On the distribution of test statistic using Srivastava's skewness and kurtosis

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Abstract

In this paper, we consider the multivariate normality test based on the sample measures of multivariate skewness and kurtosis defined by Srivastava (1984). Jarque and Bera (1987) proposed the test statistic by using both the univariate sample skewness and kurtosis as the univariate normality test. For the multivariate case, Koizumi, Okamoto and Seo (2009) proposed test statistics by using Srivastava's sample skewness and kurtosis, which are asymptotically distributed as χ^2-distribution. However, they did not derive variance of their test statistics. We propose a new test statistic using variance of the test statistic derived by Koizumi, Okamoto and Seo (2009). In order to evaluate accuracy of proposed test statistic, the numerical results by Monte Carlo simulation for some selected values of parameters are presented.

Key Words and Phrases: multivariate skewness; multivariate kurtosis; Jarque-Bera test; test for multivariate normality.

1 Introduction

In statistical analysis, the test for normality is an important problem. This problem has been considered by many authors. Shapiro and Wilk (1965) derived test statistic using order statistic. This is called Shapiro-Wilk test as the univariate normality test. Multivariate extensions of the Shapiro-Wilk test were proposed by Malkovich and Afifi (1973), Royston (1983), Srivastava

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and Hui (1987) and so on. Mardia (1970) and Srivastava (1984) gave different definitions of the multivariate sample skewness and kurtosis, and discussed test statistics using these measures for assessing multivariate normality. Mardia (1974) derived expectations and variances of multivariate sample skewness and kurtosis, and discussed their asymptotic distributions using expectation and variance of sample skewness or kurtosis. Okamoto and Seo (2008) derived the improved approximate χ^2 test statistic using the multivariate sample skewness of Srivastava (1984). Approximate accuracy of upper percentile of this test statistic is better than that of Srivastava (1984) especially for small sample size N. Test statistics using the multivariate sample kurtosis of Srivastava (1984) were discussed by Seo and Ariga (2006).

For univariate sample case, Jarque and Bera (1987) proposed the bivariate test using univariate sample skewness and kurtosis. The improved Jarque-Bera test statistics have been considered by many authors. Mardia and Foster (1983) proposed the test statistics using Mardia's sample skewness and kurtosis. Test statistics using Srivastava's sample skewness and kurtosis which are asymptotically distributed as χ^2-distribution were proposed by Koizumi, Okamoto and Seo (2009). However, for small N, there is the difference between the distribution of their statistics and χ^2-distribution. Thus, it seems that the multivariate normality test cannot be carried out correctly. Koizumi, Okamoto and Seo (2009) did not derive variance of their test statistics. Our purpose is to propose a new test statistic by using variance of the test statistic derived by Koizumi, Okamoto and Seo (2009). We investigate accuracy of variances, upper percentiles, type I error and power for multivariate Jarque-Bera test statistics via a Monte Carlo simulation for selected values of parameters.

2 Test statistics using multivariate measures of sample skewness and kurtosis

2.1 Srivastava's measures of multivariate skewness and kurtosis

Let $\boldsymbol{x}$ be a p-dimensional random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma = \Gamma D_\lambda \Gamma'$, where $\Gamma = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p)$ is an orthogonal matrix and $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Note that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of Σ. Then Srivastava (1984) defined the population measures of multivariate skewness and kurtosis as follows:

$$\beta_{1,p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{\mathrm{E}[(y_i-\theta_i)^3]}{\lambda_i^{\frac{3}{2}}}\right\}^2,$$
$$\beta_{2,p} = \frac{1}{p}\sum_{i=1}^{p}\frac{\mathrm{E}[(y_i-\theta_i)^4]}{\lambda_i^2},$$

respectively, where $y_i = \boldsymbol{\gamma}_i'\boldsymbol{x}$ and $\theta_i = \boldsymbol{\gamma}_i'\boldsymbol{\mu}$ $(i = 1, 2, \ldots, p)$. We note that $\beta_{1,p}^2 = 0$, $\beta_{2,p} = 3$ under a multivariate normal population.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be samples of size N from a multivariate population. Let $\overline{\boldsymbol{x}}$ and $S = HD_\omega H'$ be sample mean vector and sample covariance matrix as follows:

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_j,$$
$$S = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j-\overline{\boldsymbol{x}})(\boldsymbol{x}_j-\overline{\boldsymbol{x}})',$$

respectively, where $H = (\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_p)$ is an orthogonal matrix and $D_\omega = \mathrm{diag}\,(\omega_1, \omega_2, \ldots, \omega_p)$. We note that $\omega_1, \omega_2, \ldots, \omega_p$ are the eigenvalues of S. Then Srivastava (1984) defined the sample measures of multivariate skewness and kurtosis as follows:

$$b_{1,p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{1}{\omega_i^{\frac{3}{2}}}\sum_{j=1}^{N}\frac{(y_{ij}-\overline{y}_i)^3}{N}\right\}^2,$$
$$b_{2,p} = \frac{1}{p}\sum_{i=1}^{p}\frac{1}{\omega_i^2}\sum_{j=1}^{N}\frac{(y_{ij}-\overline{y}_i)^4}{N},$$

respectively, where $y_{ij} = \boldsymbol{h}_i'\boldsymbol{x}_j$ $(i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, N)$, $\overline{y}_i = N^{-1}\sum_{j=1}^{N} y_{ij}$ $(i = 1, 2, \ldots, p)$.

We note that

$$\omega_i = \boldsymbol{h}_i'S\boldsymbol{h}_i = \frac{1}{N}\sum_{j=1}^{N}(y_{ij}-\overline{y}_i)^2,\ i = 1, 2, \ldots, p.$$

Then we can write

$$b_{1,p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{m_{3i}}{m_{2i}^{\frac{3}{2}}}\right\}^2,$$

$$b_{2,p} = \frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2},$$

where $m_{\nu i} = N^{-1}\sum_{j=1}^{N}(y_{ij} - \overline{y}_i)^{\nu}$. Note that $\boldsymbol{h}_i \to \boldsymbol{\gamma}_i$ with probability one.

