GMANOVA Model

Hirokazu Yanagihara

Department of Social Systems and Management
Graduate School of Systems and Information Engineering
University of Tsukuba
1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan
E-mail : yanagi@sk.tsukuba.ac.jp

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Abstract

This paper discusses the conditions for robustness to the nonnormality of three test statistics for a general multivariate linear hypothesis, which were proposed under the normal assumption in a generalized multivariate analysis of variance (GMANOVA) model. Although generally the second terms in the asymptotic expansions of the mean and variance of the test statistics consist of skewness and kurtosis of an unknown population's distribution, we find conditions where the skewness and kurtosis disappear from the second term in the asymptotic expansions under any distributions. When such conditions are satisfied, the Bartlett correction and the modified Bartlett correction in the normal case improve the chi-square approximation even under nonnormality. By using these conditions, it is possible to see whether the used test statistic is robust to nonnormality or not.

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1. Introduction

Over the past few decades, many authors have investigated the influences of nonnormality in several statistical test procedures proposed under the normal assumption, and we have been able to obtain many results from these studies. For the test size (or the significant level), it is well known that Hotelling's two-sample test is robust to nonnormality and Hotelling's one-sample test is not. Chase and Bulge (1971), and Everett (1979) obtained this fact through numerical experiments. Figure 1 shows the numerical results of actual test sizes of the two test statistics in the case of sample size $n = 20$ and the dimension of observation $p = 2$ when we use six distributions as the population distributions (i.e., the true distributions). The six distributions used are normal, Laplace, uniform, skew-Laplace, chi-square and log-normal distributions (for details of the setting of true distributions in numerical studies, see Appendix A.1). This figure shows actual test sizes under those true distributions when we believe that the data came from the normal distribution, i.e., we use the F-distribution as the null distribution. From this figure, we can say that the difference of the actual test sizes of the two-sample test by the difference in distributions is small, but the one-sample test is not. In the case of the one-sample test, as the skewness of the true distributions becomes large, the differences between the actual and nominal test sizes grow. If the 10% test size becomes the 15 % or 25 % test size, which we believe happens, we cannot say that this will be used.

Insert Figure 1 around here

There is a large difference in the influence of nonnormality of two test statistics; however, the statistical model in both tests is essentially a multivariate linear model. What is the difference between these two? In this paper, we give the theoretical answer for this question, but not the numerical answer, which has not yet been obtained. There is a serious difference between the two tests in the second term of an asymptotic expansion of the mean of the test statistic. Although there are skewnesses of the true distribution in the second term of such an expansion of the one-sample test statistic, there is no skewness to that of the one-sample test statistic. In this paper, we set the test statistic as robust to nonnormality when there are not cumulants denoting nonnormality in the second term of an asymptotic expansion of its moments.

We extend the considered test statistic to the test statistic for a general multivariate linear hypothesis in the generalized multivariate analysis of variance (GMANOVA) model (Potthoff & Roy, 1964) from Hotelling's one- and two-sample test statistics, and search for conditions where the cumulants denoting nonnormality of the true distribution disappear from the second term of an asymptotic expansion of the mean of the test statistic. In addition, there is a condition where the cumulants disappear from the second term in an asymptotic expansion of the variance of the test statistic. These conditions are made clearer through the coefficients of an asymptotic expansion of the null distribution of the test statistic. By using these conditions, it is possible to see whether or not the used test statistic is robust to nonnormality. Moreover, when these conditions are satisfied, the Bartlett correction (Bartlett, 1937) or the modified Bartlett correction (Fujikoshi, 2000) in the normal case can give an improvement of the chi-square approximation under any distributions. It is very important to improve the chi-square approximation without estimating skewnesses and kurtosis, because it is difficult to obtain good estimators of skewness and kurtosis without an adequate sample size. Especially, the bias of the ordinary estimator of kurtosis proposed by Mardia (1970) becomes large unless the sample size is huge (see Yanagihara, 2006).

The present paper is organized as follows. In Section 2, we state asymptotic expansions, obtained by Yanagihara (2001), to the null distributions of the three test statistics in a GMANOVA model. By using the coefficients of these expansions, the conditions under which the test statistics are robust to nonnormality become clear in Section 3. Then, we confirm that when these conditions are satisfied, the Bartlett correction and the modified Bartlett correction in the normal case can give an improvement of the chi-square approximation under any distributions. We conclude our discussion in Section 4. Technical details are provided in the Appendix.

2. Asymptotic Expansions of Null Distributions of Test Statistics in the GMANOVA Model

The GMANOVA model proposed by Potthoff and Roy (1964) is defined by

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\Xi}\boldsymbol{X}' + \boldsymbol{\mathcal{E}}\boldsymbol{\Sigma}^{1/2}, \tag{2.1}$$

where $\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)'$ is an $n \times p$ observation matrix of response variables, $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)'$ is an $n \times k$ between-individuals design matrix of explanatory variables with the full rank k $(< n)$, $\boldsymbol{X}$ is a $p \times q$ within-individuals design matrix of explanatory variables with the full rank q $(\leq p)$, $\boldsymbol{\Xi}$ is a $k \times q$ unknown parameter matrix, and $\boldsymbol{\mathcal{E}} = (\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n)'$ is an $n \times p$ unobserved error matrix. It is assumed that $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$ are independent random vectors from $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_p)'$, and the true distribution of $\boldsymbol{\varepsilon}$ is distributed following an unknown distribution with a mean $\mathrm{E}[\boldsymbol{\varepsilon}] = \boldsymbol{0}_p$ and covariance matrix $\mathrm{Cov}[\boldsymbol{\varepsilon}] = \boldsymbol{I}_p$, where $\boldsymbol{0}_p$ is a $p \times 1$ vector, all of whose elements are 0. This model can frequently be applied to the analysis of growth curve data, and therefore it is also called the growth curve model.

We consider testing for a general linear hypothesis as

$$H_0 : \boldsymbol{C}\boldsymbol{\Xi}\boldsymbol{D} = \boldsymbol{O}_{c \times d}, \tag{2.2}$$

where $\boldsymbol{C}$ is a known $c \times k$ matrix with the rank c $(\leq k)$, $\boldsymbol{D}$ is a known $q \times d$ matrix with the rank d $(\leq q)$, and $\boldsymbol{O}_{c \times d}$ is a $c \times d$ matrix, all of whose elements are 0. Let $\boldsymbol{S}_h$ and $\boldsymbol{S}_e$ be the variation matrices due to the hypothesis and the error respectively, i.e.,

$$\boldsymbol{S}_h = (\boldsymbol{C}\widehat{\boldsymbol{\Xi}}\boldsymbol{D})'(\boldsymbol{C}\boldsymbol{R}\boldsymbol{C}')^{-1}(\boldsymbol{C}\widehat{\boldsymbol{\Xi}}\boldsymbol{D}), \quad \boldsymbol{S}_e = \boldsymbol{D}'(\boldsymbol{X}'\boldsymbol{S}^{-1}\boldsymbol{X})^{-1}\boldsymbol{D},$$

where

$$\begin{aligned}\widehat{\boldsymbol{\Xi}} &= (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{Y}\boldsymbol{S}^{-1}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{S}^{-1}\boldsymbol{X})^{-1},\\ \boldsymbol{R} &= (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\left(\boldsymbol{I}_n + \boldsymbol{Y}\boldsymbol{S}^{-1}\boldsymbol{Y}'\right)\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1} - \widehat{\boldsymbol{\Xi}}(\boldsymbol{X}'\boldsymbol{S}^{-1}\boldsymbol{X})\widehat{\boldsymbol{\Xi}}',\end{aligned}$$

and $\boldsymbol{S} = \boldsymbol{Y}'(\boldsymbol{I}_n - \boldsymbol{P}_{\boldsymbol{A}})\boldsymbol{Y}$. Here, $\boldsymbol{P}_{\boldsymbol{A}}$ is the projection matrix to the linear space $\mathbb{R}(\boldsymbol{A})$ generated by the column vectors of $\boldsymbol{A}$, i.e., $\boldsymbol{P}_{\boldsymbol{A}} = \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'$. Then, the three criteria proposed for testing (2.2) under normality, in particular, are as follows.

$$\begin{aligned}&(i)\ \text{the likelihood ratio statistic (LR):}\\ &\qquad T_{\mathrm{LR}} = -\{n-k-(p-q)\}\log(|\boldsymbol{S}_e|/|\boldsymbol{S}_e + \boldsymbol{S}_h|),\\ &(ii)\ \text{the Lawley-Hotelling trace criterion (HL):}\\ &\qquad T_{\mathrm{HL}} = \{n-k-(p-q)\}\mathrm{tr}(\boldsymbol{S}_h\boldsymbol{S}_e^{-1}),\\ &(iii)\ \text{the Bartlett-Nanda-Pillai trace criterion (BNP):}\\ &\qquad T_{\mathrm{BNP}} = \{n-k-(p-q)\}\mathrm{tr}\left\{\boldsymbol{S}_h(\boldsymbol{S}_h + \boldsymbol{S}_e)^{-1}\right\},\end{aligned} \tag{2.3}$$

(for these results, see e.g., von Rosen, 1991; Fujikoshi, 1993; Srivastava & von Rosen, 1999). By changing the known coefficient matrices for hypotheses $\boldsymbol{C}$ and $\boldsymbol{D}$, it is possible to express several linear hypotheses. For these examples, see Kshirsagar and Smith (1995, p. 40) and Yanagihara (2001).

