Multiple comparisons among mean vectors when the dimension is larger than the total sample size

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Abstract

We consider pairwise multiple comparisons and multiple comparisons with a control among mean vectors for high-dimensional data under the multivariate normality. For such cases, the statistics based on Dempster trace criterion are given, and also their approximate upper percentiles are derived by using Bonferroni's inequality. Finally, the accuracy of their approximate values is evaluated by Monte Carlo simulation.

Key words and phrases: Asymptotic expansion; Bonferroni's inequality; Comparison with a control; Dempster trace criterion; High-dimensional data; Monte Carlo Simulation; Pairwise comparison.

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1 Introduction

Testing equality of means or mean vectors is one of the important themes of statistical procedures. So many procedures have been studied, for example, ANOVA, MANOVA and multiple comparisons among means or mean vectors.

In this paper, we consider the multiple comparisons among mean vectors. Simultaneous confidence procedures for multiple comparisons among mean vectors have been widely studied by many authors, e.g., Seo (1995). In order to construct the simultaneous confidence intervals, it is required to obtain the upper percentiles of $T^2_{\max}$-type statistics. In general, it is difficult to obtain them exactly even under the multivariate normality. So by using Bonferroni's inequality, its approximate upper percentiles are discussed by Siotani (1959) and Seo and Siotani (1992).

On the other hand, recently, high-dimensional data have been increasingly encountered in many applications of statistics and most prominently in biological and financial studies, and many authors have studied statistical procedures for such cases. However, when the dimension is larger than the total sample size, since the sample covariance matrix becomes singular, it will be impossible to define Hotelling's T^2-type statistic. Dempster (1958, 1960) has proposed Dempster trace criterion for one and two sample problems. Also Dempster trace criterion for multivariate linear hypothesis have been discussed by Fujikoshi, Himeno and Wakaki (2004), Himeno (2007) and many other authors.

This paper discusses the case of high-dimension. For high-dimensional data, we propose statistics for multiple comparisons among mean vectors instead of $T^2_{\max}$-type statistics. Further, the approximations for the upper percentiles of them is provided on the basis of Bonferroni's inequality.

In this paper, we consider pairwise multiple comparisons and multiple comparisons with a control among mean vectors when the dimension is larger than the total sample size.

The organization of this paper is as follows. In Section 2, we describe multiple comparisons among mean vectors. In Section 3, for pairwise multiple comparisons, the statistic based on Dempster trace criterion is proposed, and the first order Bonferroni approximation is given. The statistic for comparisons with a control is also proposed, and the first order Bonferroni approximation is derived as well as preceding section in Section 4. Finally, the accuracy of the approximations are evaluated by Monte Carlo simulation in Section 5.

2 Multiple comparisons among mean vectors

Consider pairwise multiple comparisons and multiple comparisons with a control among mean vectors. Let $\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)}, \ldots, \boldsymbol{x}_N^{(i)}$ $(i = 1, 2, \ldots, k)$ be N independent sample vectors that have the p-dimensional normal distribution with mean vector $\boldsymbol{\mu}^{(i)}$ and common covariance matrix Σ. Let the i-th sample mean vector, the i-th sample covariance matrix and the pooled sample covariance matrix be

$$
\begin{aligned}
\overline{\boldsymbol{x}}^{(i)} &= \frac{1}{N}\sum_{j=1}^{N} \boldsymbol{x}_j^{(i)}, \\
S^{(i)} &= \frac{1}{N-1}\sum_{j=1}^{N} (\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})', \\
S &= \frac{1}{k}\sum_{i=1}^{k} S^{(i)},
\end{aligned}
$$

respectively.

In general, the simultaneous confidence intervals for pairwise multiple comparisons among mean vectors with the confidence level $1 - \alpha$ are given by

$$
\boldsymbol{a}'(\boldsymbol{\mu}^{(\ell)} - \boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(\ell)} - \overline{\boldsymbol{x}}^{(m)}) \pm t_{\mathrm{p}}\sqrt{2\boldsymbol{a}'S\boldsymbol{a}/N}\right], \qquad \forall \boldsymbol{a} \in \mathbb{R}^p - \{\mathbf{0}\}, \quad 1 \le \ell < m \le k, \tag{1}
$$

where $\mathbb{R}^p - \{\mathbf{0}\}$ is the set of any nonnull real p-dimensional vectors and the value

t_{p} (> 0) satisfies as follows:

$$\Pr\{T^2_{\max\cdot\mathrm{p}} > t^2_{\mathrm{p}}\} = \alpha,$$

where

$$\begin{aligned} T^2_{\max\cdot\mathrm{p}} &= \max_{1\le\ell<m\le k}\{T^2_{\ell m}\}, \\ T^2_{\ell m} &= \frac{N}{2}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})' S^{-1}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)}), \\ \boldsymbol{y}^{(i)} &= \overline{\boldsymbol{x}}^{(i)} - \boldsymbol{\mu}^{(i)}, \quad i = 1, 2, \ldots, k. \end{aligned}$$

On the other hand, letting the first population be a control, the simultaneous confidence intervals for comparisons with a control among mean vectors are given by

$$\begin{aligned} \boldsymbol{a}'(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(m)}) \pm t_{\mathrm{c}}\sqrt{2\boldsymbol{a}'S\boldsymbol{a}/N}\right], \\ \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\}, \quad 2 \le m \le k, \end{aligned}$$

and the value t_{c} (> 0) satisfies as follows:

$$\Pr\{T^2_{\max\cdot\mathrm{c}} > t_{\mathrm{c}}\} = \alpha,$$

where

$$\begin{aligned} T^2_{\max\cdot\mathrm{c}} &= \max_{2\le m\le k}\{T^2_{1m}\}, \\ T^2_{1m} &= \frac{N}{2}(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(m)})' S^{-1}(\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(m)}). \end{aligned}$$

However, because it is difficult to obtain the exact values of t_{p} and t_{c}, the approximate procedure for their upper percentiles is discussed by using the asymptotic expansion (see Siotani (1959) and Seo and Siotani (1992)).