Now, we discuss under the hypothesis of multivariate normality. For large N, we get

$$y_{ij} = \boldsymbol{h}_i'\boldsymbol{x}_j \sim N(\boldsymbol{\gamma}_i'\boldsymbol{\mu}, \lambda_i).$$

Thus, $y_{i1}, y_{i2}, \ldots, y_{iN}$ are asymptotically independently normally distributed. Hence, for large N

$$\mathrm{E}[m_{\nu i}^k m_{2i}^{-\nu k/2}]\mathrm{E}[m_{2i}^{\nu k/2}] = \mathrm{E}[m_{\nu i}^k].$$

These results are numerically checked for large N. Note that " = " means "nearly equal" because large sample size is needed. In order to avoid the dependence of y_{ij} and $\overline{y}_i$, let $\overline{y}_i^{(\alpha)}$ be a mean defined on the subset of $y_{i1}, y_{i2}, \ldots, y_{iN}$, that is

$$\overline{y}_i^{(\alpha)} = \frac{1}{N-1}\sum_{j=1, j\neq\alpha}^{N} y_{ij}.$$

Therefore, $y_{i\alpha}$ is independent of $\overline{y}_i^{(\alpha)}$. In order to obtain the expectations of $b_{1,p}^2$ and $b_{2,p}$, we put

$$\overline{y}_i^{(\alpha)} = \frac{1}{\sqrt{N-1}}Z_i.$$

Since Z_i is distributed as standard normal distribution for large N, the odd order moments equal zero and

$$\mathrm{E}[Z_i^{2k}] = (2k-1)\cdots 5\cdot 3\cdot 1,\ k = 1, 2, \ldots.$$

Srivastava (1984) obtained the following lemma:

Lemma 1 (Srivastava (1984)). *For large N, the asymptotic expectations of $b_{1,p}$ and $b_{1,p}^2$, and asymptotic expectation and variance of $b_{2,p}$ when the population is $N_p(\boldsymbol{\mu}, \Sigma)$ are given by*

$$\mathrm{E}[b_{1,p}] = 0, \quad \mathrm{E}[b_{1,p}^2] = \frac{6}{N}, \quad \mathrm{E}[b_{2,p}] = 3, \quad \mathrm{Var}[b_{2,p}] = \frac{24}{Np}.$$

By using Lemma 1, Srivastava (1984) derived the following theorem:

Theorem 1 (Srivastava (1984)). *For large N, test statistics for assessing multivariate normality using the sample measure of multivariate skewness or kurtosis are distributed as follows:*

$$\frac{Np}{6} b_{1,p}^2 \to \chi_p^2,$$
$$\sqrt{\frac{Np}{24}}(b_{2,p} - 3) \to N(0, 1),$$

respectively.

Further, Seo and Ariga (2006) gave expectation of the multivariate sample kurtosis. Okamoto and Seo (2008) gave the expectation of multivariate sample skewness without using asymptotic expansion, and derived improved approximate χ^2 test statistic. Koizumi, Okamoto and Seo (2009) derived variance of multivariate sample kurtosis.

Lemma 2. *For large N, the expectation of $b_{1,p}^2$, and expectation and variance of $b_{2,p}$ when the population is $N_p(\boldsymbol{\mu}, \Sigma)$ are given by*

$$\mathrm{E}[b_{1,p}^2] = \frac{6(N-2)}{(N+1)(N+3)},$$
$$\mathrm{E}[b_{2,p}] = \frac{3(N-1)}{N+1},$$
$$\mathrm{Var}[b_{2,p}] = \frac{24N(N-2)(N-3)}{p(N+1)^2(N+3)(N+5)},$$

respectively.

Okamoto and Seo (2008) derived the test statistic using $b_{1,p}^2$, and Koizumi, Okamoto and Seo (2009) derived the test statistic using $b_{2,p}$ for assessing multivariate normality and gave the following theorem:

Theorem 2. *For large N,*

$$\frac{(N+1)(N+3)}{6(N-2)} b_{1,p}^2 \to \chi_p^2,$$
$$\frac{\sqrt{p(N+3)(N+5)}\left\{(N+1)b_{2,p} - 3(N-1)\right\}}{\sqrt{24N(N-2)(N-3)}} \to N(0, 1),$$

respectively.

2.2 The multivariate Jarque-Bera test statistics

Jarque and Bera (1987) proposed the test statistic using univariate sample skewness and kurtosis for normality test and gave the following theorem:

Theorem 3 (Jarque and Bera (1987)). *Let $x_1, x_2, \ldots, x_N$ be samples of size N from a univariate population. And let $\overline{x} = N^{-1}\sum_{j=1}^{N} x_j$ and $s^2 = N^{-1}\sum_{j=1}^{N}(x_j - \overline{x})^2$ be the sample mean and the sample covariance, respectively. The univariate sample skewness is $\sqrt{b_1} = m_3/m_2^{\frac{3}{2}}$ and sample kurtosis is $b_2 = m_4/m_2^2$, where $m_r = N^{-1}\sum_{j=1}^{N}(x_j - \overline{x})^r$. Then the test statistic using univariate sample skewness and kurtosis is given by*

$$JB = N\left\{\frac{(\sqrt{b_1})^2}{6} + \frac{(b_2-3)^2}{24}\right\}.$$

It holds that JB statistic is asymptotically distributed as χ_2^2-distribution under normality.

On the other hand, Koizumi, Okamoto and Seo (2009) proposed test statistics for multivariate normality. From Theorem 1 and Theorem 2, Koizumi, Okamoto and Seo (2009) proposed test statistics as follows:

$$MJB = Np\left\{\frac{b_{1,p}^2}{6} + \frac{(b_{2,p}-3)^2}{24}\right\} \to \chi_{p+1}^2 \qquad (N \to \infty),$$

$$MJB^* = \frac{pb_{1,p}^2}{\mathrm{E}[b_{1,p}^2]} + \frac{(b_{2,p} - \mathrm{E}[b_{2,p}])^2}{\mathrm{Var}[b_{2,p}]} \to \chi_{p+1}^2 \qquad (N \to \infty),$$

respectively.

MJB^* statistic is improved so that accuracy of upper percentile for approximate test statistic is better than that of MJB statistic for small N. However, especially for small N, it seems that there is difference between the distribution of MJB^* statistic and χ^2-distribution. Hence, we propose a new test statistic to be closer to the upper percentile of χ^2-distribution by using MJB^* statistic.

2.3 A new multivariate Jarque-Bera test statistic using variance of MJB^*

In this subsection, we propose a new multivariate Jarque-Bera test statistic using test statistic MJB^* derived by Koizumi, Okamoto and Seo (2009). We consider variance $\mathrm{Var}[MJB^*]$ under normality which is not derived by Koizumi, Okamoto and Seo (2009). Now, expectation

$\mathrm{E}[MJB^*]$ is $p+1$. We obtain variance $\mathrm{Var}[MJB^*]$ as follows:

$$\mathrm{Var}[MJB^*] = \mathrm{Var}[T_1] + \mathrm{Var}[T_2] + 2\mathrm{Cov}[T_1, T_2],$$

where

$$T_1 = \frac{p b_{1,p}^2}{\mathrm{E}[b_{1,p}^2]}, \quad T_2 = \frac{(b_{2,p} - \mathrm{E}[b_{2,p}])^2}{\mathrm{Var}[b_{2,p}]}.$$

