Testing independence by step-wise multiple comparison procedure

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Abstract

In this paper, we consider testing independence among components of random vector in multivariate normal population. For testing independence, we use the modified likelihood ratio test statistic which is improved an approximation to χ^2 distribution of the likelihood ratio test statistic. In order to perform simultaneous tests for independence among components of random vector, we use a step-down multiple comparison procedure based on closed testing procedure proposed by Marcus, Peritz and Gabriel (1976). Moreover, we construct a step-up multiple comparison procedure for testing independence simultaneously. Finally, we perform Monte Carlo simulations and present numerical results.

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1. Introduction

Let $\boldsymbol{x}$ be a random vector from p-dimensional multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. We consider the following partitions:

$$\begin{aligned} \boldsymbol{x} &= (x_1, \boldsymbol{x}'_{(2)})', \\ \Sigma &= \begin{bmatrix} \sigma_{11} & \boldsymbol{\sigma}'_{12} \\ \boldsymbol{\sigma}_{12} & \Sigma_{22} \end{bmatrix}, \end{aligned}$$

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where $\boldsymbol{x}_{(2)} = (x_2, x_3, \ldots, x_p)'$. Then we consider testing independence between x_1 and $\boldsymbol{x}_{(2)}$, i.e., testing

$$H_0 : \boldsymbol{\sigma}_{12} = \mathbf{0} \quad \text{vs.} \quad H_1 : \boldsymbol{\sigma}_{12} \neq \mathbf{0}.$$

When H_0 is rejected, it may be required to test independence between x_1 and x_i ($i = 2, 3, \ldots, p$). Here, we use the step-wise multiple comparison procedure.

Multiple comparison procedure is known as a procedure for testing multiple hypotheses simultaneously. Many authors have been studied single-step and step-wise multiple comparison procedures. For example, single-step procedures among several mean components have been considered by Scheffé (1953), Tukey (1953), Dunnett (1955) and so on. On the other hand, step-down procedure which is one of the step-wise procedures has been discussed by Peritz (1970) and so on. Moreover, on the basis of closed testing procedure, a step-down procedure has proposed by Marcus, Peritz and Gabriel (1976). Also, step-up procedure is known as another step-wise procedure (see, e.g., Dunnett and Tamhane (1992)). In this paper, we propose the simultaneous test procedures for independence among components of random vector under the multivariate normal population by step-down and step-up multiple comparison procedures.

This paper is organized as follows. Section 2 describes the likelihood ratio test statistic for testing independence and χ^2 distribution which is the asymptotic distribution. Moreover, we give the modified likelihood ratio test statistic which is improved by an approximation to χ^2 distribution. Section 3 proposes a step-down multiple comparison procedure based on closed testing procedure for the simultaneous tests of independence. Section 4 also proposes a step-up procedure for testing independence. Section 5 investigates the power of the proposed procedures by Monte Carlo simulations for selected parameters. Section 5 also shows numerical example to illustrate our procedures. Finally, we conclude this paper and address the direction for the future studies.

2. Testing independence

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be N independent observations from a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. Let

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i, \quad A = \sum_{i=1}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})'.$$

We partition A as

$$A = \begin{bmatrix} a_{11} & \boldsymbol{a}_{12}' \\ \boldsymbol{a}_{12} & A_{22} \end{bmatrix},$$

which is similar to Σ. Then we obtain the likelihood ratio for the hypotheses H_0 vs. H_1

$$\Lambda = \frac{|A|^{\frac{N}{2}}}{a_{11}^{\frac{N}{2}} |A_{22}|^{\frac{N}{2}}}.$$

We can also express the distribution function of the statistic $-2\log\Lambda$ as

$$\Pr\{-2\log\Lambda \le c\} = \Pr\{\chi^2_{p-1} \le c\} + O(N^{-1}).$$

Moreover, the modified likelihood ratio test statistic has the following distribution.

$$\Pr\{-2\eta\log\Lambda \le c\} = \Pr\{\chi^2_{p-1} \le c\} + O(N^{-2}),$$

where

$$\eta = 1 - \frac{p+3}{2N}.$$

Therefore, both $-2\log\Lambda$ and $-2\eta\log\Lambda$ are asymptotically distributed as χ^2 distribution with $p-1$ degrees of freedom.