3 Pairwise comparisons for high-dimensional case

We first discuss pairwise comparisons in the situation such that the dimension is larger than the total sample size. Since S becomes singular for such cases, it will be impossible to define $T^2_{\ell m}$. So instead of $T^2_{\ell m}$, we use Dempster trace criterion expressed as

$$\frac{n\mathrm{tr}S_h^{(\ell,m)}}{\mathrm{tr}S_e}, \tag{2}$$

where

$$\begin{aligned} S_h^{(\ell,m)} &= \frac{N}{2}(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})(\boldsymbol{y}^{(\ell)} - \boldsymbol{y}^{(m)})', \\ S_e &= \sum_{i=1}^{k}\sum_{j=1}^{N}(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})', \end{aligned}$$

and $n = k(N-1)$.

In order to investigate its asymptotic behavior when $p > n$, we set the high-dimensional framework,

$$\mathrm{A1}: \quad n \to \infty, \quad p \to \infty, \quad \frac{p}{n} \to \gamma \in (0, \infty),$$

where $n = k(N-1)$. Further, we assume that

$$\mathrm{A2}: \quad \mathrm{tr}\Sigma^i = O(p), \quad i = 1, 2, \ldots, 8.$$

Also we define $c_i = \mathrm{tr}\Sigma^i/p$.

Next we provide lemma for the estimators of c_i's.

Lemma 1. *The unbiased consistent estimators of $c_i' s$ $(i = 1, 2, 3, 4)$ can be obtained as*

$$\begin{aligned} \hat{c}_1 &= \frac{\mathrm{tr}S_e}{np}, \\ \hat{c}_2 &= \frac{n^2}{(n+2)(n-1)p}\left\{\frac{\mathrm{tr}S_e^2}{n^2} - \frac{(\mathrm{tr}S_e)^2}{n^3}\right\}, \\ \hat{c}_3 &= \frac{1}{n^3 p}\left\{\mathrm{tr}S_e^3 - \frac{3}{n}\mathrm{tr}S_e^2\mathrm{tr}S_e + \frac{2}{n^2}(\mathrm{tr}S_e)^3\right\}, \\ \hat{c}_4 &= \frac{1}{n^4 p}\left\{b_1\mathrm{tr}S_e^4 + b_2\mathrm{tr}S_e^3\mathrm{tr}S_e + b_3(\mathrm{tr}S_e^2)^2 + b_4\mathrm{tr}S_e^2(\mathrm{tr}S_e)^2 + b_5(\mathrm{tr}S_e)^4\right\}, \end{aligned}$$

where

$$
\begin{aligned}
b_1 &= \frac{n^5(n^2+n+2)}{(n+6)(n+4)(n+1)(n-2)(n-3)(n^2+n-2)}, \\
b_2 &= -\frac{4n^4(n^2+n+2)}{(n+6)(n+4)(n+1)(n-2)(n-3)(n^2+n-2)}, \\
b_3 &= \frac{n^4(2n^2+3n-6)}{(n+6)(n+4)(n+1)(n-2)(n-3)(n^2+n-2)}, \\
b_4 &= -\frac{2n^4(5n+6)}{(n+6)(n+4)(n+1)(n-2)(n-3)(n^2+n-2)}, \\
b_5 &= -\frac{n^3(5n+6)}{(n+6)(n+4)(n+2)(n+1)(n-1)(n-2)(n-3)}.
\end{aligned}
$$

For the proof, see Appendix A.

By using the Dempster trace criterion stated in (2), we propose the following statistic:

$$
\begin{aligned}
D^2_{\max\cdot\mathrm{p}} &= \max_{1\le \ell<m\le k}\{D^2_{\ell m}\}, \\
D^2_{\ell m} &= \frac{p}{\hat{\sigma}}\left(\frac{n\mathrm{tr}S_h^{(\ell,m)}}{\mathrm{tr}S_e}-1\right),
\end{aligned}
$$

where

$$
\hat{\sigma} = \sqrt{\frac{2p\hat{c}_2}{\hat{c}_1^2}}.
$$

Then the simultaneous confidence intervals for pairwise multiple comparisons among mean vectors in high-dimensional data are given by

$$
\begin{aligned}
\boldsymbol{a}'(\boldsymbol{\mu}^{(\ell)}-\boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(\ell)}-\overline{\boldsymbol{x}}^{(m)}) \pm d_{\mathrm{p}}\sqrt{2\boldsymbol{a}'(\mathrm{tr}S)I_p\boldsymbol{a}/N}\right], \\
\forall \boldsymbol{a}\in\mathbb{R}-\{\boldsymbol{0}\}, \quad 1\le \ell<m\le k,
\end{aligned}
\tag{3}
$$