We note that

$$\begin{aligned}
\mathrm{Var}[T_1] &= \frac{p^2}{(\mathrm{E}[b_{1,p}^2])^2}\mathrm{Var}[b_{1,p}^2],\\
\mathrm{Var}[T_2] &= \frac{1}{(\mathrm{Var}[b_{2,p}])^2}\Big[\mathrm{E}[b_{2,p}^4] - 4\mathrm{E}[b_{2,p}]\mathrm{E}[b_{2,p}^3] + \mathrm{E}[b_{2,p}^2]\big\{8(\mathrm{E}[b_{2,p}])^2 - \mathrm{E}[b_{2,p}^2]\big\} - 4(\mathrm{E}[b_{2,p}])^4\Big],\\
\mathrm{Cov}[T_1, T_2] &= \frac{p}{\mathrm{E}[b_{1,p}^2]\mathrm{Var}[b_{2,p}]}\big\{\mathrm{Cov}[b_{1,p}^2, b_{2,p}^2] - 2\mathrm{E}[b_{2,p}]\mathrm{Cov}[b_{1,p}^2, b_{2,p}]\big\},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Cov}[b_{1,p}^2, b_{2,p}] &= \mathrm{E}[b_{1,p}^2 b_{2,p}] - \mathrm{E}[b_{1,p}^2]\mathrm{E}[b_{2,p}],\\
\mathrm{Cov}[b_{1,p}^2, b_{2,p}^2] &= \mathrm{E}[b_{1,p}^2 b_{2,p}^2] - \mathrm{E}[b_{1,p}^2]\mathrm{E}[b_{2,p}^2].
\end{aligned}$$

Therefore, we derive the moments $\mathrm{E}[b_{2,p}^3]$, $\mathrm{E}[b_{2,p}^4]$, $\mathrm{E}[b_{1,p}^2 b_{2,p}]$ and $\mathrm{E}[b_{1,p}^2 b_{2,p}^2]$.

Now, we consider expectation $\mathrm{E}[b_{2,p}^3]$. Then we have

$$\begin{aligned}
\mathrm{E}[b_{2,p}^3] &= \mathrm{E}\left[\left\{\frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2}\right\}^3\right]\\
&= \frac{1}{p^3}\left\{p\mathrm{E}\left[\frac{m_{4i}^3}{m_{2i}^6}\right] + 3p(p-1)\mathrm{E}\left[\frac{m_{4i}^2 m_{4j}}{m_{2i}^4 m_{2j}^2}\right] + p(p-1)(p-2)\mathrm{E}\left[\frac{m_{4i}m_{4j}m_{4k}}{m_{2i}^2 m_{2j}^2 m_{2k}^2}\right]\right\},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{E}[m_{4i}] &= \left(1 - \frac{1}{N}\right)^4 \mathrm{E}[C_{i\alpha}^4],\\
\mathrm{E}[m_{4i}^2] &= \frac{1}{N^2}\left(1 - \frac{1}{N}\right)^8 \Big\{N\mathrm{E}[C_{i\alpha}^8] + N(N-1)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4]\Big\},\\
\mathrm{E}[m_{4i}^3] &= \frac{1}{N^3}\left(1 - \frac{1}{N}\right)^{12} \Big\{N\mathrm{E}[C_{i\alpha}^{12}] + 3N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4] + N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4]\Big\},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}[m_{2i}^2] &= \frac{1}{N^2}\left(1-\frac{1}{N}\right)^4 \Big\{N\mathrm{E}[C_{i\alpha}^4] + N(N-1)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2]\Big\},\\
\mathrm{E}[m_{2i}^4] &= \frac{1}{N^4}\left(1-\frac{1}{N}\right)^8 \Big\{N\mathrm{E}[C_{i\alpha}^8] + 4N(N-1)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^2]\\
&\quad + 3N(N-1)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4] + 6N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2 C_{i\gamma}^2]\\
&\quad + N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2]\Big\},\\
\mathrm{E}[m_{2i}^6] &= \frac{1}{N^6}\left(1-\frac{1}{N}\right)^{12} \Big\{N\mathrm{E}[C_{i\alpha}^{12}] + 6N(N-1)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^2]\\
&\quad + 15N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4] + 15N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^2 C_{i\gamma}^2]\\
&\quad + 10N(N-1)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^6] + 60N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4 C_{i\gamma}^2]\\
&\quad + 20N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2]\\
&\quad + 15N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4]\\
&\quad + 45N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^2 C_{i\delta}^2]\\
&\quad + 15N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2]\\
&\quad + N(N-1)(N-2)(N-3)(N-4)(N-5)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2]\Big\}
\end{aligned}
$$

and $C_{i\alpha} = y_{i\alpha} - \overline{y}_i^{(\alpha)}$. It is easy to calculate expectation because $y_{i\alpha}$ and $\overline{y}_i^{(\alpha)}$ are independent. Also, we calculate with $\lambda_i = 1$ because $b_{1,p}^2$ and $b_{2,p}$ are hardly influenced by Σ for large N.

After a great deal of calculation of expectation, we obtain

$$
\begin{aligned}
\mathrm{E}[m_{4i}] &= \frac{3(N-1)^2}{N^2},\\
\mathrm{E}[m_{4i}^2] &= \frac{3(N-1)(3N^3+23N^2-63N+45)}{N^4},\\
\mathrm{E}[m_{4i}^3] &= \frac{27(N-1)(N^5+27N^4+226N^3-1098N^2+1725N-945)}{N^6},\\
\mathrm{E}[m_{2i}^2] &= \frac{(N-1)(N+1)}{N^2},\\
\mathrm{E}[m_{2i}^4] &= \frac{(N-1)(N+1)(N+3)(N+5)}{N^4},\\
\mathrm{E}[m_{2i}^6] &= \frac{(N-1)(N+1)(N+3)(N+5)(N+7)(N+9)}{N^6},
\end{aligned}
$$

and we can obtain the expectation for $b_{2,p}^3$ as

$$
\mathrm{E}[b_{2,p}^3] = 27\big\{p^2N^7 + p(21p+8)N^6 + (137p^2+80p+64)N^5 + (197p^2-176p-640)N^4
$$

$$
\begin{aligned}
&- (693p^2 + 1664p - 2112)N^3 - (809p^2 - 4776p + 2560)N^2 \\
&+ 3(697p^2 - 1008p + 256)N - 945p^2\big\} \\
&\times \frac{1}{p^2(N+1)^3(N+3)(N+5)(N+7)(N+9)}.
\end{aligned}
$$