In order to simplify the results of the asymptotic expansions in Yanagihara (2001), we state some notations of the moments and cumulants of the true distribution, and some projection matrices. Let $\mu_{i_1\cdots i_m}$ be an mth multivariate moment of $\boldsymbol{\varepsilon}$ defined by

$$\mu_{i_1\cdots i_m} = \mathrm{E}[\varepsilon_{i_1}\cdots\varepsilon_{i_m}].$$

Similarly, the corresponding mth multivariate cumulant of $\boldsymbol{\varepsilon}$ is denoted by $\kappa_{i_1\cdots i_m}$, e.g.,

$$\kappa_{abc} = \mu_{abc}, \quad \kappa_{abcd} = \mu_{abcd} - \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc},$$

where δ_{ab} is the Kronecker delta, i.e., $\delta_{aa} = 1$ and $\delta_{ab} = 0$ for $a \neq b$. Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be independent random vectors from $\boldsymbol{\varepsilon}$. We define the multivariate skewness and kurtosis of the transformed $\boldsymbol{\varepsilon}$ by the symmetric $p \times p$ matrices $\boldsymbol{H}$, $\boldsymbol{L}$ and $\boldsymbol{M}$, whose (a, b)th elements are h_{ab}, l_{ab} and m_{ab}, respectively, as

$$\begin{aligned}
\kappa(\boldsymbol{H}, \boldsymbol{M}) &= \mathrm{E}_{\boldsymbol{e}_1}\left[(\boldsymbol{e}_1'\boldsymbol{H}\boldsymbol{e}_1)(\boldsymbol{e}_1'\boldsymbol{M}\boldsymbol{e}_1)\right] - \{\mathrm{tr}(\boldsymbol{H})\mathrm{tr}(\boldsymbol{M}) + 2\mathrm{tr}(\boldsymbol{H}\boldsymbol{M})\} \\
&= \sum_{a,b,c,d}^{p} \kappa_{abcd} h_{ab} m_{cd}, \\
\gamma_1^2(\boldsymbol{H}, \boldsymbol{L}, \boldsymbol{M}) &= \mathrm{E}_{\boldsymbol{e}_1}\mathrm{E}_{\boldsymbol{e}_2}\left[(\boldsymbol{e}_1'\boldsymbol{H}\boldsymbol{e}_2)(\boldsymbol{e}_1'\boldsymbol{L}\boldsymbol{e}_2)(\boldsymbol{e}_1'\boldsymbol{M}\boldsymbol{e}_2)\right] \\
&= \sum_{a,b,c,d,e,f}^{p} \kappa_{abc}\kappa_{def} h_{ad} l_{be} m_{cf}, \\
\gamma_2^2(\boldsymbol{H}, \boldsymbol{L}, \boldsymbol{M}) &= \mathrm{E}_{\boldsymbol{e}_1}\mathrm{E}_{\boldsymbol{e}_2}\left[(\boldsymbol{e}_1'\boldsymbol{H}\boldsymbol{e}_1)(\boldsymbol{e}_1'\boldsymbol{L}\boldsymbol{e}_2)(\boldsymbol{e}_2'\boldsymbol{M}\boldsymbol{e}_2)\right] \\
&= \sum_{a,b,c,d,e,f}^{p} \kappa_{abc}\kappa_{def} h_{ab} l_{cd} m_{ef},
\end{aligned}$$

where the notation $\sum_{a_1,\ldots,a_j}^{p}$ means $\sum_{a_1=1}^{p} \cdots \sum_{a_j=1}^{p}$. Then, the ordinary multivariate skewnesses and kurtosis (see e.g., Mardia, 1970, and Isogai, 1983) can be expressed as

$$\kappa_4^{(1)} = \kappa(\boldsymbol{I}_p, \boldsymbol{I}_p), \quad \kappa_{3,3}^{(1)} = \gamma_1^2(\boldsymbol{I}_p, \boldsymbol{I}_p, \boldsymbol{I}_p), \quad \kappa_{3,3}^{(2)} = \gamma_2^2(\boldsymbol{I}_p, \boldsymbol{I}_p, \boldsymbol{I}_p).$$

Let an $n \times n$ idempotent matrix $\boldsymbol{B}$ whose (i, j)th element is b_{ij} be given by

$$\boldsymbol{B} = \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{C}'\{\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{C}'\}^{-1}\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'.$$

Then, the matrices $\boldsymbol{B}_{(d)}$ and $\boldsymbol{B}_{(3)}$ are defined by b_{ij} as $\boldsymbol{B}_{(d)} = \mathrm{diag}(b_{11}, \ldots, b_{nn})$ and $\boldsymbol{B}_{(3)} = [b_{ij}^3]$, which means the (i, j)th element of $\boldsymbol{B}_{(3)}$ is b_{ij}^3. The projection matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are given by

$$\begin{aligned}
\boldsymbol{\Psi} &= \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{D}\{\boldsymbol{D}'(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{D}\}\boldsymbol{D}'(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1/2}, \\
\boldsymbol{\Phi} &= \boldsymbol{I}_p - \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{\Sigma}^{-1/2}.
\end{aligned}$$

By using these notations, asymptotic expansions of the null distributions of the test statistics in (2.3), obtained by Yanagihara (2001), are expressed as in the following lemma.

Lemma 2.1. *Suppose that the between-individuals design matrix $\boldsymbol{A}$ and the true distribution of the error matrix $\boldsymbol{\mathcal{E}}$ satisfy the following five assumptions:*

1. $\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \|\boldsymbol{a}_j\|^4 < \infty$, *where* $\|\cdot\|$ *denotes the Euclidean norm.*
2. $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$, *where* λ_n *is the smallest eigenvalue of* $\boldsymbol{A}'\boldsymbol{A}$,
3. *For some constant* $\delta \in (0,\ 1/2]$, $M_n = \mathrm{O}(n^{1/2-\delta})$, *where* $M_n = \max_{j=1,\ldots,n} \|\boldsymbol{a}_j\|$,
4. $\mathrm{E}[\|\boldsymbol{\varepsilon}\|^8] < \infty$,
5. *The Cramér's condition for the joint distribution of* $\boldsymbol{\varepsilon}$ *and* $\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'$ *holds.*

Then, the null distributions of the three test statistics in (2.3) *are expanded as*

$$\mathrm{P}(T \le x) = G_{cd}(x) + \frac{1}{n}\sum_{j=0}^{3} \beta_j G_{cd+2j}(x) + \mathrm{o}(n^{-1}), \tag{2.4}$$

where $G_f(x)$ *is the distribution function of the chi-square distribution with* f *degrees of freedom, and the coefficients* β_j *are given by*

$$\begin{aligned}
\beta_0 = &\frac{1}{8}m_4^{(1)}\kappa(\boldsymbol{\Psi},\boldsymbol{\Psi}) - \frac{1}{24}\left\{2m_{3,3}^{(1)} + 3(c-2)(c+1)m_{3,1}^{(1)}\right\}\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi}) \\
&-\frac{1}{8}\left\{m_{3,3}^{(2)} - 2m_{3,1}^{(1)} - (c-2)m_{1,1}^{(1)}\right\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi}) \\
&-\frac{1}{2}\left\{2m_{3,1}^{(1)} - (2c+1)m_{1,1}^{(1)}\right\}\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi}) \\
&-\frac{1}{2}\left\{m_{3,1}^{(1)} - cm_{1,1}^{(1)}\right\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi}) \\
&-\frac{1}{2}\left\{m_{3,1}^{(1)} - (c+1)m_{1,1}^{(1)}\right\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Psi}) \\
&-\frac{1}{2}m_{1,1}^{(1)}\left\{3\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Phi}) + 2\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Phi}) + \gamma_2^2(\boldsymbol{\Phi},\boldsymbol{\Psi},\boldsymbol{\Phi})\right\} \\
&+\frac{1}{4}cd(c-d-1), \\
\beta_1 = &-\frac{1}{4}m_4^{(1)}\kappa(\boldsymbol{\Psi},\boldsymbol{\Psi}) + \frac{1}{8}\left\{2m_{3,3}^{(1)} - 4m_{3,1}^{(1)} + (3c^2+c-6)m_{1,1}^{(1)}\right\}\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi}) \\
&+\frac{1}{8}\left\{3m_{3,3}^{(2)} - 2(3c+2)m_{3,1}^{(1)} + (c+6)m_{1,1}^{(1)}\right\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left\{4m_{3,1}^{(1)}-(4c+5)m_{1,1}^{(1)}\right\}\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi})\\
&+\left\{m_{3,1}^{(1)}-(c+1)m_{1,1}^{(1)}\right\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi}) \qquad (2.5)\\
&+\frac{1}{2}\{2m_{3,1}^{(1)}-(2c+3)m_{1,1}^{(1)}\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Psi})\\
&+\frac{1}{2}m_{1,1}^{(1)}\{3\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Phi})+2\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Phi})+\gamma_2^2(\boldsymbol{\Phi},\boldsymbol{\Psi},\boldsymbol{\Phi})\}\\
&-\frac{1}{2}cd\{c+r(c+d+1)\},\\
\beta_2 &= \frac{1}{8}m_4^{(1)}\kappa(\boldsymbol{\Psi},\boldsymbol{\Psi})-\frac{1}{8}\left\{2m_{3,3}^{(1)}-8m_{3,1}^{(1)}+(c+2)(3c-1)m_{1,1}^{(1)}\right\}\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi})\\
&-\frac{1}{8}\left\{3m_{3,3}^{(2)}-6(3c+4)m_{3,1}^{(1)}+5(c+2)m_{1,1}^{(1)}\right\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi})\\
&-\left\{m_{3,1}^{(1)}-(c+2)m_{1,1}^{(1)}\right\}\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi})\\
&-\frac{1}{2}\left\{m_{3,1}^{(1)}-(c+2)m_{1,1}^{(1)}\right\}\left\{\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi})+\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Psi})\right\}\\
&+\frac{1}{4}cd(c+d+1)(1+2r),\\
\beta_3 &= \frac{1}{24}\left\{2m_{3,3}^{(1)}-12m_{3,1}^{(1)}+3(c+1)(c+2)m_{1,1}^{(1)}\right\}\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi})\\
&+\frac{1}{8}\left\{m_{3,3}^{(2)}-2(c+2)m_{3,1}^{(1)}+3(c+2)m_{1,1}^{(1)}\right\}\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi}).
\end{aligned}
$$