where $d_{\mathrm{p}}^2 = 1+(\hat{\sigma}/p)z_{\mathrm{p}}$ and z_{p} satisfies

$$
\Pr\{D^2_{\max\cdot\mathrm{p}} > z_{\mathrm{p}}\} = \alpha.
$$

We consider the expected length of (3). Let $L = d_{\mathrm{p}}^2 2\boldsymbol{a}'(\mathrm{tr}S)I_p\boldsymbol{a}/N$. Under the framework A1 and the assumption A2,

$$
\mathrm{E}[L] \to 2\gamma c_1(\boldsymbol{a}'\boldsymbol{a}).
$$

Thus, the expected length stated in (3) converges to some positive constant. In particular, p/n ratio is smaller, the expected length becomes shorter.

In order to construct the actual simultaneous confidence intervals stated in (3) with the confidence level $1-\alpha$, it is required to find the upper 100α percentiles of $D^2_{\max\cdot\mathrm{p}}$ statistic. However, it is difficult to find the exact value. So we give an approximation for z_{p} based on Bonferroni's inequality.

By Bonferroni's inequality for $\Pr\{D^2_{\max\cdot\mathrm{p}} > z_{\mathrm{p}}\}$,

$$\Pr\{D^2_{\max\cdot\mathrm{p}} > z_{\mathrm{p}}\} < \sum_{\ell=1}^{k-1}\sum_{m=\ell+1}^{k}\Pr\{D^2_{\ell m} > z_{\mathrm{p}}\}.$$

Therefore, the first order Bonferroni approximation $z_{1\cdot\mathrm{p}}$ is defined as

$$\sum_{\ell=1}^{k-1}\sum_{m=\ell+1}^{k}\Pr\{D^2_{\ell m} > z_{1\cdot\mathrm{p}}\} = \alpha.$$

It should be noted that $z_{1\cdot\mathrm{p}}$ is overestimated and essentially the upper percentile of $D^2_{\ell m}$ statistic. Therefore, it is important to study the distribution of $D^2_{\ell m}$.

At first, by using the same idea as Fujikoshi, Himeno and Wakaki (2004), we have the following theorem for the distribution of $D^2_{\ell m}$:

Theorem 2. *Under the framework A1 and the assumption A2, it holds that*

$$D^2_{\ell m} \xrightarrow{d} N(0,1).$$

Further, expanding the distribution of $D^2_{\ell m}$, we obtain as follows:

$$\begin{aligned}\Pr\{D^2_{\ell m} \le z\} \;=\; & \Phi(z) - \phi(z)\left\{\frac{1}{\sqrt{p}}\frac{\sqrt{2}c_3}{3\sqrt{c_2^3}}h_2(z) + \frac{1}{p}\left(\frac{c_4}{2c_2^2}h_3(z) + \frac{c_3^2}{9c_2^3}h_5(z)\right)\right.\\ & \left. + \frac{1}{2n}h_1(z)\right\} + O(p^{-\frac{3}{2}}),\end{aligned}$$

where $\Phi(z)$ and $\phi(z)$ are the distribution function of the standard normal distribution and the density function of the standard normal distribution, respectively, and $h_i(z)$ are the Hermite polynomials given by

$$h_1(z) = z, \quad h_2(z) = z^2, \quad h_3(z) = z^3 - 3z, \quad h_5(z) = z^5 - 10z^3 + 15z.$$

Also, by Cornish-Fisher expansion, its upper 100α percentile can be expanded as

$$z_{\ell m}(\alpha) = z_\alpha + \frac{1}{\sqrt{p}}\frac{\sqrt{2}c_3}{3\sqrt{c_2^3}}(z_\alpha^2 - 1) + \frac{1}{p}\left\{\frac{c_4}{2c_2^2}z_\alpha(z_\alpha^2 - 3) - \frac{2c_3^2}{9c_2^3}z_\alpha(2z_\alpha^2 - 5)\right\} + \frac{1}{2n}z_\alpha,$$

where z_α is the upper 100α percentile of the standard normal distribution. Since c_i's contain the unknown parameter Σ, it is required to estimate c_i's. Therefore, by using Lemma 1, we have the following theorem:

Theorem 3. *Under the framework A1 and the assumption A2, the distribution of $D_{\ell m}^2$ can be expanded as*

$$\begin{aligned}\Pr\left\{D_{\ell m}^2 \le z\right\} &= \Phi(z) - \phi(z)\left\{\frac{1}{\sqrt{p}}\frac{\sqrt{2}\hat{c}_3}{3\sqrt{\hat{c}_2^3}}h_2(z) + \frac{1}{p}\left(\frac{\hat{c}_4}{2\hat{c}_2^2}h_3(z) + \frac{\hat{c}_3^2}{9\hat{c}_2^3}h_5(z)\right)\right.\\ &\quad \left. + \frac{1}{2n}h_1(z)\right\} + O(p^{-\frac{3}{2}}),\end{aligned}$$

and also its upper 100α percentile can be expanded as

$$z_{\ell m}(\alpha) = z_\alpha + \frac{1}{\sqrt{p}}\frac{\sqrt{2}\hat{c}_3}{3\sqrt{\hat{c}_2^3}}(z_\alpha^2 - 1) + \frac{1}{p}\left\{\frac{\hat{c}_4}{2\hat{c}_2^2}z_\alpha(z_\alpha^2 - 3) - \frac{2\hat{c}_3^2}{9\hat{c}_2^3}z_\alpha(2z_\alpha^2 - 5)\right\} + \frac{1}{2n}z_\alpha.$$