In order to evaluate the accuracy of the obtained statistics, the Monte Carlo simulation of the upper percentiles of the statistics is implemented for selected parameters. As the numerical examination, we carry out 10^6 replications.

Each value is calculated for the following combinations of parameter values: $p = 3, 5, 10$, $N = 10, 20, 40, 80, 200$ $(p < N)$ and $\alpha = 0.01, 0.05, 0.1$.

Tables 1-3 list numerical results of the simulation. It can be observed from some numerical results that $-2\eta\log\Lambda$ is closer to the upper percentiles of χ^2 distribution.

3. Step-down multiple comparison procedure based on closed testing procedure

In this section, we construct the step-down multiple comparison procedure based on the closed testing procedure for the simultaneous tests for independence.

Let M_q be the family of subsets $\{2, \ldots, p\}$ with cardinal number q. For $m = \{\ell_1, \ell_2, \ldots, \ell_q\} \in M_q$ ($\ell_1 < \ell_2 < \cdots < \ell_q$), let $\Sigma^{(q,m)}$ be the covariance matrix of $q+1$-dimensional random vector $(x_1, \boldsymbol{x}_{(2)}^{(q,m)\prime})'$, where $\boldsymbol{x}_{(2)}^{(q,m)} = (x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_q})'$. $\Sigma^{(q,m)}$ is partitioned as

$$\Sigma^{(q,m)} = \begin{bmatrix} \sigma_{11} & \boldsymbol{\sigma}_{12}^{(q,m)\prime} \\ \boldsymbol{\sigma}_{12}^{(q,m)} & \Sigma_{22}^{(q,m)} \end{bmatrix}.$$

Then the hypothesis for independence between x_1 and $\boldsymbol{x}_{(2)}^{(q,m)}$ is set as

$$H_0^{(q,m)} : \boldsymbol{\sigma}_{12}^{(q,m)} = \mathbf{0} \quad \text{vs.} \quad H_1^{(q,m)} : \boldsymbol{\sigma}_{12}^{(q,m)} \neq \mathbf{0}.$$

Here, let $A^{(q,m)}$ be submatrix of A and

$$A^{(q,m)} = \begin{bmatrix} a_{11} & \boldsymbol{a}_{12}^{(q,m)\prime} \\ \boldsymbol{a}_{12}^{(q,m)} & A_{22}^{(q,m)} \end{bmatrix},$$

which is similar to $\Sigma^{(q,m)}$. Then we obtain the likelihood ratio for the hypotheses $H_0^{(q,m)}$ vs. $H_1^{(q,m)}$ is

$$\Lambda^{(q,m)} = \frac{|A^{(q,m)}|^{\frac{N}{2}}}{a_{11}^{\frac{N}{2}} |A_{22}^{(q,m)}|^{\frac{N}{2}}}.$$

When $q = p-1$, we write $H_0^{(q,m)} = H_0$ and $\Lambda^{(q,m)} = \Lambda$. The statistic $-2\log\Lambda^{(q,m)}$ is asymptotically distributed as χ^2 distribution with q degrees of freedom under the null hypothesis $H_0^{(q,m)}$. In this case, we give the modified likelihood ratio test statistic $-2\tau\log\Lambda^{(q,m)}$, where $\tau = 1 - (q+4)/2N$.

Let F_q be the set consisting of all hypotheses $H_0^{(q,m)}$. Moreover, let $F = \cup_{q=1}^{p-1} F_q$. Then the family F is closed. Using the likelihood ratio test statistic, Imada (2010) proposed the step-down multiple comparison procedure based on closed testing procedure for F. In this paper, we carry out the step-down test for all hypotheses in F by using the modified likelihood ratio test statistic as follows:

Step 1. We test hypothesis H_0.

Case 1. If $-2\tau \log \Lambda > \chi^2_{p-1}(\alpha)$, we reject H_0 and go to Step 2.

Case 2. If $-2\tau \log \Lambda \leq \chi^2_{p-1}(\alpha)$, we retain all hypotheses in F and stop the test.

Step 2. We test all hypotheses $H_0^{(p-2,m)}$ in F_{p-2}.