Here, the constant r is determined by the test statistics as

$$
r=\begin{cases} -1/2 & (\textit{when } T \textit{ is } T_{\mathrm{LR}}) \\ 0 & (\textit{when } T \textit{ is } T_{\mathrm{HL}}) \\ -1 & (\textit{when } T \textit{ is } T_{\mathrm{BNP}}) \end{cases}
$$

and the coefficients m's are defined by the between-individuals design matrix $\boldsymbol{A}$ and the coefficient matrix for hypothesis $\boldsymbol{C}$ as

$$
\begin{aligned}
&m_{1,1}^{(1)}=\frac{1}{n}\mathbf{1}_n'\boldsymbol{B}\mathbf{1}_n, \qquad m_4^{(1)}=n\,\mathrm{tr}\left(\boldsymbol{B}_{(d)}^2\right)-c(c+2),\\
&m_{3,1}^{(1)}=\mathbf{1}_n'\boldsymbol{B}_{(d)}\boldsymbol{B}\mathbf{1}_n,\ \ m_{3,3}^{(1)}=n\mathbf{1}_n'\boldsymbol{B}_{(3)}\mathbf{1}_n,\ m_{3,3}^{(2)}=n\mathbf{1}_n'\boldsymbol{B}_{(d)}\boldsymbol{B}\boldsymbol{B}_{(d)}\mathbf{1}_n,
\end{aligned}
\qquad (2.6)
$$

where $\mathbf{1}_n$ is an $n\times 1$ vector, all of whose elements are 1.

Notice that

$$
\begin{aligned}
m_{1,1}^{(1)} &= \frac{1}{n^2}\sum_{i,j}^{n} \boldsymbol{u}_i'\boldsymbol{u}_j, & m_4^{(1)} &= \frac{1}{n}\sum_{i=1}^{n} \|\boldsymbol{u}_i\|^4 - c(c+2), \\
m_{3,1}^{(1)} &= \frac{1}{n^2}\sum_{i,j}^{n} \|\boldsymbol{u}_i\|^2(\boldsymbol{u}_i'\boldsymbol{u}_j), & m_{3,3}^{(1)} &= \frac{1}{n^2}\sum_{i,j}^{n} (\boldsymbol{u}_i'\boldsymbol{u}_j)^3, \\
m_{3,3}^{(2)} &= \frac{1}{n^2}\sum_{i,j}^{n} \|\boldsymbol{u}_i\|^2\|\boldsymbol{u}_j\|^2(\boldsymbol{u}_i'\boldsymbol{u}_j),
\end{aligned}
\tag{2.7}
$$

where

$$
\boldsymbol{u}_i = \sqrt{n}\{\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{C}'\}^{-1/2}\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{a}_i, \quad (i = 1, \ldots, n). \tag{2.8}
$$

Therefore, the coefficients β_j $(j = 0, 1, 2, 3)$ are determined by the skewnesses and kurtosis of $\boldsymbol{\varepsilon}$ and sample cumulants of $\boldsymbol{u}_i$ $(i = 1, \ldots, n)$. From Lemma 2.1, we see that the second term in the asymptotic expansion of the null distribution of the test statistic in (2.3) depends on the skewnesses and kurtosis of the true distribution. Therefore, we can generally say that the second terms in expansions of the mean and variance of the test statistic in (2.3) consist of the skewnesses and kurtosis of the true distribution. However, in the next section, we show that there are conditions where the skewnesses and kurtosis disappear from the second terms of the expansions.

3. Conditions for Robustness to Nonnormality

3.1. Conditions for the mean

In this sub-section, we consider conditions where the cumulants denoting nonnormality of the true distribution disappear from the second term of an asymptotic expansion of the mean of the test statistic in (2.3). Notice that

$$
\int_0^\infty x^s dG_f(x) = f(f+2) \times \cdots \times (f + 2(s-1)).
$$

From the formula (2.4) in Lemma 2.1, we obtain an asymptotic expansion of the mean of the test statistic in (2.3) as

$$\mathrm{E}[T] = cd + \frac{1}{n}\sum_{j=0}^{3}\beta_j(cd+2j) + \mathrm{o}(n^{-1}).$$

By using the relation $\sum_{j=0}^{3}\beta_j = 0$, we can write a concrete form of the expansion of the mean as

$$\begin{aligned}
\mathrm{E}[T] = cd + \frac{1}{n}m_{1,1}^{(1)}\big[&\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi}) + \gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Psi}) \\
&+3\left\{\gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi}) + \gamma_1^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Phi})\right\} \\
&+2\left\{\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Psi}) + \gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Phi},\boldsymbol{\Phi})\right\} \\
&+\gamma_2^2(\boldsymbol{\Psi},\boldsymbol{\Psi},\boldsymbol{\Phi}) + \gamma_2^2(\boldsymbol{\Phi},\boldsymbol{\Psi},\boldsymbol{\Phi})\big] \\
&+\frac{1}{n}cd\left\{r(c+d+1)+d+1\right\} + \mathrm{o}(n^{-1}) \\
= cd\left(1+\frac{\eta_1}{n}\right) + \mathrm{o}(n^{-1}).&
\end{aligned} \tag{3.1}$$

From this equation, we can see that η_1 in (3.1) is independent of the cumulants denoting nonnormality if the coefficient $m_{1,1}^{(1)}$ is 0 or all the $\gamma_j^2(*,*,*)$ $(j=1,2)$ are 0. It means that a location of the null distribution of the test statistic is not easily moved due to the nonnormality of the true distribution. Then, the test statistic becomes robust to nonnormality. Notice that $m_{1,1}^{(1)} = 0$ is equivalent to

$$\frac{1}{n}\boldsymbol{1}_n'\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{C}'\{\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{C}'\}^{-1}\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{1}_n = 0.$$

Since $\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{C}'$ is a positive definite matrix, the above equation can be rewritten as $\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{1}_n = \boldsymbol{0}_c$. Therefore, we obtain the conditions for robustness to nonnormality in the following theorem.

Theorem 3.1. *If a considered test satisfies either of the following, Condition 1 or 2:*

- *Condition 1:* $\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{1}_n = \boldsymbol{0}_c$,
- *Condition 2: All the* $\gamma_j^2(*,*,*)$ $(j=1,2)$ *are 0,*

the second term of the asymptotic expansion of the mean of the test statistic T becomes independent of the cumulants denoting nonnormality of the true distribution, i.e.,

$$\mathrm{E}[T] = cd\left[1 + \frac{1}{n}\{r(c+d+1)+d+1\}\right] + \mathrm{o}(n^{-1}). \tag{3.2}$$

Then, the location of the null distribution of T is not easily moved due to nonnormality of the true distribution.

From Appendix A.2, we can see that all the $\gamma_j^2(*,*,*)$ are 0 if the true distribution is an orthant symmetric (Efron, 1969), which is not a skewed distribution. Therefore, we obtain the following corollary (the proof is given in Appendix A.2).

Corollary 3.1.*If the true distribution is an orthant symmetric, then Condition 2 holds.*

It is well known that the elliptical distribution is an orthant symmetric. Many authors have reported that a test statistic is robust when the distribution of observation is the elliptical distribution, e.g., Efron (1969), Eaton and Efron (1970), Dawid (1977), Kariya (1981a, 1981b) and Khartri (1988). Robustness to nonnormality, which is caused by Condition 2, corresponds to their results.