Hence, we have the moments of skewness and kurtosis as follows:

$$
\begin{aligned}
\mathrm{E}[b_{2,p}^4] &= \mathrm{E}\left[\left\{\frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2}\right\}^4\right] \\
&= \frac{1}{p^4}\left\{p\mathrm{E}\left[\frac{m_{4i}^4}{m_{2i}^8}\right] + 4p(p-1)\mathrm{E}\left[\frac{m_{4i}^3 m_{4j}}{m_{2i}^6 m_{2j}^2}\right] + 3p(p-1)\mathrm{E}\left[\frac{m_{4i}^2 m_{4j}^2}{m_{2i}^4 m_{2j}^4}\right]\right. \\
&\quad + 6p(p-1)(p-2)\mathrm{E}\left[\frac{m_{4i}^2 m_{4j} m_{4k}}{m_{2i}^4 m_{2j}^2 m_{2k}^2}\right] \\
&\quad \left. + p(p-1)(p-2)(p-3)\mathrm{E}\left[\frac{m_{4i} m_{4j} m_{4k} m_{4l}}{m_{2i}^2 m_{2j}^2 m_{2k}^2 m_{2l}^2}\right]\right\}, \\
\mathrm{E}[b_{1,p}^2 b_{2,p}] &= \mathrm{E}\left[\left(\frac{1}{p}\sum_{i=1}^{p}\frac{m_{3i}^2}{m_{2i}^3}\right)\left(\frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2}\right)\right] \\
&= \frac{1}{p}\left\{\mathrm{E}\left[\frac{m_{3i}^2 m_{4i}}{m_{2i}^5}\right] + (p-1)\mathrm{E}\left[\frac{m_{3i}^2 m_{4j}}{m_{2i}^3 m_{2j}^2}\right]\right\}, \\
\mathrm{E}[b_{1,p}^2 b_{2,p}^2] &= \mathrm{E}\left[\left(\frac{1}{p}\sum_{i=1}^{p}\frac{m_{3i}^2}{m_{2i}^3}\right)\left(\frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2}\right)^2\right] \\
&= \frac{1}{p^2}\left\{\mathrm{E}\left[\frac{m_{3i}^2 m_{4i}^2}{m_{2i}^7}\right] + (p-1)\mathrm{E}\left[\frac{m_{3i}^2 m_{4j}^2}{m_{2i}^3 m_{2j}^4}\right] + 2(p-1)\mathrm{E}\left[\frac{m_{3i}^2 m_{4i} m_{4j}}{m_{2i}^5 m_{2j}^2}\right]\right. \\
&\quad \left. + (p-1)(p-2)\mathrm{E}\left[\frac{m_{3i}^2 m_{4j} m_{4k}}{m_{2i}^3 m_{2j}^2 m_{2k}^2}\right]\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{E}[m_{4i}^4] &= \frac{1}{N^4}\left(1-\frac{1}{N}\right)^{16}\Big\{N\mathrm{E}[C_{i\alpha}^{16}] + 4N(N-1)\mathrm{E}[C_{i\alpha}^{12}C_{i\beta}^4] + 3N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^8] \\
&\quad + 6N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4 C_{i\gamma}^4] + N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4 C_{i\delta}^4]\Big\}, \\
\mathrm{E}[m_{3i}^2] &= \frac{1}{N^2}\left(1-\frac{1}{N}\right)^{6}\Big\{N\mathrm{E}[C_{i\alpha}^6] + N(N-1)\mathrm{E}[C_{i\alpha}^3 C_{i\beta}^3]\Big\}, \\
\mathrm{E}[m_{2i}^3] &= \frac{1}{N^3}\left(1-\frac{1}{N}\right)^{6}\Big\{N\mathrm{E}[C_{i\alpha}^6] + 3N(N-1)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2] \\
&\quad + N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[m_{2i}^5] = \frac{1}{N^5}\left(1-\frac{1}{N}\right)^{10} \Big\{ & N\mathrm{E}[C_{i\alpha}^{10}] + 5N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^2] + 10N(N-1)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4] \\
& + 10N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^2 C_{i\gamma}^2] + 15N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^2] \\
& + 10N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2] \\
& + N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2]\Big\}, \\
\mathrm{E}[m_{2i}^7] = \frac{1}{N^7}\left(1-\frac{1}{N}\right)^{14} \Big\{ & N\mathrm{E}[C_{i\alpha}^{14}] + 7N(N-1)\mathrm{E}[C_{i\alpha}^{12} C_{i\beta}^2] + 21N(N-1)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^4] \\
& + 21N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^2 C_{i\gamma}^2] + 35N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^6] \\
& + 105N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4 C_{i\gamma}^2] \\
& + 35N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2] \\
& + 70N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^6 C_{i\gamma}^2] + 105N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4 C_{i\gamma}^4] \\
& + 210N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4 C_{i\gamma}^2 C_{i\delta}^2] \\
& + 35N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2] \\
& + 105N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4 C_{i\delta}^2] \\
& + 105N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2] \\
& + 21N(N-1)(N-2)(N-3)(N-4)(N-5)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2] \\
& + N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2 C_{i\eta}^2]\Big\}, \\
\mathrm{E}[m_{2i}^8] = \frac{1}{N^8}\left(1-\frac{1}{N}\right)^{16} \Big\{ & N\mathrm{E}[C_{i\alpha}^{16}] + 8N(N-1)\mathrm{E}[C_{i\alpha}^{14} C_{i\beta}^2] \\
& + 28N(N-1)\mathrm{E}[C_{i\alpha}^{12} C_{i\beta}^4] + 28N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^{12} C_{i\beta}^2 C_{i\gamma}^2] \\
& + 56N(N-1)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^6] + 168N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^4 C_{i\gamma}^2] \\
& + 56N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2] \\
& + 35N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^8] + 280N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^6 C_{i\gamma}^2] \\
& + 210N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4 C_{i\gamma}^4] \\
& + 420N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^4 C_{i\gamma}^2 C_{i\delta}^2] \\
& + 70N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2]
\end{aligned}
$$

$$
\begin{aligned}
&+280N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^6 C_{i\gamma}^4] \\
&+280N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^6 C_{i\gamma}^2 C_{i\delta}^2] \\
&+840N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4 C_{i\gamma}^4 C_{i\delta}^2] \\
&+560N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2] \\
&+56N(N-1)(N-2)(N-3)(N-4)(N-5)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2] \\
&+105N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4 C_{i\delta}^4] \\
&+420N(N-1)(N-2)(N-3)(N-4)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^4 C_{i\delta}^2 C_{i\epsilon}^2] \\
&+210N(N-1)(N-2)(N-3)(N-4)(N-5)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2] \\
&+28N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2 C_{i\eta}^2] \\
&+N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)(N-7) \\
&\times \mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2 C_{i\gamma}^2 C_{i\delta}^2 C_{i\epsilon}^2 C_{i\zeta}^2 C_{i\eta}^2 C_{i\theta}^2]\Big\}, \\
\mathrm{E}[m_{3i}^2 m_{4i}] = {}& \frac{1}{N^3}\left(1-\frac{1}{N}\right)^{10}\Big\{N\mathrm{E}[C_{i\alpha}^{10}] + 2N(N-1)\mathrm{E}[C_{i\alpha}^7 C_{i\beta}^3] + N(N-1)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4] \\
&+N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^3 C_{i\gamma}^3]\Big\}, \\
\mathrm{E}[m_{3i}^2 m_{4i}^2] = {}& \frac{1}{N^4}\left(1-\frac{1}{N}\right)^{14}\Big\{N\mathrm{E}[C_{i\alpha}^{14}] + 2N(N-1)\mathrm{E}[C_{i\alpha}^{11} C_{i\beta}^3] \\
&+2N(N-1)\mathrm{E}[C_{i\alpha}^{10} C_{i\beta}^4] + N(N-1)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^6] \\
&+N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^8 C_{i\beta}^3 C_{i\gamma}^3] + 2N(N-1)\mathrm{E}[C_{i\alpha}^7 C_{i\beta}^7] \\
&+4N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^7 C_{i\beta}^4 C_{i\gamma}^3] + N(N-1)(N-2)\mathrm{E}[C_{i\alpha}^6 C_{i\beta}^4 C_{i\gamma}^4] \\
&+N(N-1)(N-2)(N-3)\mathrm{E}[C_{i\alpha}^4 C_{i\beta}^4 C_{i\gamma}^3 C_{i\delta}^3]\Big\}.
\end{aligned}
$$