Case 1. If $-2\tau \log \Lambda^{(p-2,m)} > \chi^2_{p-2}(\alpha)$, we reject $H_0^{(p-2,m)}$.

Case 2. If $-2\tau \log \Lambda^{(p-2,m)} \leq \chi^2_{p-2}(\alpha)$, we retain $H_0^{(p-2,m)}$ and all hypotheses implied by $H_0^{(p-2,m)}$.

If all hypotheses in $\cup_{q=1}^{p-3} F_q$ are retained, we finish the test. Otherwise, we go to Step 3.

Step 3. We test all hypotheses in F_{p-3} which are not retained in Step 2.

We repeat similar judgments till Step $p-1$ at the maximum.

From a principle of closed testing procedure, it should be noted that the maximum type I FWE (familywise error rate) of this step-down multiple comparison procedure is not greater than α.

4. Step-up multiple comparison procedure

In this section, we construct the step-up procedure for testing independence. We consider testing the following hypotheses:

$$H_{1i} : \sigma_{1i} = 0 \quad \text{vs.} \quad H_{1i}^A : \sigma_{1i} \neq 0, \quad i = 2, 3, \ldots, p.$$

The likelihood ratio for H_{1i} is given by

$$\Lambda_{1i} = \frac{|A_{1i}|^{\frac{N}{2}}}{a_{11}^{\frac{N}{2}} a_{ii}^{\frac{N}{2}}},$$

where A_{1i} is submatrix of A in the following form:

$$A_{1i} = \begin{bmatrix} a_{11} & a_{1i} \\ a_{1i} & a_{ii} \end{bmatrix}.$$

Therefore, the modified likelihood ratio test statistic $-2\eta \log \Lambda_{1i}$ is asymptotically distributed as χ^2 distribution with 1 degree of freedom under the hypothesis H_{1i}.

Here, we define $L_{1i} \equiv -2\eta \log \Lambda_{1i}$ and let

$$L_{12}^{(2)} \leq L_{13}^{(3)} \leq \cdots \leq L_{1p}^{(p)},$$

be ordered statistics obtained by calculating $L_{12}, L_{13}, \ldots, L_{1p}$ based on observations. $H_{12}^{(2)}, H_{13}^{(3)}, \ldots, H_{1p}^{(p)}$ denote the corresponding hypotheses. When all hypotheses are true, we determine the critical values of the step-up procedure $c_2, c_3, \ldots, c_p$ satisfying

$$\Pr\{(L_{12}, L_{13}, \ldots, L_{1m}) \leq (c_2, c_3, \ldots, c_m)\} = 1 - \alpha, \tag{1}$$

and

$$c_2 \leq c_3 \leq \cdots \leq c_p, \tag{2}$$

for each $m = 2, 3, \ldots, p$, where

$$(L_{12}, L_{13}, \ldots, L_{1m}) \leq (c_2, c_3, \ldots, c_m)$$

implies $L_{12}^{(2)} \leq c_2, L_{13}^{(3)} \leq c_3, \ldots, L_{1m}^{(m)} \leq c_m$. Then we test $H_{12}, H_{13}, \ldots, H_{1p}$ as follows:

Step 1. We test hypothesis $H_{12}^{(2)}$.

Case 1. If $L_{12}^{(2)} > c_2$, we reject $H_{12}^{(2)}, H_{13}^{(3)}, \ldots, H_{1p}^{(p)}$ and stop the test.

Case 2. If $L_{12}^{(2)} \leq c_2$, we retain $H_{12}^{(2)}$ and go to Step 2.

Step 2. We test hypothesis $H_{13}^{(3)}$.

Case 1. If $L_{13}^{(3)} > c_3$, we reject $H_{13}^{(3)}, H_{14}^{(4)}, \ldots, H_{1p}^{(p)}$ and stop the test.

Case 2. If $L_{13}^{(3)} \leq c_3$, we retain $H_{13}^{(3)}$ and go to Step 2.

Step 3. We test hypothesis $H_{14}^{(4)}$.

We repeat similar judgments till Step $p-1$ at the maximum.

Then we have the following Theorem.

Theorem 1. *The maximum type I FWE of this step-up multiple comparison procedure is not greater than* α.