Although Condition 2 depends on the unknown true distribution, Condition 1 only depends on the setting of the hypothesis testing. By using Condition 1, we can judge whether a used test statistic is robust to nonnormality or not. Moreover, we can consider two types of the between-individuals design matrix $\boldsymbol{A}$. When the matrix $\boldsymbol{A}$ has a segment, i.e., $\boldsymbol{A}$ is given by $\mathbf{1}_n$ and any $n \times (k-1)$ matrix $\boldsymbol{A}^+$ with the rank $k-1$ is given as $\boldsymbol{A} = (\mathbf{1}_n\ \boldsymbol{A}^+)$, then we call such a design matrix the type I between-individuals design matrix. On the other hand, when the matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1}_{n_1} & \cdots & \mathbf{0}_{n_1} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{n_k} & \cdots & \mathbf{1}_{n_k} \end{pmatrix}, \tag{3.3}$$

where $\sum_{i=1}^{k} n_i = n$, then we call such a design matrix the type II between-individuals design matrix. Through these matrices, we can obtain settings which satisfy Condition 1 in the following corollary (the proof is given in Appendix A.3).

COROLLARY 3.2. *We consider the two cases of the between-individuals design matrix $\boldsymbol{A}$, i.e., $\boldsymbol{A}$ is type I or type II. Then, Condition 1 holds if the following conditions of the coefficient matrix for hypothesis $\boldsymbol{C}$ are satisfied.*

1. *In the case that $\boldsymbol{A}$ is type I, if $\boldsymbol{C} = (\boldsymbol{0}_c\ \boldsymbol{C}^{+})$, where $\boldsymbol{C}^{+}$ is any $c \times (k-1)$ matrix with the rank c, Condition 1 holds.*

2. *In the case that $\boldsymbol{A}$ is type II, if the equation $\boldsymbol{C}\boldsymbol{1}_k = \boldsymbol{0}_c$ is satisfied, Condition 1 holds.*

From Corollary 3.2, we can see that Hotelling's two-sample test satisfies Condition 1 and Hotelling's one-sample test does not. Even if the between-individuals design matrix $\boldsymbol{A}$ is not type I or II, the test can be made to satisfy Condition 1 through the transformations $\boldsymbol{A} \to (\boldsymbol{1}_n\ \boldsymbol{A})$ and $\boldsymbol{C} \to (\boldsymbol{0}_c\ \boldsymbol{C})$. Furthermore, we can see that testing for equality, e.g., equality for the means between k-groups, i.e., $H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_k$, satisfies Condition 1, because matrix $\boldsymbol{A}$ is type II and $\boldsymbol{C}\boldsymbol{1}_k = \boldsymbol{0}_c$ in this case. Practical tests are tests of the equality for parameters, e.g., the testing equality for several means in a multi-way MANOVA model; therefore, it seems that our result is useful for actual use.

If the mean of the test statistic T is expanded as (3.1), then the Bartlett correction can be obtained as

$$\tilde{T} = \left(1 - \frac{\hat{\eta}_1}{n}\right) T,$$

where $\hat{\eta}_1$ is a consistent estimator of η_1. Therefore, in general, it is necessary to estimate skewnesses in order to use the Bartlett correction. However, from Theorem 3.1, we can see that the Bartlett correction in the normal case, which is the transformation by the constant coefficient, can improve a chi-square approximation, even if the true distribution is not the normal distribution; this is because η_1 is independent of the cumulants denoting nonnormality. Therefore, we obtain the following corollary.

COROLLARY 3.3. *Let $\tilde{T}$ denote the test statistics adjusted by the Bartlett correction in the normal case as*

$$\tilde{T} = \begin{cases} \left(1 + \dfrac{c-d-1}{2n}\right) T_{\mathrm{LR}} \\ \left(1 - \dfrac{d+1}{n}\right) T_{\mathrm{HL}} \\ \left(1 + \dfrac{c}{n}\right) T_{\mathrm{BNP}} \end{cases} .$$

Suppose that either Condition 1 or 2 holds. Then, the asymptotic expansion of the mean of $\tilde{T}$ becomes $\mathrm{E}[\tilde{T}] = cd + \mathrm{o}(n^{-1})$. This naturally implies that the Bartlett correction in the normal case can improve the chi-square approximation even under nonnormality.

Before concluding this sub-section, we verify the validity of our theorem through a numerical experiment. The true distributions considered are described in Appendix A.1. We dealt with the test for a given matrix in the case of $p = 2$ and $n = 30$. The $p \times (p-1)$ within-individuals design matrix $\boldsymbol{X}$ was generated from the uniform (-1,1) distribution, and the coefficient matrix for the hypothesis was $\boldsymbol{D} = \boldsymbol{I}_q$. We prepared two test statistics, T_0 and T_1, specified by the between-individuals matrix $\boldsymbol{A}$ and the coefficient matrix for hypothesis $\boldsymbol{C}$, and adjusted versions of T_0 and T_1 by the Bartlett correction in the normal case as follows:

T_0: $\boldsymbol{A} = \boldsymbol{A}^+$ and $\boldsymbol{C} = \boldsymbol{I}_3$,

$\tilde{T}_0$: the adjusted version of T_0 by the Bartlett correction in the normal case,

T_1: $\boldsymbol{A} = (\mathbf{1}_n\ \boldsymbol{A}^+)$ and $\boldsymbol{C} = (\mathbf{0}_3\ \boldsymbol{I}_3)$,

$\tilde{T}_1$: the adjusted version of T_1 by the Bartlett correction in the normal case,

where an $n \times 3$ matrix $\boldsymbol{A}^+$ was generated from the uniform (0,1) distribution. It is easy to see that T_1 satisfies Condition 1, and T_0 and T_1 satisfy Condition 2, when the true distributions are 1, 2 and 3. Table 1 shows the actual test sizes of these test statistics when we used the chi-square distribution as the null distribution. The average and standard deviation (S.D.) of actual sizes for every true distribution are also shown in the table. From this table, we can see that the difference of the actual test size by the difference of the true distribution is small when either Condition 1 or 2 holds, because the S.D. becomes small. In these cases, the Bartlett correction in the normal improves a chi-square approximation. From the numerical results, we suggest that the test statistic satisfying Condition 1 should be adjusted by the Bartlett correction in the normal case.

Insert Table 1 around here

3.2. Condition for the variance

In this sub-section, we consider a condition where the cumulants denoting nonnormality of the true distribution disappear simultaneously from the second terms of asymptotic expansions of both the mean and variance of the test statistic in (2.3). From the formula (2.4) in Lemma 2.1, we obtain the asymptotic expansion of the second moment of the test statistic in (2.3) as

$$\mathrm{E}[T^2] = cd(cd+2) + \frac{1}{n}\sum_{j=0}^{3} \beta_j (cd+2j)(cd+2+2j) + \mathrm{o}(n^{-1}).$$

By using the relation $\sum_{j=0}^{3} \beta_j = 0$, we can write a concrete form of the expansion of the second moment as

$$\mathrm{E}[T^2] = cd(cd+2) + \frac{1}{n}\Big[m_4^{(1)} \kappa(\boldsymbol{\Psi}, \boldsymbol{\Psi})$$

$$
\begin{aligned}
&-2\left\{m_{3,1}^{(1)}-(cd+2c+6)m_{1,1}^{(1)}\right\}\gamma_1^2(\mathbf{\Psi},\mathbf{\Psi},\mathbf{\Psi})\\
&-2\left\{2m_{3,1}^{(1)}-(cd+2c+6)m_{1,1}^{(1)}\right\}\gamma_2^2(\mathbf{\Psi},\mathbf{\Psi},\mathbf{\Psi})\\
&-2\left\{4m_{3,1}^{(1)}-(cd+4c+10)m_{1,1}^{(1)}\right\}\gamma_1^2(\mathbf{\Psi},\mathbf{\Psi},\mathbf{\Phi})\\
&-2\left\{2m_{3,1}^{(1)}-(cd+2c+6)m_{1,1}^{(1)}\right\}\left\{\gamma_2^2(\mathbf{\Psi},\mathbf{\Psi},\mathbf{\Phi})+\gamma_2^2(\mathbf{\Psi},\mathbf{\Phi},\mathbf{\Psi})\right\}\\
&+2(cd+2)m_{1,1}^{(1)}\left\{3\gamma_1^2(\mathbf{\Psi},\mathbf{\Phi},\mathbf{\Phi})+2\gamma_2^2(\mathbf{\Psi},\mathbf{\Phi},\mathbf{\Phi})+\gamma_2^2(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Phi})\right\}\Big]\\
&+\frac{2}{n}cd\left[(cd+2)\left\{r(c+d+1)+d+1\right\}+(c+d+1)(1+2r)\right]+\mathrm{o}(n^{-1})\\
=\ & cd(cd+2)\left(1+\frac{\eta_2}{n}\right)+\mathrm{o}(n^{-1}). \qquad (3.4)
\end{aligned}
$$

It is easy to see that the coefficient $m_{3,1}^{(1)}$ becomes 0 if $m_{1,1}^{(1)}=0$. Therefore, from these equations, if either the coefficient $m_{1,1}^{(1)}=0$ or all $\gamma_j^2(*,*,*)=0$ $(j=1,2)$ holds and also the equation $m_4^{(1)}=0$ holds simultaneously, η_1 in (3.1) and η_2 in (3.4) become independent of the cumulants denoting nonnormality of the true distribution. This means that the location and dispersion of the null distribution of the test statistic are not easily changed due to nonnormality of the true distribution. Notice that the coefficient $m_4^{(1)}$ expresses the sample kurtosis of $\boldsymbol{u}_i$ $(i=1,\dots,n)$ in (2.8). Therefore, we obtain the condition for robustness to nonnormality in the following theorem.