After a great deal of calculation of expectation, we obtain

$$
\begin{aligned}
\mathrm{E}[m_{4i}^4] &= \frac{27(N-1)}{N^8} \times (3N^7 + 171N^6 + 4111N^5 + 42871N^4 - 308727N^3 + 776385N^2 \\
&\quad - 897435N + 405405), \\
\mathrm{E}[m_{3i}^2] &= \frac{6(N-1)(N-2)}{N^3}, \\
\mathrm{E}[m_{2i}^3] &= \frac{(N-1)(N+1)(N+3)}{N^3},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[m_{2i}^5] &= \frac{(N-1)(N+1)(N+3)(N+5)(N+7)}{N^5}, \\
\mathrm{E}[m_{2i}^7] &= \frac{(N-1)(N+1)(N+3)(N+5)(N+7)(N+9)(N+11)}{N^7}, \\
\mathrm{E}[m_{2i}^8] &= \frac{(N-1)(N+1)(N+3)(N+5)(N+7)(N+9)(N+11)(N+13)}{N^8}, \\
\mathrm{E}[m_{3i}^2 m_{4i}] &= \frac{18(N-1)(N-2)(N^2+22N-35)}{N^5}, \\
\mathrm{E}[m_{3i}^2 m_{4i}^2] &= \frac{18(N-1)(N-2)(3N^4+164N^3+3346N^2-11700N+10395)}{N^7},
\end{aligned}
$$

and we can obtain the moments as

$$
\begin{aligned}
\mathrm{E}[b_{2,p}^4] = 27\big\{ & 3p^3N^{12} + 12p^2(13p+4)N^{11} + 2p(1659p^2+984p+416)N^{10} \\
& + 12(3069p^3+2424p^2+1504p+960)N^9 \\
& + (221565p^3+165312p^2+46016p-85248)N^8 \\
& + 8(78663p^3-3996p^2-105568p-34368)N^7 \\
& + 12(6687p^3-276552p^2-155312p+249984)N^6 \\
& - 8(435261p^3+584736p^2-1781584p+278496)N^5 \\
& - 3(1164721p^3-7721536p^2-2766528p+7525120)N^4 \\
& + 12(770105p^3+1570500p^2-7438208p+4324608)N^3 \\
& + 18(330851p^3-4060968p^2+5142144p-1830912)N^2 \\
& - 540(28283p^3-72072p^2+36608p-7680)N + 6081075p^3\big\} \\
& \times \frac{1}{p^3(N+1)^4(N+3)^2(N+5)^2(N+7)(N+9)(N+11)(N+13)}, \\
\mathrm{E}[b_{1,p}^2 b_{2,p}] = {} & \frac{18(N-2)\{pN^3+(11p+12)N^2+(23p-36)N-35p\}}{p(N+1)^2(N+3)(N+5)(N+7)}, \\
\mathrm{E}[b_{1,p}^2 b_{2,p}^2] = {} & \frac{18(N-2)}{p^2(N+1)^3(N+3)^2(N+5)(N+7)(N+9)(N+11)} \times \big\{3p^2N^7 + p(99p+80)N^6 \\
& + (1203p^2+1544p+1440)N^5 + (6315p^2+5920p-6048)N^4 \\
& + (10737p^2-22160p-12768)N^3 - 3(4853p^2+22480p-20704)N^2 \\
& - 9(3887p^2-10824p+2752)N + 31185p^2\big\}.
\end{aligned}
$$

Thus, we can obtain

$$\mathrm{Cov}[b_{1,p}^2, b_{2,p}] = \frac{216N(N-2)(N-3)}{p(N+1)^2(N+3)(N+5)(N+7)},$$

$$\mathrm{Cov}[b_{1,p}^2, b_{2,p}^2] = \frac{432N(N-2)(N-3)}{p^2(N+1)^3(N+3)^2(N+5)(N+7)(N+9)(N+11)} \times \left\{3pN^4 + 6(11p+10)N^3 + 24(17p-3)N^2 + 2(207p-374)N - 891p + 344\right\},$$

$$\mathrm{Cov}[T_1, T_2] = \frac{12(15N^3 - 18N^2 - 187N + 86)}{(N-2)(N+7)(N+9)(N+11)},$$

and we can derive variance $\mathrm{Var}[MJB^*]$ as follows:

$$\begin{aligned}\mathrm{Var}[MJB^*] = &\frac{2}{pN(N-2)(N-3)(N+5)(N+7)(N+9)(N+11)(N+13)} \\ &\times \left\{p(p+1)N^8 + 2(29p^2+110p+135)N^7 + (859p^2+3055p+702)N^6\right. \\ &+ 2(1058p^2 - 217p - 7272)N^5 - (21665p^2 + 71105p + 38844)N^4 \\ &- 2(13471p^2 + 10792p - 96183)N^3 + 3(44759p^2 + 130587p + 134898)N^2 \\ &\left.+ 90(767p - 6222)N + 81000\right\}, \ (N \neq 2, 3). \qquad (1)\end{aligned}$$

By using variance $\mathrm{Var}[MJB^*]$, we propose a new multivariate Jarque-Bera test statistic as follows:

$$MJB^{**} = cMJB^* + (1-c)(p+1),$$

where

$$c = \left[\frac{\mathrm{Var}[T_1] + \mathrm{Var}[T_2] + 2\mathrm{Cov}[T_1, T_2]}{2(p+1)}\right]^{-\frac{1}{2}}.$$

MJB^{**} statistic is asymptotically distributed as χ^2_{p+1}-distribution. c is a constant and satisfies the following expression:

$$
\begin{aligned}
c^2 = {} & p(p+1)N(N-2)(N-3)(N+5)(N+7)(N+9)(N+11)(N+13) \\
& \times \big\{p(p+1)N^8 + 2(29p^2+110p+135)N^7 + (859p^2+3055p+702)N^6 \\
& + 2(1058p^2-217p-7272)N^5 - (21665p^2+71105p+38844)N^4 \\
& - 2(13471p^2+10792p-96183)N^3 + 3(44759p^2+130587p+134898)N^2 \\
& + 90(767p-6222)N + 81000\big\}^{-1},
\end{aligned}
$$

where $c > 0,\ c \to 1\ (N \to \infty)$ and $N \neq 2, 3$.