Proof. Suppose that $H_{12}, H_{13}, \ldots, H_{1k}$ are true and $H_{1,k+1}, H_{1,k+2}, \ldots, H_{1p}$ are false ($k \leq p$). Then we show that the probability that all $H_{12}, H_{13}, \ldots, H_{1k}$ are retained is not less than $1 - \alpha$.

Let

$$L_{1i_2}^{(i_2)} \leq L_{1i_3}^{(i_3)} \leq \cdots \leq L_{1i_k}^{(i_k)}$$

be ordered statistics of $L_{12}, L_{13}, \ldots, L_{1k}$ and we define the event E as

$$E : L_{1i_2}^{(i_2)} \leq c_2, L_{1i_3}^{(i_3)} \leq c_3, \ldots, L_{1i_k}^{(i_k)} \leq c_k.$$

By (1),

$$\Pr\{E\} = 1 - \alpha.$$

We show $L_{1m}^{(m)} \leq c_m$ for $2 \leq m \leq i_k$ under E. If $m \leq i_2$,

$$L_{1m}^{(m)} \leq L_{1i_2}^{(i_2)} \leq c_2 \leq c_m.$$

Next, we assume $i_h < m \leq i_{h+1}$ $(2 \leq h \leq k-1)$. Then we obtain

$$L_{1m}^{(m)} \leq L_{1i_{h+1}}^{(i_{h+1})} \leq c_{h+1}.$$

$h + 1 \leq m$ because $h \leq i_h < m$. $L_{1m}^{(m)} \leq c_m$ since $c_{h+1} \leq c_m$. Therefore, $H_{12}, H_{13}, \ldots, H_{1k}$ are retained until Step i_k, that is, $H_{12}, H_{13}, \ldots, H_{1k}$ are retained under E and its probability is not less than $1 - \alpha$. ■

To use this procedure, it is required to find the values $c_2, c_3, \ldots, c_p$. However, it is difficult to find the exact values $c_2, c_3, \ldots, c_p$. So, in this paper, we use Bonferroni's equality in order to determine the values $c_2, c_3, \ldots, c_p$.

We define the events E_i as

$$E_i : L_{1i}^{(i)} \leq c_i, \ i = 2, 3, \ldots, m$$

for each $m = 2, 3, \ldots, p$. Then we can rewrite the probability of (1) as follows:

$$\Pr\left\{\bigcap_{i=2}^{m} E_i\right\} = 1 - \alpha.$$

By Bonferroni's inequality for $\Pr\{\cap_{i=2}^{m} E_i\}$, it holds

$$\begin{aligned} \Pr\left\{\bigcap_{i=2}^{m} E_i\right\} &= 1 - \Pr\left\{\bigcup_{i=2}^{m} E_i^c\right\} \\ &\geq 1 - \sum_{i=2}^{m} \Pr\{E_i^c\}, \end{aligned}$$

that is,

$$\sum_{i=2}^{m} \Pr\{E_i^c\} \leq \alpha.$$

Therefore, we determine the critical values of the step-up procedure $c_2, c_3, \ldots, c_p$ satisfying

$$\sum_{i=2}^{m} \Pr\{L_{1i}^{(i)} > c_i\} \leq \alpha$$

and the inequality of (2).

5. Numerical examinations

In this section, we compare the efficiency of the proposed procedures in terms of power. We give some numerical results of the power of procedures by Monte Carlo simulation. The Monte Carlo simulations are made from 10^6 trials for selected values of parameters.

Tables 4-6 list the simulation results for the case where $\alpha = 0.05$; $p = 4$; $N = 10, 20, 30$; and $\Sigma = \Sigma_1, \Sigma_2, \Sigma_3$, that is,

$$\text{(i)}\ \Sigma_1 = \begin{bmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{bmatrix}, \quad -\frac{1}{3} < \rho < 1,$$

$$\text{(ii)}\ \Sigma_2 = \begin{bmatrix} 1 & \rho & \rho & 0 \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ 0 & \rho & \rho & 1 \end{bmatrix}, \quad \frac{1}{8}(1 - \sqrt{17}) < \rho < \frac{1}{8}(1 + \sqrt{17}),$$

$$\text{(iii)}\ \Sigma_3 = \begin{bmatrix} 1 & \rho & 0 & 0 \\ \rho & 1 & \rho & 0 \\ 0 & \rho & 1 & \rho \\ 0 & 0 & \rho & 1 \end{bmatrix}, \quad \frac{1}{2}(1 - \sqrt{5}) < \rho < \frac{1}{2}(-1 + \sqrt{5}).$$