Further, from Theorem 2, we can obtain the following corollary:

Corollary 4. *Under the framework A1 and the assumption A2, the first order Bonferroni approximate upper 100α percentiles of $D_{\max\cdot\mathrm{p}}^2$, that is, $z_{1\cdot\mathrm{p}} \equiv z_{1\cdot\mathrm{p}}(\alpha)$ is given by*

$$z_{1\cdot\mathrm{p}} = z_{\alpha_\mathrm{p}} + \frac{1}{\sqrt{p}}\frac{\sqrt{2}\hat{c}_3}{3\sqrt{\hat{c}_2^3}}(z_{\alpha_\mathrm{p}}^2 - 1) + \frac{1}{p}\left\{\frac{\hat{c}_4}{2\hat{c}_2^2}z_{\alpha_\mathrm{p}}(z_{\alpha_\mathrm{p}}^2 - 3) - \frac{2\hat{c}_3^2}{9\hat{c}_2^3}z_{\alpha_\mathrm{p}}(2z_{\alpha_\mathrm{p}}^2 - 5)\right\} + \frac{1}{2n}z_{\alpha_\mathrm{p}},$$

where $\alpha_\mathrm{p} = \alpha/K$ and $K = k(k-1)/2$.

Therefore, by using $z_{1\cdot\mathrm{p}}$, we obtain the following approximate simultaneous confidence intervals for high-dimensional data:

$$\begin{aligned}\boldsymbol{a}'(\boldsymbol{\mu}^{(\ell)} - \boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(\ell)} - \overline{\boldsymbol{x}}^{(m)}) \pm d_{1\cdot\mathrm{p}}\sqrt{2\boldsymbol{a}'(\mathrm{tr}S)I_p\boldsymbol{a}/N}\right],\\ \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\}, \quad 1 \le \ell < m \le k,\end{aligned}$$

where $d_{1\cdot\mathrm{p}}^2 = 1 + (\hat{\sigma}/p)z_{1\cdot\mathrm{p}}$.

4 Approximation for comparisons with a control

In this section, we consider comparisons with a control for high-dimensional data. In this case, on the basis of Dempster trace criterion, we propose the following $D^2_{\max\cdot\mathrm{c}}$ statistic:

$$\begin{aligned} D^2_{\max\cdot\mathrm{c}} &= \max_{2\le m\le k}\{D^2_{1m}\}, \\ D^2_{1m} &= \frac{p}{\hat{\sigma}}\left(\frac{n\mathrm{tr}S_h^{(1,m)}}{\mathrm{tr}S_e}-1\right). \end{aligned}$$

By Bonferroni inequality for $\Pr\{D^2_{\max\cdot\mathrm{c}} > z_{\mathrm{c}}\}$,

$$\Pr\{D^2_{\max\cdot\mathrm{c}} > z_{\mathrm{c}}\} < \sum_{m=2}^{k}\Pr\{D^2_{1m} > z_{\mathrm{c}}\}.$$

Therefore, the first order Bonferroni approximation $z_{1\cdot\mathrm{c}}$ is defined as

$$\sum_{m=2}^{k}\Pr\{D^2_{1m} > z_{1\cdot\mathrm{c}}\} = \alpha.$$

By the same way as pairwise comparisons, under the framework A1 and the assumption A2, the first order Bonferroni approximate upper 100α percentiles of $D^2_{\max\cdot\mathrm{c}}$, that is, $z_{1\cdot\mathrm{c}} \equiv z_{1\cdot\mathrm{c}}(\alpha)$ is given by

$$z_{1\cdot\mathrm{c}} = z_{\alpha_{\mathrm{c}}} + \frac{1}{\sqrt{p}}\frac{\sqrt{2}\hat{c}_3}{3\sqrt{\hat{c}_2^3}}(z^2_{\alpha_{\mathrm{c}}}-1) + \frac{1}{p}\left\{\frac{\hat{c}_4}{2\hat{c}_2^2}z_{\alpha_{\mathrm{c}}}(z^2_{\alpha_{\mathrm{c}}}-3) - \frac{2\hat{c}_3^2}{9\hat{c}_2^3}z_{\alpha_{\mathrm{c}}}(2z^2_{\alpha_{\mathrm{c}}}-5)\right\} + \frac{1}{2n}z_{\alpha_{\mathrm{c}}},$$

where $\alpha_{\mathrm{c}} = \alpha/(k-1)$. Therefore, by using $z_{1\cdot\mathrm{c}}$, we give the following approximate simultaneous confidence intervals for high-dimensional data:

$$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)}-\boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\overline{\boldsymbol{x}}^{(1)}-\overline{\boldsymbol{x}}^{(m)}) \pm d_{1\cdot\mathrm{c}}\sqrt{2\boldsymbol{a}'(\mathrm{tr}S)I_p\boldsymbol{a}/N}\right],$$
$$\forall\boldsymbol{a}\in\mathbb{R}^p-\{\boldsymbol{0}\},\quad 2\le m\le k,$$

where $d^2_{1\cdot\mathrm{c}} = 1 + (\hat{\sigma}/p)z_{1\cdot\mathrm{c}}$.