Although $\mathrm{Var}[MJB^*] = 2(p+1) + O(N^{-1})$, variance of MJB^{**} is improved as $\mathrm{Var}[MJB^{**}] = 2(p+1)$.

3 Simulation studies

Accuracy of variances, upper percentiles, type I error and power of multivariate Jarque-Bera test statistics MJB, MJB^* and MJB^{**} is evaluated via a Monte Carlo simulation study. Simulation parameters are as follows: $p = 3,\ 10,\ 20,\ 30,\ N = 20,\ 50,\ 100,\ 200,\ 400,\ 800$ and significance level $\alpha = 0.05$. When we derive power, each element of sample is generated using χ^2_5-distribution. As a numerical experiment, we carry out 1,000,000 replications.

Table 1 and Figure 1 give variance derived in this paper and simulated values derived by Monte Carlo simulation. In Table 1, "MJB^*" is values calculated by using (1). MJB^* and simulated values are almost same for all parameters.

Table 2 and Figure 2 give variance of MJB^{**}. In Table 2, "MJB^{**}" represents variance $\mathrm{Var}[MJB^{**}] = 2(p+1)$ and "*Simulation*" is simulated variance of test statistic MJB^{**} derived by Monte Carlo simulation. We proposed MJB^{**} such that the variance of MJB^{**} coincides with $2(p+1)$. It can be seen from Table 2 and Figure 2 that MJB^{**} has almost the same variance as χ^2_{p+1}-distribution for all parameters.

Table 3 and Figure 3 give values of the upper 5 percentiles of MJB, MJB^* and MJB^{**}. When N is small, they show that difference of MJB^{**} and χ^2_{p+1}-distribution is smaller than that of χ^2_{p+1}-distribution and MJB^*.

Table 4 and Figure 4 give values of type I error of MJB, MJB^* and MJB^{**}. They show that MJB^{**} is closer to 0.05 than others when N is small. We note that if type I error is smaller than 0.05, the test using its test statistic is conservative.

Table 5 and Figure 5 give values of power of MJB, MJB^* and MJB^{**}. It can be seen from Table 5 and Figure 5 that MJB, MJB^* and MJB^{**} have almost the same power for all parameters.

In conclusion, the multivariate Jarque-Bera test statistic MJB^{**} proposed in this paper is useful for the multivariate normality test.

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Table 1. Variance of MJB^*.

p	N	MJB^*	$Simulation$	p	N	MJB^*	$Simulation$
3	20	17.6	18.3	10	20	34.3	36.8
	50	15.6	15.8		50	31.5	31.8
	100	12.9	13.1		100	28.0	28.0
	200	10.8	10.8		200	25.4	25.4
	400	9.5	9.4		400	23.8	23.8
	800	8.8	8.8		800	22.9	23.0
20	50	55.8	56.4	30	50	80.2	81.6
	100	50.7	50.7		100	73.5	73.6
	200	46.9	46.8		200	68.5	68.5
	400	44.6	44.7		400	65.4	65.5
	800	43.3	43.4		800	63.8	63.7

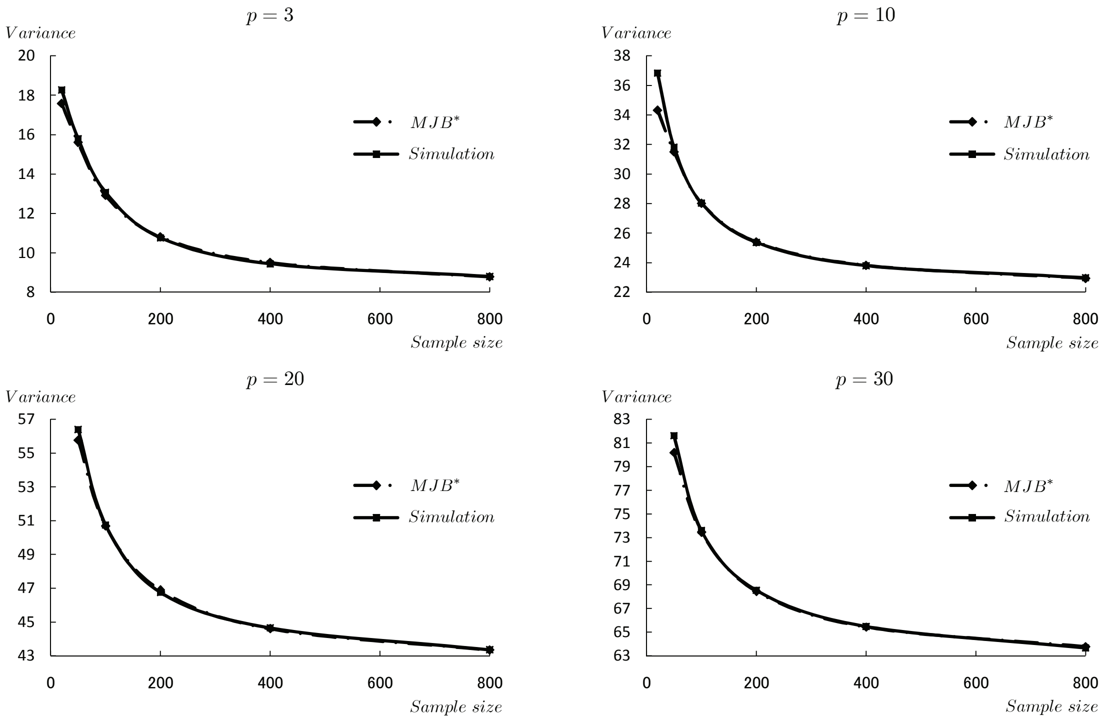

Figure 1. Variance of MJB^* for $p = 3,\ 10,\ 20,\ 30$.

Table 2. Variance of MJB^{**}.

p	N	MJB^{**}	$Simulation$	p	N	MJB^{**}	$Simulation$
3	20	8.0	8.3	10	20	22.0	23.6
	50	8.0	8.1		50	22.0	22.2
	100	8.0	8.1		100	22.0	22.0
	200	8.0	8.0		200	22.0	22.0
	400	8.0	7.9		400	22.0	22.0
	800	8.0	8.0		800	22.0	22.0
20	50	42.0	42.5	30	50	62.0	63.1
	100	42.0	42.1		100	62.0	62.1
	200	42.0	41.9		200	62.0	62.1
	400	42.0	42.0		400	62.0	62.0
	800	42.0	42.0		800	62.0	61.9