Tables 4-6 give the following four procedures as follows:

SD : the step-down multiple comparison procedure based on closed testing procedure,

SU1 : the step-up multiple comparison procedure by using the critical value $c_i = \chi_1^2(\alpha/2^{i-1})$,

SU2 : the step-up multiple comparison procedure by using the critical value $c_i = \chi_1^2(3\alpha/4^{i-1})$,

BI : the procedure based on Bonferroni's inequality by using the critical value $c \equiv c_i = \chi_1^2(\alpha/(p-1))$,

for $i = 2, 3, \ldots, p$, where $\chi_p^2(\alpha)$ is the upper 100α percentile of χ^2 distribution with p degree of freedom.

It can be observed from Table 4 that the power of SU2 procedure is greater than the others when N is small. When N is large, SD procedure becomes better. Also, it can be observed from Tables 4-6 that the power of SU1 and SU2 results in worse when the number of independent variables increases. From Tables 4-6, it should be noted that the power of SD procedure tends to greater than the other three procedures without the effect of covariance structure.

Next, we apply our procedures to raw data to illustrate our procedures. We consider the observations from the boys group of Dental data studied at times 8, 10, 12 and 14 ages (Potthoff and Roy (1964)). Its dimensionality and sample size are $p = 4$, $N = 14$. First, we apply the step-down procedure based on closed testing procedure.

Step 1. We test hypothesis H_0.

Since $-2\eta \log \Lambda = 21.477 > 7.815 = \chi_3^2(0.05)$, H_0 is rejected.

Step 2. We test all hypotheses $H_0^{(2,m)}$ in F_2.

Since $-2\eta \log \Lambda^{(2,\{2,3\})} = 19.559 > 5.991 = \chi_2^2(0.05)$, $H_0^{(2,\{2,3\})}$ is rejected.

Since $-2\eta \log \Lambda^{(2,\{2,4\})} = 8.726 > 5.991 = \chi^2_2(0.05)$, $H_0^{(2,\{2,4\})}$ is rejected.

Since $-2\eta \log \Lambda^{(2,\{3,4\})} = 22.221 > 5.991 = \chi^2_2(0.05)$, $H_0^{(2,\{3,4\})}$ is rejected.

Step 3. We test all hypotheses in F_1 which are not retained in Step 2.

Since $-2\eta \log \Lambda^{(1,\{2\})} = 6.909 > 5.991 = \chi^2_1(0.05)$, $H_0^{(1,\{2\})}$ is rejected.

Since $-2\eta \log \Lambda^{(1,\{3\})} = 20.311 > 5.991 = \chi^2_1(0.05)$, $H_0^{(1,\{3\})}$ is rejected.

Since $-2\eta \log \Lambda^{(1,\{4\})} = 6.355 > 5.991 = \chi^2_1(0.05)$, $H_0^{(1,\{4\})}$ is rejected.

Therefore, there exists the correlation between x_1 and each of x_2, x_3, x_4.

As the second procedure, we apply the step-up procedure by using the critical value $c_i = \chi^2_1(\alpha/2^{i-1})$. We obtain the values of L_{12}, L_{13} and L_{14} are 7.223 ($\equiv L_{13}^{(3)}$), 20.311 ($\equiv L_{14}^{(4)}$) and 6.355 ($\equiv L_{12}^{(2)}$), respectively.

Step 1. We test hypothesis $H_{12}^{(2)}$.

Since $L_{12}^{(2)} = 6.355 > 4.328 = \chi^2_1(0.025)$, $H_{12}^{(2)}, H_{13}^{(3)}, H_{14}^{(4)}$ are rejected.

Therefore, there exists the correlation between x_1 and each of x_2, x_3, x_4.