5 Numerical examinations

We evaluate the accuracy of the approximation by Monte Carlo simulation. Monte Carlo simulation for the upper percentiles of $D^2_{\max\cdot\mathrm{p}}$ and $D^2_{\max\cdot\mathrm{c}}$ statistics is implemented from 10^5 trials for selected values of parameters and covariance structures.

Tables 1-6 and Tables 7-12 list the simulated and approximate values of the upper percentiles of $D^2_{\max\cdot\mathrm{p}}$ statistic and those of $D^2_{\max\cdot\mathrm{c}}$ statistic, respectively, for the combinations of the following parameters: $\alpha = 0.1, 0.05, 0.01$, $k = 3, 6$, $M \equiv Nk = 60, 120$ and $p = 60, 90, 120, 150, 200$ $(M \leq p)$. For the covariance structures, we set as follows: identity matrix $(\Sigma = I_p)$, AR(1) model $(\Sigma = (\rho^{|i-j|})$ with $\rho = 0.2)$ and AR(1) model $(\Sigma = (\rho^{|i-j|})$ with $\rho = 0.5)$.

Tables 1-6 list the results for the case of pairwise comparisons. In Tables 1-6, $P(z_{\alpha_\mathrm{p}})$ and $P(z_{1\cdot\mathrm{p}})$ denote $\Pr\{D^2_{\max\cdot\mathrm{p}} < z_{\alpha_\mathrm{p}}\}$ and $\Pr\{D^2_{\max\cdot\mathrm{p}} < z_{1\cdot\mathrm{p}}\}$, respectively. It should be noted that z^*_p is a simulated value of the upper percentiles of $D^2_{\max\cdot\mathrm{p}}$ statistic, that is, $\Pr\{D^2_{\max\cdot\mathrm{p}} > z^*_\mathrm{p}\} = \alpha$.

It can be observed from Tables 1 and 2 that $P(z_{1\cdot\mathrm{p}})$ is always larger than $P(z_{\alpha_\mathrm{p}})$ and $z_{1\cdot\mathrm{p}}$ is almost conservative. And then increasing the number of population, $P(z_{1\cdot\mathrm{p}})$ becomes large. It should be also noted that $P(z_{1\cdot\mathrm{p}})$ does not depend on the total sample size and the dimension. Also, from Tables 3-6, it can be observed that there is the same tendency as identity matrix.

On the other hand, Tables 7-12 list the results for the case of comparisons with a control. In Tables 7-12, $P(z_{\alpha_\mathrm{c}})$ and $P(z_{1\cdot\mathrm{c}})$ denote $\Pr\{D^2_{\max\cdot\mathrm{c}} < z_{\alpha_\mathrm{c}}\}$ and $\Pr\{D^2_{\max\cdot\mathrm{c}} < z_{1\cdot\mathrm{c}}\}$, respectively. It should be noted that z^*_c is a simulated value of the upper percentiles of $D^2_{\max\cdot\mathrm{c}}$ statistic, that is, $\Pr\{D^2_{\max\cdot\mathrm{c}} > z^*_\mathrm{c}\} = \alpha$.

Also, for comparisons with a control, it can be observed from Tables 7-12 that there is the same tendency as pairwise comparisons.

6 Conclusions

We considered multiple comparisons among mean vectors based on Dempster trace criterion when the dimension is larger than the total sample size. In Section 3, the statistic for pairwise comparison was proposed in such cases, and its asymptotic distribution and approximate upper percentile were derived. Also the approximation for comparison with a control is discussed in Section 4. Further, we evaluated the accuracy of their approximations by Monte Carlo simulation in Section 5. Through the numerical examinations, $z_{1\cdot \mathrm{p}}$ and $z_{1\cdot \mathrm{c}}$ are always better than $z_{\alpha_{\mathrm{p}}}$ and $z_{\alpha_{\mathrm{c}}}$, respectively. Also the covariance structure has little effect on $P(z_{1\cdot \mathrm{p}})$ and $P(z_{1\cdot \mathrm{c}})$. Therefore, the approximations obtained by asymptotic expansion are useful for pairwise multiple comparisons and multiple comparisons with a control among mean vectors in high-dimensional data.

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Appendix A Proof of Lemma 1

In this appendix, we prove the unbiased consistent estimators of c_i's $(i = 1, 2, 3, 4)$. We start with some preliminary results.