THEOREM 3.2. *If a considered test satisfies either Condition 1 or 2 in Theorem 3.1, and furthermore, if the following condition also holds:*

- *Condition 3: The sample kurtosis of* $\boldsymbol{u}_i$ $(i=1,\dots,n)$ *is* 0.

Then, the second terms of asymptotic expansions of the mean and variance of the test statistic T become independent of the cumulants denoting nonnormality of the true distribution, i.e.,

$$
\begin{aligned}
\mathrm{E}[T]&=cd\left[1+\frac{1}{n}\left\{r(c+d+1)+d+1\right\}\right]+\mathrm{o}(n^{-1}),\\
\mathrm{Var}[T]&=2cd\left[1+\frac{1}{n}\left\{4r(c+d+1)+c+3(d+1)\right\}\right]+\mathrm{o}(n^{-1}).
\end{aligned}
\qquad (3.5)
$$

Consequently, the location and dispersion of the null distribution of the test statistic are not easily changed due to nonnormality of the true distribution.

From Wakaki et al. (2003), if the between-individuals design matrix $\boldsymbol{A}$ is type II and the equation $\boldsymbol{C}\mathbf{1}_k = \mathbf{0}_c$ holds, then the coefficient $m_4^{(1)}$ is rewritten as $m_4^{(1)} = \sum_{j=1}^k n/n_j - k^2 - 2k + 2$. Therefore, we obtain the following corollary.

Corollary 3.4. *Suppose that the between-individuals design matrix $\boldsymbol{A}$ is type II and equation $\boldsymbol{C}\mathbf{1}_k = \mathbf{0}_c$ holds. If the ratio of sample sizes of k-groups $n_1, \ldots, n_k$ is satisfied by the following equation, $\sum_{j=1}^k n/n_j = k^2 + 2k - 2$, then Conditions 1 and 3 hold simultaneously.*

This result means that the testing equality for the means of k-groups becomes robust to nonnormality by adjusting each sample size. In the ANOVA model, this fact was reported by Box and Watson (1962), and Box and Draper (1975). They called such a design matrix a robust design matrix. Table 2 shows the robust design of our test statistic in the cases of $k = 4, 5$ and 6. If the ratio of sample sizes is satisfied as in Table 2, then the test statistic becomes more robust to nonnormality.

Insert Table 2 around here

If the mean and variance of the test statistic T are expanded as (3.1) and (3.4), then the modified Bartlett correction proposed by Fujikoshi (2000), which not only brings the mean close to an asymptotic one but also brings the variance close to an asymptotic one, is given as

$$\begin{aligned}
&(1) \quad \tilde{T}' = (n\hat{\nu}_1 + \hat{\nu}_1) \log\left(1 + \frac{T}{n\hat{\nu}_1}\right), && \hat{\nu}_1 > 0,\ n\hat{\nu}_1 + \hat{\nu}_2 > 0,\\
&(2) \quad \tilde{T}' = T + \frac{T}{n}\left(\frac{\hat{\nu}_2}{\hat{\nu}_1} - \frac{T}{2\hat{\nu}_1}\right), && \hat{\nu}_1 < 0,\ n\hat{\nu}_1 + \hat{\nu}_2 < 0,\\
&(3) \quad \tilde{T}' = (n\hat{\nu}_1 + \hat{\nu}_2)\left\{1 - \exp\left(-\frac{T}{n\hat{\nu}_1}\right)\right\}, && \text{for any } \hat{\nu}_1,\ n \text{ and } \hat{\nu}_2,
\end{aligned}$$

where

$$\hat{\nu}_1 = \frac{2}{\hat{\eta}_2 - 2\hat{\eta}_1}, \quad \hat{\nu}_2 = \frac{(cd+2)\hat{\eta}_2 - 2(cd+4)\hat{\eta}_1}{2(\hat{\eta}_2 - 2\hat{\eta}_1)},$$

and $\hat{\eta}_1$ and $\hat{\eta}_2$ are consistent estimators of η_1 and η_2, respectively. In general, it is necessary to estimate skewnesses and kurtosis in order to use the modified Bartlett correction. However, from Theorem 3.2, we can see that the modified Bartlett correction in the normal case, which is the transformation by a known monotone function, can improve the chi-square approximation, even if the true distribution is not the normal distribution. This is because η_1 and η_2 are independent of the cumulants denoting nonnormality of the true distribution. Therefore, we obtain the following corollary.

COROLLARY 3.5. *Let $\tilde{T}'$ denote the test statistics adjusted by the modified Bartlett corrections in the normal case as*

$$\tilde{T}' = \begin{cases} \dfrac{cd+2}{c+d+1}\left\{n + \dfrac{1}{2}(c-d-1)\right\}\log\left\{1 + \dfrac{c+d+1}{n(cd+2)}T_{\mathrm{HL}}\right\} \\ T_{\mathrm{BNP}} + \dfrac{1}{2n}\left(c-d-1+\dfrac{c+d+1}{cd+2}T_{\mathrm{BNP}}\right)T_{\mathrm{BNP}} \end{cases}.$$

Suppose that either Condition 1 or 2 holds, and also Condition 3 holds simultaneously, then the mean and variance of $\tilde{T}'$ are expanded as $\mathrm{E}[\tilde{T}'] = cd + \mathrm{o}(n^{-1})$ *and* $\mathrm{Var}[\tilde{T}'] = 2cd + \mathrm{o}(n^{-1})$. *This naturally implies that the modified Bartlett correction in the normal case can improve the chi-square approximation even under nonnormality.*

From equations (3.1) and (3.4), when either Condition 1 or 2 holds, and also Condition 3 holds, the mean and the second moment of T_{LR} in the normal case are expanded as

$$\mathrm{E}[T_{\mathrm{LR}}] = cd\left\{1 - \frac{1}{2n}(c-d-1)\right\} + \mathrm{o}(n^{-1}),$$
$$\mathrm{E}[T_{\mathrm{LR}}^2] = cd(cd+2)\left\{1 - \frac{1}{n}(c-d-1)\right\} + \mathrm{o}(n^{-1}).$$

These expansions make $\eta_2 = 2\eta_1$ in the LR test statistic. Therefore, we can see that the modified Bartlett correction in the LR test statistic cannot be defined in the normal case. This is because the Bartlett correction of the LR test statistic not only brings a mean close to an asymptotic one, but also it brings the null distribution close to the chi-square distribution, when the true distribution is also the normal distribution (see e.g., Barndorff-Nielsen & Cox, 1984; Barndorff-Nielsen & Hall, 1988). However, although the modified Bartlett correction cannot be defined, T_{LR} adjusted by the Bartlett correction in the normal case also can improve the chi-square approximation, as in the modified Bartlett correction in the following corollary (the proof is given in Appendix A.4).

COROLLARY 3.6. *Let $\tilde{T}_{\mathrm{LR}}$ be the version of T_{LR} adjusted by the Bartlett correction in the normal case. Suppose that either Condition 1 or 2 holds, and also Condition 3 holds simultaneously. Then, $\tilde{T}_{\mathrm{LR}}$ can improve the chi-square approximation even under nonnormality, as in the modified Bartlett correction, i.e., $\mathrm{E}[\tilde{T}_{\mathrm{LR}}] = cd + \mathrm{o}(n^{-1})$ and $\mathrm{Var}[\tilde{T}_{\mathrm{LR}}] = 2cd + \mathrm{o}(n^{-1})$.*

Before concluding this sub-section, we verify the validity of our theorem through a numerical experiment. The error distributions considered are described in Appendix A.1. We dealt with the test for equality of $\boldsymbol{\Xi}$ in the case of $p = 2$, $k = 4$ and $n = 32$. The $p \times (p-1)$ within-individuals design matrix $\boldsymbol{X}$ was generated from the uniform (-1,1) distribution, and the coefficient matrices for hypothesis were $\boldsymbol{C} = (\boldsymbol{I}_{k-1} \ \ -\boldsymbol{1}_{k-1})$ and $\boldsymbol{D} = \boldsymbol{I}_q$. We prepared the three test statistics specified by the between-individuals design matrix $\boldsymbol{A}$, which is type II in (3.3), as follows:

Case 1 : $n_1 = n_2 = n_3 = n_4 = 8$ ($m_4^{(1)} = -6.0$).

Case 2 : $n_1 = n_2 = 4$, $n_3 = 8$ and $n_4 = 16$ ($m_4^{(1)} = 0.0$).