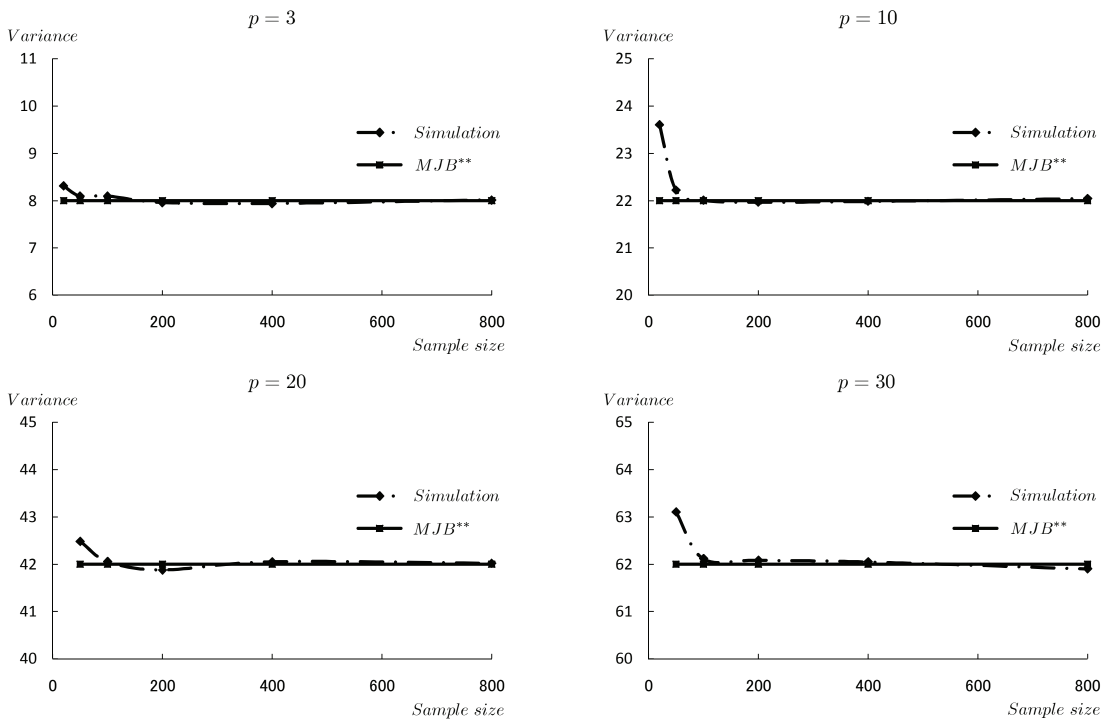

Figure 2. Variance of MJB^{**} for $p = 3,\ 10,\ 20,\ 30$.

Table 3. The upper 5 percentiles of MJB, MJB^{*} and MJB^{**}.

p	N	MJB	MJB^{*}	MJB^{**}	χ^2_{p+1}	p	N	MJB	MJB^{*}	MJB^{**}	χ^2_{p+1}
3	20	6.8	11.2	8.9	9.5	10	20	15.0	22.5	20.2	19.7
	50	8.4	10.6	8.7	9.5		50	17.9	21.4	19.7	19.7
	100	9.0	10.2	8.9	9.5		100	18.9	20.8	19.7	19.7
	200	9.3	9.9	9.0	9.5		200	19.3	20.3	19.7	19.7
	400	9.4	9.7	9.2	9.5		400	19.5	20.0	19.7	19.7
	800	9.5	9.6	9.4	9.5		800	19.6	19.9	19.7	19.7
20	50	29.8	34.9	33.1	32.7	30	50	41.1	47.6	45.6	45.0
	100	31.3	34.0	32.9	32.7		100	43.1	46.6	45.3	45.0
	200	32.0	33.4	32.8	32.7		200	44.1	45.9	45.2	45.0
	400	32.4	33.1	32.7	32.7		400	44.6	45.5	45.1	45.0
	800	32.5	32.9	32.7	32.7		800	44.8	45.2	45.0	45.0

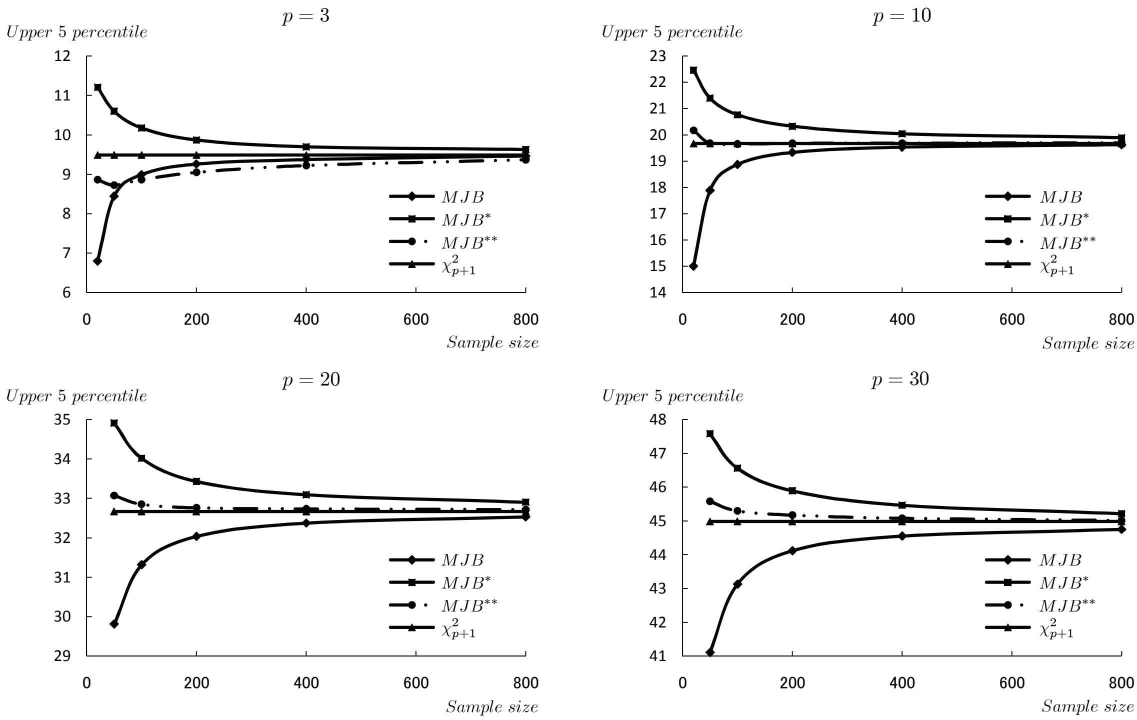

Figure 3. The upper 5 percentiles of MJB, MJB^{*} and MJB^{**} for $p=3,\ 10,\ 20,\ 30$.