In conclusion, the step-down procedure based on closed testing procedure which is proposed by this paper is useful for testing independence in terms of the power. Also, it should be noted that the power of step-up procedure depends on the critical values c_i's. However, it is defficult to find the exact values c_i's, and it is left as a future problem.

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References

[1] Dunnett, C. W. (1955). A multiple comparison procedure for comparing several treatments with a control. *Journal of the American Statistical Association*, **50**, 1096–1121.

[2] Dunnett, C. W. and Tamhane, A. C. (1992). A step-up multiple test procedure. *Journal of the American Statistical Association*, **87**, 162–170.

[3] Imada, T. (2010). Step down multiple comparison procedure for testing the independence of a specified normal random variable and each of several normal random variables. *Proceedings of Japan-Korea Special Conference of Statistics*, 217–220.

[4] Marcus, R., Peritz, E. and Gabriel, K. R. (1976). On closed testing procedures with special reference to ordered analysis of variance. *Biometrika*, **63**, 655–660.

[5] Muirhead, R. J. (1982). *Aspects of Multivariate Statistical Theory.* New York: Wiley.

[6] Peritz, E. (1970). A note on multiple comparison. Unpublished manuscript, Hebrew University, Israel.

[7] Potthoff, R. F. and Roy, S, N. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems. *Biometrika*, **51**, 313–326.

[8] Scheffé, H. (1953). A method of judging all contrasts in the analysis of variance. *Biometrika*, **40**, 87–104.

[9] Tukey, J. W. (1953). *The problem of multiple comparisons.* Unpublished manuscript, New Jersey: Department of Statistics, Princeton University.

Table 1: The upper percentiles of $-2\log\Lambda$ and $-2\eta\log\Lambda$ for $p = 3$

p	N	α	$-2\log\Lambda$	$-2\eta\log\Lambda$	χ^2_{p-1}
3	10	0.01	13.136	9.195	9.210
		0.05	8.539	5.977	5.991
		0.1	6.563	4.594	4.605
	20	0.01	10.818	9.195	9.210
		0.05	7.065	6.005	5.991
		0.1	5.426	4.612	4.605
	40	0.01	9.973	9.225	9.210
		0.05	6.469	5.984	5.991
		0.1	4.979	4.606	4.605
	80	0.01	9.597	9.237	9.210
		0.05	6.229	5.996	5.991
		0.1	4.786	4.606	4.605
	200	0.01	9.379	9.238	9.210
		0.05	6.084	5.993	5.991
		0.1	4.674	4.604	4.605

Table 2: The upper percentiles of $-2\log\Lambda$ and $-2\eta\log\Lambda$ for $p = 5$

p	N	α	$-2\log\Lambda$	$-2\eta\log\Lambda$	χ^2_{p-1}
5	10	0.01	23.066	13.839	13.277
		0.05	16.379	9.827	9.488
		0.1	13.366	8.019	7.779
	20	0.01	16.679	13.343	13.277
		0.05	11.907	9.525	9.488
		0.1	9.767	7.813	7.779
	40	0.01	14.733	13.259	13.277
		0.05	10.551	9.496	9.488
		0.1	8.640	7.776	7.779
	80	0.01	13.999	13.299	13.277
		0.05	9.989	9.490	9.488
		0.1	8.184	7.775	7.779
	200	0.01	13.545	13.274	13.277
		0.05	9.696	9.502	9.488
		0.1	7.945	7.786	7.779

Table 3: The upper percentiles of $-2\log\Lambda$ and $-2\eta\log\Lambda$ for $p=10$

p	N	α	$-2\log\Lambda$	$-2\eta\log\Lambda$	χ^2_{p-1}
10	20	0.01	33.802	22.816	21.666
		0.05	26.237	17.710	16.919
		0.1	22.700	15.322	14.684
	40	0.01	26.098	21.857	21.666
		0.05	20.357	17.049	16.919
		0.1	17.662	14.792	14.684
	80	0.01	23.616	21.697	21.666
		0.05	18.450	16.951	16.919
		0.1	16.021	14.719	14.684
	200	0.01	22.378	21.651	21.666
		0.05	17.484	16.916	16.919
		0.1	15.177	14.683	14.684