Case 3 : $n_1 = 3$, $n_2 = n_3 = 4$ and $n_4 = 21$ ($m_4^{(1)} = 6.2$).

Moreover, we adjusted each test statistic by the Bartlett correction and the modified Bartlett correction. It is easy to see that the three test statistics satisfy Condition 1 and also Condition 3 when $\boldsymbol{A}$ is Case 2. Tables 3, 4 and 5 show actual test sizes of the LR test statistic, HL test statistic and BNP test statistic, respectively. The average and S.D. of actual test sizes for every true distribution are also shown in these tables. From the tables, we can see that the difference of the actual test size by the difference of the error distribution is small when Condition 3 holds, because S.D. becomes small. In this case, the modified Bartlett correction in the normal improves the chi-square approximation.

Insert Tables 3, 4 and 5 around here

4. Conclusion

In this paper, we find the conditions for robustness to nonnormality. If either Condition 1 or 2 holds, the cumulants denoting nonnormality of the true distribution disappear from the second term in the asymptotic expansion of the mean of the test statistic T. Then, the location of the null distribution of T is not easily moved due to nonnormality of the true distribution, and we use the Bartlett correction without estimating several cumulants. Especially, since Condition 1 does not depend on the unknown true distribution, checking whether or not Condition 1 holds can determine whether the test statistic used is robust to nonnormality. Moreover, from the simple transformation, we can make the test statistic the robust one. Therefore, it is a very useful Condition 1.

If either Condition 1 or 2 holds, and also Condition 3 holds simultaneously, the cumulants denoting nonnormality of the true distribution disappear simultaneously from the second terms of the asymptotic expansions of the mean and variance of the test statistic T. Then, the location and dispersion of the null distribution of T are not easily changed due to nonnormality of the true distribution, and we use the modified Bartlett correction without estimating several cumulants. Especially, by adjusting the ratio of sample sizes in the k-groups, we make the test statistic for the equality of several means more robust to nonnormality. However, we do not, of course, suggest that one would deliberately seek unequal sample sizes to lessen the effect of nonnormality in the test to compare means. The reduction in precision, with which comparison among the means could be made and an increase in sensitivity to variance inequalities would result, would certainly not be worth the small increase in robustness to nonnormality (see Box & Watson, 1962). Moreover, such unequal sample sizes do not make an optimal design for estimating variance (see Guiard et al., 2000).

Nevertheless, if Condition 1 holds, the test statistic becomes robust to nonnormality. Therefore, we encourage using the between-individuals design matrix having the segment, and adjusting its test statistic by the Bartlett correction in the normal case.

Actually, our conditions are depend on only the between-individuals design matrix $\boldsymbol{A}$ and the coefficient matrix for hypothesis $\boldsymbol{C}$. Therefore, the robustness to nonnormality of the test in a GMANOVA model is independent of the within-individuals design matrix $\boldsymbol{X}$ and the coefficient matrix for hypothesis $\boldsymbol{D}$. Consequently, our conditions essentially coincide with conditions of testing $H_0 : \boldsymbol{C\Xi} = \boldsymbol{O}_{c\times p}$ in a multivariate linear model, which were similer to obtained by Wakaki et al. (2002).

Appendix

A.1. Used Distributions as True Distributions

In this sub-section, we describe the used distributions as the true distributions for the numerical studies in our paper. The elements of $\boldsymbol{\varepsilon}$ are independently and identically distributed, and generated from the following six distributions:

1. *Normal Distribution*: $\varepsilon_j \sim \mathrm{N}(0,1)$, ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} = 0$ and $\kappa_4^{(1)} = 0$).

2. *Laplace Distribution*: ε_j is generated from a Laplace distribution with mean 0 and standard deviation 1 ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} = 0$ and $\kappa_4^{(1)} = 3p$).

3. *Uniform Distribution*: ε_j is generated from the uniform (-5,5) distribution divided by the standard deviation $5/\sqrt{3}$ ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} = 0$ and $\kappa_4^{(1)} = -1.2p$).

4. *Skew-Laplace Distribution*: ε_j is generated from a skew-Laplace distribution with location parameter 0, dispersion parameter 1 and skew parameter 1 standardized by mean 3/4 and standard deviation $\sqrt{23}/4$ ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} \approx 1.12p$ and $\kappa_4^{(1)} \approx 3.26p$).

5. *Chi-Square Distribution*: ε_j is generated from a chi-square distribution with 2 degrees of freedom standardized by mean 2 and standard deviation 2 ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} = 2p$ and $\kappa_4^{(1)} = 6p$).

6. *Log-Normal Distribution*: ε_j is generated from a lognormal distribution such that $\log \varepsilon_j \sim \mathrm{N}(0,1)$, standardized by mean $e^{1/2}$ and standard deviation $\sqrt{e(e-1)}$ ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} \approx 6.18p$ and $\kappa_4^{(1)} \approx 110.94p$).

The skew-Laplace distribution was proposed by Balakrishnan and Ambagaspitiya (1994) (for the probability density function, see e.g., Yanagihara & Yuan, 2005). It is easy to see that distributions 1, 2 and 3 are symmetric distributions, and distributions 4, 5 and 6 are skewed distributions.

A.2. Proof of Corollary 3.1

In this sub-section, we show the proof of Corollary 3.1. If the true distribution of $\boldsymbol{\varepsilon}$ is an orthant symmetric, the third multivariate moment of $\boldsymbol{\varepsilon}$ is given by

$$\mu_{abc} = \mathrm{E}[\varepsilon_a \varepsilon_b \varepsilon_c] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varepsilon_a \varepsilon_b \varepsilon_c f(\varepsilon_1, \ldots, \varepsilon_p) d\boldsymbol{\varepsilon},$$

where $f(\cdot)$ is a density function of $\boldsymbol{\varepsilon}$. Since the distribution of $\boldsymbol{\varepsilon}$ is an orthant symmetric, the equation $f(\varepsilon_1, \ldots, \varepsilon_l, \ldots, \varepsilon_p) = f(\varepsilon_1, \ldots, -\varepsilon_l, \ldots, \varepsilon_p)$ holds for any ε_ls. Let $g(\varepsilon_a, \varepsilon, \varepsilon_b)$ be the marginal distribution of ε_a, ε_b and ε_c. Therefore,

$$\mu_{abc} = 3\int_0^\infty \int_0^\infty \int_0^\infty (\varepsilon_a\varepsilon_b\varepsilon_c - \varepsilon_a\varepsilon_b\varepsilon_c)\, g(\varepsilon_a, \varepsilon_b, \varepsilon_c) d\varepsilon_a d\varepsilon_b d\varepsilon_c = 0.$$

Notice that $\mu_{abc} = \kappa_{abc}$. We obtain $\kappa_{abc} = 0$ under an orthant symmetric distribution. Since all $\gamma_j^2(*, *, *)$ $(j = 1, 2)$ consist of κ_{abc}, it implies that all $\gamma_j^2(*, *, *) = 0$ $(j = 1, 2)$ under an orthant symmetric distribution.

A.3. Proof of Corollary 3.2

In this sub-section, we show the proof of Corollary 3.2. For the proof of Corollary 3.2, it is sufficient to confirm $\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{1}_n = \boldsymbol{0}_c$. When the between-individuals design matrix $\boldsymbol{A}$ is type I, from the general formula of an inverse matrix (see e.g., Siotani et al., 1985, p. 592), $(\boldsymbol{A}'\boldsymbol{A})^{-1}$ is given by

$$(\boldsymbol{A}'\boldsymbol{A})^{-1} = \frac{1}{w_1}\begin{pmatrix} 1 & -\boldsymbol{1}_n'\boldsymbol{A}_1(\boldsymbol{A}_1'\boldsymbol{A}_1)^{-1} \\ -(\boldsymbol{A}_1'\boldsymbol{A}_1)^{-1}\boldsymbol{A}_1'\boldsymbol{1}_n & \boldsymbol{W}_1 \end{pmatrix},$$

where $w_1 = n - \boldsymbol{1}_n'\boldsymbol{P}_{\boldsymbol{A}_1}\boldsymbol{1}_n$ and

$$\boldsymbol{W}_1 = w_1(\boldsymbol{A}_1'\boldsymbol{A}_1)^{-1} + (\boldsymbol{A}_1'\boldsymbol{A}_1)^{-1}\boldsymbol{A}_1'\boldsymbol{1}_n\boldsymbol{1}_n'\boldsymbol{A}_1(\boldsymbol{A}_1'\boldsymbol{A}_1)^{-1}.$$

If $\boldsymbol{C} = (\boldsymbol{0}_c\ \boldsymbol{C}_1)$, where $\boldsymbol{C}_1$ is any $c \times (k-1)$ matrix with the rank c, then,

$$\begin{aligned}\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{1}_n &= -\frac{1}{w_1}\boldsymbol{C}_1\left\{(\boldsymbol{A}_1'\boldsymbol{A}_1)^{-1}\boldsymbol{A}_1'\boldsymbol{1}_n\boldsymbol{1}_n' - \boldsymbol{W}_1\boldsymbol{A}_1'\right\}\boldsymbol{1}_n \\ &= -\frac{1}{w_1}\left(n - w_1 - \boldsymbol{1}_n'\boldsymbol{P}_{\boldsymbol{A}_1}\boldsymbol{1}_n\right)\boldsymbol{C}_1(\boldsymbol{A}_1'\boldsymbol{A}_1)^{-1}\boldsymbol{A}_1'\boldsymbol{1}_n = \boldsymbol{0}_c.\end{aligned}$$

On the other hand, when the between-individuals design matrix $\boldsymbol{A}$ is type II, $(\boldsymbol{A}'\boldsymbol{A})^{-1} = \mathrm{diag}(n_1^{-1}, \ldots, n_k^{-1})$ and $\boldsymbol{A}'\boldsymbol{1}_n = (n_1, \ldots, n_k)'$. Notice that $\boldsymbol{C}\boldsymbol{1}_k = \boldsymbol{0}_c$. Therefore,

$$\boldsymbol{C}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{1}_n = \boldsymbol{C}\boldsymbol{1}_k = \boldsymbol{0}_c.$$

From these equations, we obtain the proof of Corollary 3.2.