Lemma A.1. *Let $S_e \sim W_p(\Sigma, n)$. Then the forth-moments of traces of S_e are calculated as*

$$
\begin{aligned}
\mathrm{E}[\mathrm{tr}S_e^4] &= (n^4 + 6n^3 + 21n^2 + 20n)\mathrm{tr}\Sigma^4 + (4n^3 + 12n^2 + 16n)\mathrm{tr}\Sigma^3\mathrm{tr}\Sigma \\
&\quad +(2n^3 + 5n^2 + 5n)(\mathrm{tr}\Sigma^2)^2 + (6n^2 + 6n)\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2 + n(\mathrm{tr}\Sigma)^4, \\
\mathrm{E}[\mathrm{tr}S_e^3\mathrm{tr}S_e] &= (6n^3 + 18n^2 + 24n)\mathrm{tr}\Sigma^4 + (n^4 + 3n^3 + 16n^2 + 12n)\mathrm{tr}\Sigma^3\mathrm{tr}\Sigma \\
&\quad +(6n^2 + 6n)(\mathrm{tr}\Sigma^2)^2 + (3n^3 + 3n^2 + 6n)\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2 + n^2(\mathrm{tr}\Sigma)^4, \\
\mathrm{E}[\mathrm{tr}(S_e^2)^2] &= (8n^3 + 20n^2 + 20n)\mathrm{tr}\Sigma^4 + (16n^2 + 16n)\mathrm{tr}\Sigma^3\mathrm{tr}\Sigma \\
&\quad +(n^4 + 2n^3 + 5n^2 + 4n)(\mathrm{tr}\Sigma^2)^2 + (2n^3 + 2n^2 + 8n)\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2 \\
&\quad +n^2(\mathrm{tr}\Sigma)^4, \\
\mathrm{E}[\mathrm{tr}S_e^2(\mathrm{tr}S_e)^2] &= (24n^2 + 24n)\mathrm{tr}\Sigma^4 + (8n^3 + 8n^2 + 16n)\mathrm{tr}\Sigma^3\mathrm{tr}\Sigma \\
&\quad +(2n^3 + 2n^2 + 8n)(\mathrm{tr}\Sigma^2)^2 + (3n^3 + 3n^2 + 6n)\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2 \\
&\quad +n^3(\mathrm{tr}\Sigma)^4, \\
\mathrm{E}[(\mathrm{tr}S_e)^4] &= 48n\mathrm{tr}\Sigma^4 + 32n^2\mathrm{tr}\Sigma^3\mathrm{tr}\Sigma \\
&\quad +12n^2(\mathrm{tr}\Sigma^2)^2 + 12n^3\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2 + n^4(\mathrm{tr}\Sigma)^4.
\end{aligned}
$$

For the proof, see Watamori (1990).

We are ready to prove Lemma 1.

Proof of Lemma 1. The unbiased consistent estimators, $\hat{c}_1$ and $\hat{c}_2$, are given by Srivastava (2005). Also, for $\hat{c}_3$, see Himeno (2007). So we give the proof of $\hat{c}_4$

First of all, in order to prove the unbiasedness of $\hat{c}_4$, we consider the following equation:

$$
\mathrm{E}\left[b_1\mathrm{tr}S_e^4 + b_2\mathrm{tr}S_e^3\mathrm{tr}S_e + b_3(\mathrm{tr}S_e^2)^2 + b_4\mathrm{tr}S_e^2(\mathrm{tr}S_e)^2 + b_5(\mathrm{tr}S_e)^4\right] = n^4\mathrm{tr}\Sigma^4. \qquad (4)
$$

From Lemma A.1, we can obtain the expectation stated in (3). Further, by comparing the coefficient of left-hand side with that of right-hand side, we have a simultaneous equation,

$$
\begin{aligned}
n^4 &= b_1n^4 + (6b_1 + 6b_2 + 8b_3)n^3 + (21b_1 + 18b_2 + 20b_3 + 24b_4)n^2 \\
&\quad +(20b_1 + 24b_2 + 20b_3 + 24b_4 + 48b_5)n,
\end{aligned}
$$

$$
\begin{aligned}
0 &= b_2n^4 + (4b_1 + 3b_2 + 8b_4)n^3 + (12b_1 + 16b_2 + 16b_3 + 8b_4 + 32b_5)n^2 \\
&\quad +(16b_1 + 12b_2 + 16b_3 + 16b_4)n, \\
0 &= b_3n^4 + (2b_1 + 2b_3 + 2b_4)n^3 + (5b_1 + 6b_2 + 5b_3 + 2b_4 + 12b_5)n^2 \\
&\quad +(5b_1 + 6b_2 + 4b_3 + 8b_4)n, \\
0 &= (3b_2 + 2b_3 + 3b_4 + 12b_5)n^3 + (6b_1 + 3b_2 + 2b_3 + 3b_4)n^2 \\
&\quad +(6b_1 + 6b_2 + 8b_3 + 6b_4)n, \\
0 &= b_5n^4 + b_4n^3 + b_3n^2 + b_2n^2 + b_1n.
\end{aligned}
$$

By solving this simultaneous equation, the coefficients b_i's is obtained. Therefore, the unbiasedness is proved.

Secondly, we prove the consistency of $\hat{c}_4$. By using the moments of traces of Wishart matrix in Watamori (1990), the variance of $\hat{c}_4$ is expressed as