Table 4. Type I error of MJB, MJB^{*} and MJB^{**}.

p	N	MJB	MJB^{*}	MJB^{**}	p	N	MJB	MJB^{*}	MJB^{**}
3	20	0.021	0.070	0.042	10	20	0.013	0.079	0.055
	50	0.037	0.064	0.040		50	0.032	0.070	0.050
	100	0.043	0.060	0.041		100	0.041	0.064	0.050
	200	0.046	0.056	0.043		200	0.046	0.058	0.050
	400	0.048	0.054	0.045		400	0.048	0.055	0.050
	800	0.050	0.053	0.048		800	0.049	0.053	0.050
20	50	0.027	0.072	0.054	30	50	0.023	0.072	0.055
	100	0.037	0.064	0.052		100	0.035	0.064	0.053
	200	0.043	0.058	0.051		200	0.042	0.059	0.052
	400	0.047	0.055	0.051		400	0.046	0.055	0.051
	800	0.048	0.053	0.051		800	0.048	0.052	0.050

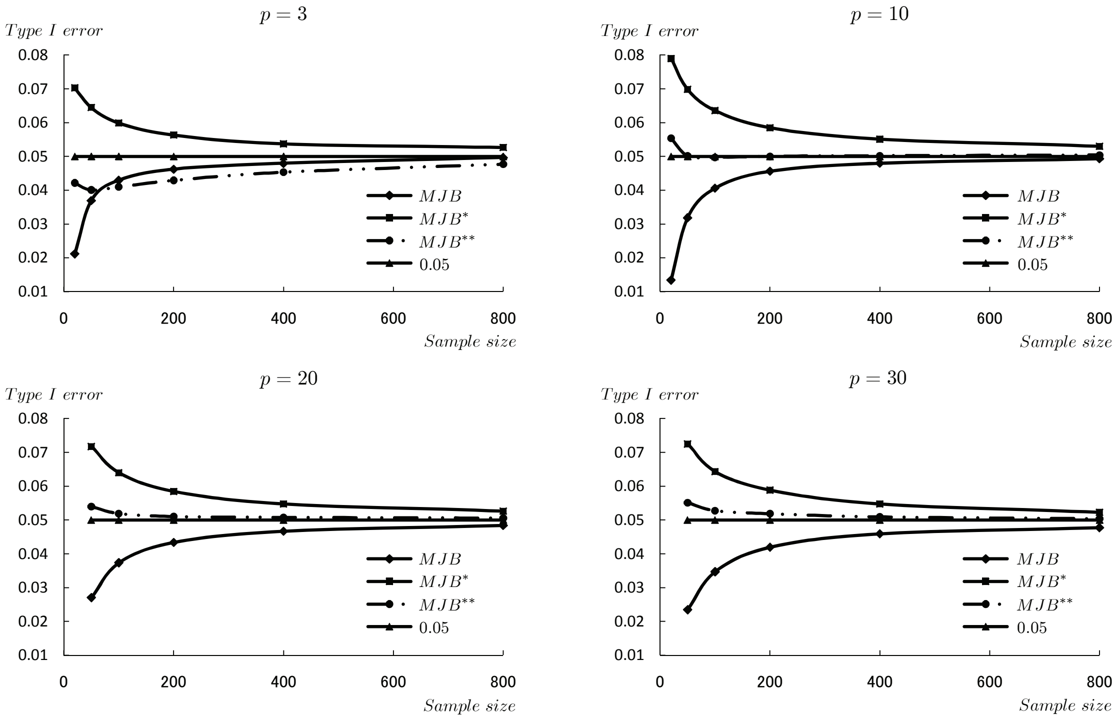

Figure 4. Type I error of MJB, MJB^{*} and MJB^{**} for $p = 3,\ 10,\ 20,\ 30$.

Table 5. Power of MJB, MJB^{*} and MJB^{**}.

p	N	MJB	MJB^{*}	MJB^{**}	p	N	MJB	MJB^{*}	MJB^{**}
3	20	0.264	0.414	0.331	10	20	0.154	0.349	0.294
	50	0.756	0.805	0.742		50	0.586	0.690	0.645
	100	0.960	0.967	0.954		100	0.903	0.929	0.916
	200	0.998	0.998	0.998		200	0.995	0.997	0.996
	400	1.000	1.000	1.000		400	1.000	1.000	1.000
	800	1.000	1.000	1.000		800	1.000	1.000	1.000
20	50	0.386	0.532	0.487	30	50	0.278	0.441	0.399
	100	0.729	0.800	0.778		100	0.563	0.667	0.640
	200	0.964	0.975	0.972		200	0.879	0.913	0.906
	400	1.000	1.000	1.000		400	0.995	0.997	0.997
	800	1.000	1.000	1.000		800	1.000	1.000	1.000

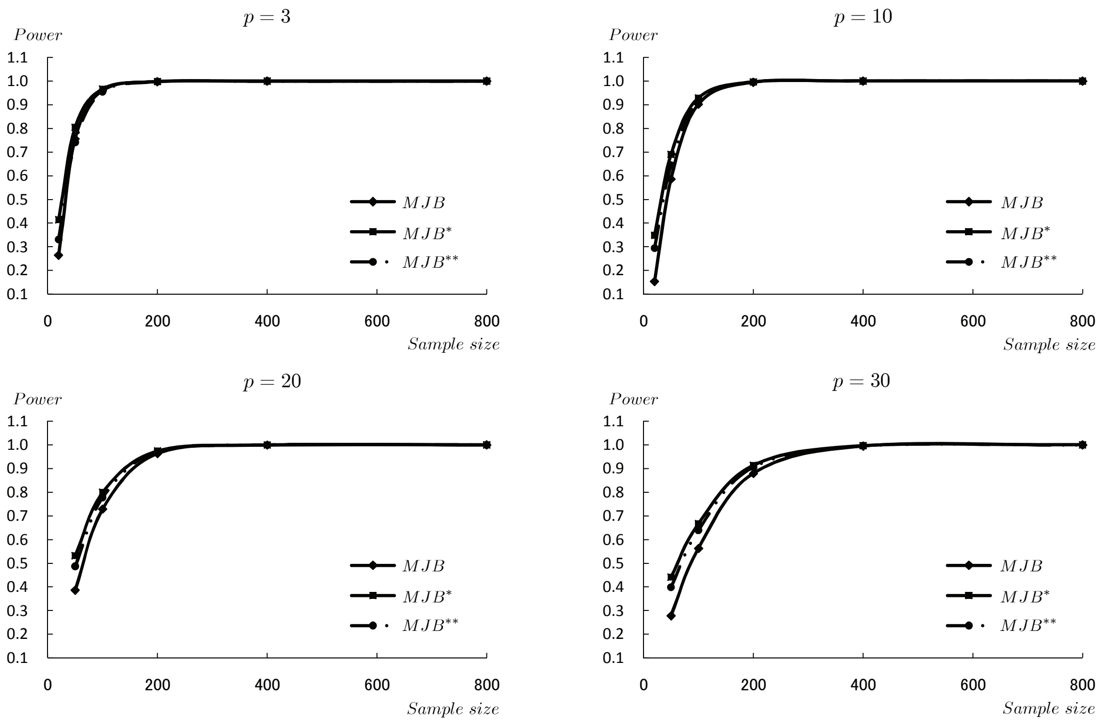

Figure 5. Power of MJB, MJB^{*} and MJB^{**} for $p = 3,\ 10,\ 20,\ 30$.