A.4. Proof of Corollary 3.6

In this sub-section, we show the proof of Corollary 3.6. From the equations (3.5), if either Condition 1 or 2 holds, and also Condition 3 holds, the variance of T_{LR} is expanded as

$$\mathrm{Var}[T_{\mathrm{LR}}] = 2cd\left\{1 - \frac{1}{n}(c-d-1)\right\} + \mathrm{o}(n^{-1}).$$

It is known that

$$\mathrm{Var}[\tilde{T}_{\mathrm{LR}}] = \left(1 + \frac{c-d-1}{2n}\right)^2 \mathrm{Var}[T_{\mathrm{LR}}] = 2cd + \mathrm{o}(n^{-1}).$$

Therefore, we obtain the proof of Corollary 3.6.

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Table 1. Actual test sizes of several test statistics for testing a given matrix

(LR)

	T_0			$\tilde{T}_0$			T_1			$\tilde{T}_1$		
	Nominal Sizes			Nominal Sizes			Nominal Sizes			Nominal Sizes		
Distribution	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
Normal	9.7	4.9	0.9	10.3	5.2	1.0	9.3	4.8	1.0	9.7	5.0	1.0
Laplace	9.1	4.6	0.7	9.5	4.8	0.8	8.8	4.5	0.9	9.3	4.7	1.0
Uniform	9.5	4.8	1.1	10.0	5.1	1.2	9.6	5.1	1.0	10.1	5.3	1.1
Skew-Laplace	10.4	5.7	1.2	11.0	5.9	1.3	9.3	4.7	0.9	9.8	4.9	1.0
Chi-Sqare	11.7	6.4	1.8	12.2	6.8	1.9	9.6	4.6	0.7	10.0	4.8	0.8
Log-Normal	16.5	10.3	4.1	17.1	10.7	4.3	8.8	4.1	0.8	9.2	4.4	0.9
Average	11.2	6.1	1.6	11.7	6.4	1.8	9.2	4.6	0.9	9.7	4.9	1.0
S.D.	2.52	1.97	1.16	2.56	2.01	1.21	0.33	0.30	0.10	0.36	0.28	0.09

(HL)

	T_0			$\tilde{T}_0$			T_1			$\tilde{T}_1$		
	Nominal Sizes			Nominal Sizes			Nominal Sizes			Nominal Sizes		
Distribution	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
Normal	13.1	7.6	2.3	11.1	6.4	1.8	12.8	7.4	2.5	10.7	6.1	1.9
Laplace	12.3	6.9	2.2	10.3	5.6	1.6	12.0	7.0	2.1	10.2	5.8	1.6
Uniform	12.5	7.2	2.5	10.8	5.9	1.9	12.7	7.6	2.4	10.9	6.3	2.0
Skew-Laplace	13.8	8.1	2.9	11.8	6.9	2.4	12.5	7.5	2.3	10.7	6.2	1.8
Chi-Square	14.9	9.5	3.5	13.2	8.0	2.8	12.8	7.3	2.0	11.0	5.9	1.5
Log-Normal	20.3	13.9	6.9	18.0	12.1	6.0	12.1	6.8	1.9	10.4	5.5	1.5
Average	14.5	8.9	3.4	12.6	7.5	2.8	12.5	7.3	2.2	10.7	6.0	1.7
S.D.	2.73	2.40	1.63	2.61	2.21	1.52	0.31	0.29	0.23	0.27	0.29	0.18

(BNP)

	T_0			$\tilde{T}_0$			T_1			$\tilde{T}_1$		
	Nominal Sizes			Nominal Sizes			Nominal Sizes			Nominal Sizes		
Distribution	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
Normal	6.7	2.5	0.1	9.1	4.0	0.3	6.1	2.5	0.1	8.6	3.8	0.3
Laplace	5.9	2.3	0.2	8.4	3.8	0.3	5.9	2.1	0.1	8.1	3.5	0.3
Uniform	6.2	2.6	0.3	8.8	4.0	0.5	6.4	2.4	0.1	8.8	3.9	0.4
Skew-Laplace	7.2	3.1	0.1	9.7	4.7	0.5	6.3	2.3	0.1	8.6	3.6	0.4
Chi-Square	8.4	3.7	0.6	11.1	5.5	1.0	6.0	2.0	0.1	8.8	3.5	0.3
Log-Normal	12.6	7.1	1.8	15.7	9.0	2.6	5.5	1.9	0.1	8.0	3.2	0.3
Average	7.9	3.5	0.5	10.5	5.2	0.9	6.1	2.2	0.1	8.5	3.6	0.4
S.D.	2.28	1.65	0.58	2.48	1.82	0.81	0.30	0.22	0.01	0.32	0.23	0.04

Table 2. Robust design to nonnormality in testing the equality

($k = 4$)		
$n_1 : n_2 : n_3 : n_4$	$1:1:2:4$	$1:2:2:5$
	$1:2:4:4$	$1:3:4:4$
	$2:5:5:10$	
($k = 5$)		
$n_1 : n_2 : n_3 : n_4 : n_5$	$1:1:1:3:3$	$1:1:3:3:3$
	$1:3:3:3:5$	$3:5:5:7:15$
	$3:5:8:12:12$	$3:7:7:10:15$
($k = 6$)		
$n_1 : n_2 : n_3 : n_4 : n_5 : n_6$	$2:3:4:5:8:8$	$2:3:5:5:5:10$
	$3:3:3:4:5:12$	$3:4:4:4:9:12$
	$3:4:6:6:6:15$	$3:6:6:8:10:15$
	$3:9:9:9:12:14$	$4:5:5:5:12:15$
	$4:5:12:12:12:15$	$4:8:10:15:16:16$
	$4:10:10:10:15:20$	$5:8:12:15:20:20$

Table 3. Actual test sizes of several test statistics for testing the equality (LR)

Nominal Size	$m_4^{(1)}$	Test Statistic	Distribution 1	2	3	4	5	6	Average	S.D.
10%	-6.0	T	9.7	9.1	9.8	9.6	8.6	8.3	9.2	0.57
		$\tilde{T}$	10.0	9.5	10.2	10.0	8.9	8.7	9.5	0.57
	0.0	T	9.7	9.5	9.7	10.0	9.8	9.2	9.7	0.24
		$\tilde{T}$	9.9	10.0	10.0	10.3	10.2	9.6	10.0	0.23
	6.2	T	10.0	10.0	9.5	9.4	10.0	9.7	9.8	0.25
		$\tilde{T}$	10.4	10.4	9.8	9.9	10.5	10.1	10.2	0.26
5%	-6.0	T	4.7	4.3	4.9	4.6	4.3	4.0	4.5	0.27
		$\tilde{T}$	4.9	4.6	5.1	4.8	4.6	4.2	4.7	0.29
	0.0	T	4.4	4.7	4.6	5.2	5.2	4.7	4.8	0.30
		$\tilde{T}$	4.8	4.9	4.8	5.5	5.5	5.0	5.1	0.29
	6.2	T	4.8	5.1	4.6	5.0	5.5	5.9	5.2	0.43
		$\tilde{T}$	5.0	5.3	4.9	5.3	5.7	6.1	5.4	0.41
1%	-6.0	T	0.9	0.7	1.0	0.8	0.7	0.7	0.8	0.11
		$\tilde{T}$	1.0	0.8	1.1	0.9	0.8	0.7	0.9	0.12
	0.0	T	0.9	0.8	1.0	1.0	1.3	1.4	1.1	0.21
		$\tilde{T}$	1.0	0.9	1.0	1.1	1.4	1.5	1.2	0.23
	6.2	T	0.8	1.0	0.7	1.4	1.9	1.8	1.3	0.46
		$\tilde{T}$	0.8	1.1	0.8	1.5	2.0	2.0	1.4	0.50

TABLE 4. Actual test sizes of several test statistics for testing the equality (HL)