$$
\begin{aligned}
\mathrm{Var}[\hat{c}_4] &= \frac{32c_8n^6}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)p} \\
&\quad + \frac{48c_4^2n^5}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{64c_3c_5n^5}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{368c_8n^5}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)p} \\
&\quad + \frac{32c_2^2c_4pn^4}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{320c_4^2n^4}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{512c_3c_5n^4}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{32c_2c_6n^4}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{608c_8n^4}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)p} \\
&\quad + \frac{8c_2^4p^2n^3}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{64c_2c_3^2pn^3}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad + \frac{176c_2^2c_4pn^3}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
&\quad - \frac{680c_4^2n^3}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{960c_3c_5n^3}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&+\frac{320c_2c_6n^3}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{6768c_8n^3}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)p}\\
&+\frac{8c_2^4p^2n^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&+\frac{256c_2c_3^2pn^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{752c_2^2c_4pn^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{5048c_4^2n^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{9088c_3c_5n^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{352c_2c_6n^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{15872c_8n^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)p}\\
&+\frac{16c_2^4p^2n}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{960c_2c_3^2pn}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{864c_2^2c_4pn}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&+\frac{5808c_4^2n}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&+\frac{10752c_3c_5n}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&-\frac{7040c_2c_6n}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&+\frac{65056c_8n}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)p}\\
&-\frac{1152c_2c_3^2p}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&+\frac{9792c_4^2}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}\\
&+\frac{18432c_3c_5}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{8832c_2c_6}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)} \\
& -\frac{6720c_8}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)p} \\
& +\frac{15360c_2c_6}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)n} \\
& -\frac{80640c_8}{(n-3)(n-2)(n-1)(n+1)(n+2)(n+4)(n+6)pn} \\
= \ & \frac{\gamma^2 c_2^4}{n^2} + o(n^{-2}).
\end{aligned}
$$

Also, from Chebyshev's inequality, we obtain

$$
\begin{aligned}
\Pr\{|\hat{c}_4 - c_4| \geq \varepsilon\} &\leq \frac{1}{\varepsilon^2}\mathrm{Var}[\hat{c}_4] \\
&= \frac{1}{\varepsilon^2}\frac{\gamma^2 c_2^4}{n^2} + o(n^{-2}),
\end{aligned}
$$

for $\forall\varepsilon > 0$. Therefore, under the framework A1 and the assumption A2, it holds that

$$
\hat{c}_4 \xrightarrow{p} c_4.
$$

□

Appendix B Asymptotic distribution of $D_{\ell m}^2$

In this appendix, we derive the asymptotic distribution of $D_{\ell m}^2$ and its upper percentiles.

B.1 Stochastic expansion

We can rewrite as

$$
D_{\ell m}^2 = \frac{1}{\sqrt{2p\hat{c}_2}}\left(\mathrm{tr}S_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\right).
$$

Letting

$$
W = n(\hat{c}_2 - c_2),
$$

we have

$$
D_{\ell m}^2 = \frac{1}{\sqrt{2pc_2}}\left(\mathrm{tr}S_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\right)\left(1 + \frac{1}{nc_2}W\right)^{-\frac{1}{2}}.
$$

It should be noted that $W = O_p(1)$. Since

$$
\left(1 + \frac{1}{nc_2}W\right)^{-\frac{1}{2}} = 1 - \frac{1}{2nc_2}W + O_p(n^{-2}),
$$

we obtain

$$D^2_{\ell m} = \frac{1}{\sqrt{2pc_2}}\Big(\mathrm{tr}S_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\Big)\Big(1 - \frac{1}{2nc_2}W + O_p(n^{-2})\Big).$$

B.2 Expansion of characteristic function

Next we consider the characteristic function of $D^2_{\ell m}$.

$$\begin{aligned}
C(t) &= \mathrm{E}[\exp(itD^2_{\ell m})] \\
&= \mathrm{E}\bigg[\exp\bigg\{\frac{it}{\sqrt{2pc_2}}\Big(\mathrm{tr}S_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\Big)\bigg\} \\
&\qquad \times\exp\bigg\{\frac{it}{\sqrt{2pc_2}}\Big(\mathrm{tr}S_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\Big)\bigg\}\Big(-\frac{1}{2nc_2}W + O_p(n^{-2})\Big)\bigg] \\
&= \mathrm{E}\bigg[\exp\bigg\{\frac{it}{\sqrt{2pc_2}}\Big(\mathrm{tr}S_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\Big)\bigg\}g(S_h^{(\ell,m)}, S_e)\bigg],
\end{aligned}$$

where

$$g(S_h^{(\ell,m)}, S_e) = 1 - \frac{it}{2nc_2\sqrt{2pc_2}}W\Big(trS_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\Big) + O_p(n^{-2}).$$

Let

$$\begin{aligned}
S_h^{(\ell,m)} &= \Sigma^{\frac{1}{2}}\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^{\frac{1}{2}}, \\
S_e &= \Sigma^{\frac{1}{2}}Z_2Z_2'\Sigma^{\frac{1}{2}},
\end{aligned}$$

where $\boldsymbol{z}_1 \equiv \boldsymbol{z}_{\ell m} \sim N_p(\mathbf{0}, I_p)$ and $Z_2 \sim N_{p\times n}(O, I_{p\times n})$. Then