Table 4: Power comparison for $\Sigma = \Sigma_1$

N	ρ	SD	SU1	SU2	BI
10	0.1	0.000	0.000	0.000	0.000
	0.2	0.001	0.000	0.001	0.000
	0.3	0.005	0.002	0.005	0.001
	0.4	0.020	0.010	0.019	0.005
	0.5	0.062	0.038	0.063	0.022
	0.6	0.161	0.117	0.170	0.078
	0.7	0.358	0.298	0.381	0.226
	0.8	0.660	0.610	0.689	0.528
	0.9	0.939	0.927	0.950	0.898
20	0.1	0.001	0.000	0.000	0.000
	0.2	0.006	0.002	0.004	0.001
	0.3	0.036	0.016	0.027	0.009
	0.4	0.135	0.077	0.113	0.050
	0.5	0.346	0.242	0.312	0.184
	0.6	0.640	0.530	0.608	0.454
	0.7	0.887	0.828	0.872	0.777
	0.8	0.988	0.978	0.985	0.967
	0.9	1.000	1.000	1.000	1.000
30	0.1	0.001	0.000	0.001	0.000
	0.2	0.015	0.005	0.010	0.003
	0.3	0.090	0.045	0.070	0.029
	0.4	0.305	0.198	0.261	0.147
	0.5	0.626	0.504	0.581	0.429
	0.6	0.887	0.820	0.865	0.769
	0.7	0.986	0.973	0.982	0.961
	0.8	1.000	0.999	1.000	0.999
	0.9	1.000	1.000	1.000	1.000

Table 5: Power comparison for $\Sigma = \Sigma_2$

N	ρ	SD	SU1	SU2	BI
10	0.1	0.001	0.000	0.000	0.000
	0.2	0.002	0.001	0.001	0.002
	0.3	0.006	0.004	0.002	0.006
	0.4	0.017	0.011	0.007	0.017
	0.5	0.055	0.034	0.023	0.049
	0.6	0.185	0.096	0.069	0.128
	0.7	–	–	–	–
	0.8	–	–	–	–
	0.9	–	–	–	–
20	0.1	0.001	0.001	0.000	0.001
	0.2	0.007	0.004	0.002	0.006
	0.3	0.033	0.020	0.014	0.028
	0.4	0.119	0.076	0.057	0.099
	0.5	0.343	0.221	0.180	0.265
	0.6	0.683	0.482	0.427	0.536
	0.7	–	–	–	–
	0.8	–	–	–	–
	0.9	–	–	–	–
30	0.1	0.002	0.001	0.001	0.001
	0.2	0.016	0.009	0.006	0.013
	0.3	0.082	0.051	0.038	0.067
	0.4	0.278	0.190	0.155	0.230
	0.5	0.629	0.467	0.413	0.520
	0.6	0.882	0.777	0.735	0.814
	0.7	–	–	–	–
	0.8	–	–	–	–
	0.9	–	–	–	–

Table 6: Power comparison for $\Sigma = \Sigma_3$

N	ρ	SD	SU1	SU2	BI
10	0.1	0.010	0.006	0.003	0.019
	0.2	0.017	0.010	0.005	0.030
	0.3	0.030	0.019	0.010	0.052
	0.4	0.057	0.037	0.019	0.092
	0.5	0.115	0.073	0.040	0.161
	0.6	0.268	0.142	0.085	0.279
	0.7	–	–	–	–
	0.8	–	–	–	–
	0.9	–	–	–	–
20	0.1	0.017	0.009	0.004	0.025
	0.2	0.039	0.023	0.012	0.056
	0.3	0.093	0.058	0.033	0.125
	0.4	0.206	0.136	0.085	0.250
	0.5	0.422	0.283	0.199	0.443
	0.6	0.686	0.515	0.407	0.676
	0.7	–	–	–	–
	0.8	–	–	–	–
	0.9	–	–	–	–
30	0.1	0.022	0.011	0.005	0.031
	0.2	0.063	0.037	0.021	0.086
	0.3	0.168	0.110	0.069	0.208
	0.4	0.374	0.265	0.187	0.415
	0.5	0.665	0.513	0.411	0.669
	0.6	0.868	0.780	0.700	0.871
	0.7	–	–	–	–
	0.8	–	–	–	–
	0.9	–	–	–	–