Nominal Size	$m_4^{(1)}$	Test Statistic	Distribution 1	2	3	4	5	6	Average	S.D.
10%	-6.0	T	12.6	12.2	12.9	12.6	11.4	11.4	12.2	0.57
		$\tilde{T}$	10.8	10.4	11.1	10.8	9.7	9.6	10.4	0.57
		$\tilde{T}'$	10.5	10.0	10.6	10.3	9.3	9.1	10.0	0.58
	0.0	T	12.7	12.5	12.6	13.0	12.5	12.4	12.6	0.18
		$\tilde{T}$	10.9	10.8	11.0	11.3	11.1	10.5	10.9	0.26
		$\tilde{T}'$	10.4	10.4	10.4	10.8	10.7	10.1	10.5	0.25
	6.2	T	13.2	13.0	12.7	12.2	13.1	12.4	12.7	0.36
		$\tilde{T}$	11.3	11.3	10.9	10.5	11.4	10.9	11.0	0.30
		$\tilde{T}'$	10.9	10.8	10.4	10.2	10.9	10.5	10.6	0.28
5%	-6.0	T	7.3	6.9	7.4	7.0	6.7	6.1	6.9	0.44
		$\tilde{T}$	5.9	5.7	6.0	5.9	5.7	5.1	5.7	0.28
		$\tilde{T}'$	5.3	5.0	5.4	5.1	4.9	4.5	5.0	0.29
	0.0	T	7.2	7.2	7.2	7.7	7.6	7.0	7.3	0.25
		$\tilde{T}$	5.9	6.0	5.8	6.6	6.5	5.9	6.1	0.31
		$\tilde{T}'$	5.2	5.4	5.2	5.8	5.8	5.3	5.4	0.27
	6.2	T	7.5	7.6	7.2	7.5	7.9	8.1	7.6	0.30
		$\tilde{T}$	6.3	6.4	6.0	6.4	6.7	7.0	6.5	0.33
		$\tilde{T}'$	5.4	5.7	5.3	5.7	6.1	6.5	5.8	0.42
1%	-6.0	T	2.3	1.8	2.3	2.1	2.0	1.8	2.0	0.18
		$\tilde{T}$	1.7	1.4	1.7	1.6	1.5	1.4	1.5	0.12
		$\tilde{T}'$	1.2	0.9	1.2	1.1	1.0	0.9	1.0	0.12
	0.0	T	2.1	2.2	2.1	2.4	2.7	2.5	2.3	0.23
		$\tilde{T}$	1.7	1.7	1.7	2.0	2.1	2.0	1.9	0.18
		$\tilde{T}'$	1.2	1.1	1.1	1.2	1.7	1.6	1.3	0.24
	6.2	T	2.3	2.2	2.0	2.7	3.4	3.6	2.7	0.62
		$\tilde{T}$	1.6	1.8	1.5	2.2	2.9	3.0	2.2	0.57
		$\tilde{T}'$	0.9	1.2	0.9	1.6	2.2	2.2	1.5	0.52

Table 5. Actual test sizes of several test statistics for testing the equality (BNP)

Nominal Size	$m_4^{(1)}$	Test Statistic	Distribution 1	2	3	4	5	6	Average	S.D.
10%	-6.0	T	6.3	6.1	6.5	6.3	6.1	5.5	6.1	0.33
		$\tilde{T}$	8.9	8.5	9.0	8.8	7.9	7.8	8.5	0.47
		$\tilde{T}'$	9.2	8.8	9.3	9.1	8.1	7.9	8.7	0.53
	0.0	T	6.4	6.5	6.3	7.1	6.8	6.4	6.6	0.27
		$\tilde{T}$	9.0	9.0	8.9	9.3	9.2	8.5	9.0	0.26
		$\tilde{T}'$	9.2	9.2	9.3	9.6	9.4	8.8	9.2	0.25
	6.2	T	6.7	6.7	6.3	6.8	7.0	7.4	6.8	0.32
		$\tilde{T}$	9.2	9.3	8.9	8.9	9.4	9.2	9.1	0.17
		$\tilde{T}'$	9.5	9.6	9.2	9.1	9.7	9.4	9.4	0.21
5%	-6.0	T	2.6	2.1	2.5	2.4	2.2	2.0	2.3	0.21
		$\tilde{T}$	3.7	3.4	4.0	3.6	3.5	3.2	3.6	0.24
		$\tilde{T}'$	4.1	3.9	4.5	4.1	3.9	3.7	4.0	0.24
	0.0	T	2.3	2.4	2.4	2.8	3.0	2.8	2.6	0.25
		$\tilde{T}$	3.7	3.7	3.7	4.2	4.3	3.8	3.9	0.24
		$\tilde{T}'$	4.1	4.2	4.1	4.7	4.8	4.3	4.4	0.27
	6.2	T	2.6	2.6	2.3	3.0	3.6	3.8	3.0	0.58
		$\tilde{T}$	3.8	4.1	3.8	4.2	4.6	5.3	4.3	0.51
		$\tilde{T}'$	4.4	4.6	4.2	4.6	5.1	5.6	4.8	0.46
1%	-6.0	T	0.2	0.2	0.2	0.1	0.2	0.1	0.2	0.03
		$\tilde{T}$	0.4	0.3	0.5	0.4	0.3	0.3	0.4	0.08
		$\tilde{T}'$	0.7	0.5	0.8	0.6	0.5	0.5	0.6	0.11
	0.0	T	0.2	0.2	0.2	0.2	0.3	0.4	0.2	0.07
		$\tilde{T}$	0.4	0.4	0.5	0.5	0.7	0.7	0.5	0.14
		$\tilde{T}'$	0.7	0.6	0.7	0.8	1.0	1.0	0.8	0.16
	6.2	T	0.1	0.2	0.1	0.4	0.5	0.7	0.3	0.22
		$\tilde{T}$	0.3	0.4	0.3	0.7	1.0	1.1	0.6	0.31
		$\tilde{T}'$	0.6	0.7	0.5	1.1	1.4	1.4	0.9	0.38

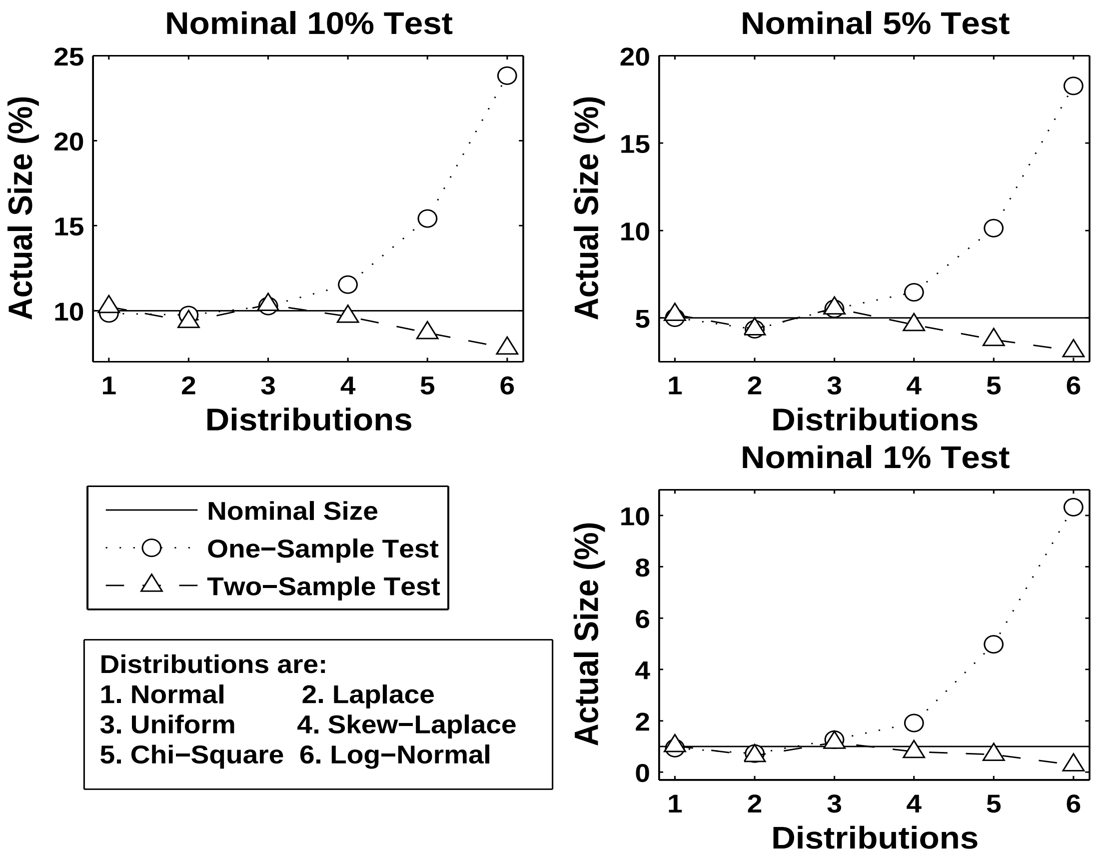

FIGURE 1. Actual test sizes of Hotelling's one- and two-sample tests