$$\begin{aligned}
C(t) &= \mathrm{E}\bigg[\exp\bigg\{\frac{it}{\sqrt{2pc_2}}\Big(\mathrm{tr}(\Sigma^{\frac{1}{2}}\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^{\frac{1}{2}}) - \frac{1}{n}\mathrm{tr}(\Sigma^{\frac{1}{2}}Z_2Z_2'\Sigma^{\frac{1}{2}})\Big)\bigg\}g(S_h^{(\ell,m)}, S_e)\bigg] \\
&= \int_{\boldsymbol{z}_1}\int_{Z_2}(2\pi)^{-\frac{(n+1)p}{2}}\mathrm{etr}\Big(-\frac{1}{2}\boldsymbol{z}_1\boldsymbol{z}_1' - \frac{1}{2}Z_2Z_2'\Big) \\
&\qquad \times\mathrm{etr}\Big(\frac{it}{\sqrt{2pc_2}}\Sigma\boldsymbol{z}_1\boldsymbol{z}_1' - \frac{it}{n\sqrt{2pc_2}}\Sigma Z_2Z_2'\Big)g(S_h^{(\ell,m)}, S_e)d\boldsymbol{z}_1dZ_2 \\
&= \int_{\boldsymbol{z}_1}\int_{Z_2}(2\pi)^{-\frac{(n+1)p}{2}}\mathrm{etr}\Big\{-\frac{1}{2}\Big(I_p - \frac{\sqrt{2}it}{\sqrt{pc_2}}\Sigma\Big)\boldsymbol{z}_1\boldsymbol{z}_1'\Big\} \\
&\qquad \times\mathrm{etr}\Big\{-\frac{1}{2}\Big(I_p + \frac{\sqrt{2}it}{n\sqrt{pc_2}}\Sigma\Big)Z_2Z_2'\Big\}g(S_h^{(\ell,m)}, S_e)d\boldsymbol{z}_1dZ_2, \\
&= \Big|I_p - \frac{\sqrt{2}it}{\sqrt{pc_2}}\Sigma\Big|^{-\frac{1}{2}}\Big|I_p + \frac{\sqrt{2}it}{n\sqrt{pc_2}}\Sigma\Big|^{-\frac{n}{2}} \\
&\qquad \times\int_{\boldsymbol{z}_1^*}\int_{Z_2^*}(2\pi)^{-\frac{(n+1)p}{2}}\mathrm{etr}\Big(-\frac{1}{2}\boldsymbol{z}_1^*\boldsymbol{z}_1^{*'} - \frac{1}{2}Z_2Z_2^{*'}\Big)g(S_h^{(\ell,m)}, S_e)d\boldsymbol{z}_1^*dZ_2^* \\
&= \Big|I_p - \frac{\sqrt{2}it}{\sqrt{pc_2}}\Sigma\Big|^{-\frac{1}{2}}\Big|I_p + \frac{\sqrt{2}it}{n\sqrt{pc_2}}\Sigma\Big|^{-\frac{n}{2}}\mathrm{E}_{(\boldsymbol{z}_1^*, Z_2^*)}[g(S_h^{(\ell,m)}, S_e)],
\end{aligned}$$

where

$$\boldsymbol{z}_1^* = \left(I_p - \frac{\sqrt{2}it}{\sqrt{pc_2}}\Sigma\right)^{\frac{1}{2}} \boldsymbol{z}_1, \quad Z_2^* = \left(I_p + \frac{\sqrt{2}it}{n\sqrt{pc_2}}\Sigma\right)^{\frac{1}{2}} Z_2.$$

$$\begin{aligned}
\log\left|I_p - \frac{\sqrt{2}it}{\sqrt{pc_2}}\Sigma\right|^{-\frac{1}{2}} &= -\frac{1}{2}\log\left|I_p - \frac{\sqrt{2}it}{\sqrt{pc_2}}\Sigma\right| \\
&= -\frac{1}{2}\bigg\{-\frac{\sqrt{2}it}{\sqrt{pc_2}}\mathrm{tr}\Sigma - \frac{1}{2}\left(\frac{\sqrt{2}it}{\sqrt{pc_2}}\right)^2 \mathrm{tr}\Sigma^2 \\
&\quad -\frac{1}{3}\left(\frac{\sqrt{2}it}{\sqrt{pc_2}}\right)^3 \mathrm{tr}\Sigma^3 - \frac{1}{4}\left(\frac{\sqrt{2}it}{\sqrt{pc_2}}\right)^4 \mathrm{tr}\Sigma^4 + O(p^{-\frac{3}{2}})\bigg\} \\
&= \frac{\sqrt{p}itc_1}{\sqrt{2c_2}} + \frac{(it)^2}{2} + \frac{\sqrt{2}(it)^3 c_3}{3\sqrt{pc_2^3}} + \frac{(it)^4 c_4}{2pc_2^2} + O(p^{-\frac{3}{2}}), \\
\log\left|I_p + \frac{\sqrt{2}it}{n\sqrt{pc_2}}\Sigma\right|^{-\frac{n}{2}} &= -\frac{n}{2}\log\left|I_p + \frac{\sqrt{2}it}{n\sqrt{pc_2}}\Sigma\right| \\
&= -\frac{n}{2}\left\{\frac{\sqrt{2}(it)}{n\sqrt{pc_2}}\mathrm{tr}\Sigma - \frac{1}{2}\left(\frac{\sqrt{2}it}{n\sqrt{pc_2}}\right)^2 \mathrm{tr}\Sigma^2 + O(p^{-\frac{7}{2}})\right\} \\
&= -\frac{\sqrt{p}itc_1}{\sqrt{2c_2}} + \frac{(it)^2}{2n} + O(p^{-\frac{3}{2}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|I_p - \frac{\sqrt{2}it}{\sqrt{pc_2}}\Sigma\right|^{-\frac{1}{2}} &= \exp\left\{\frac{\sqrt{p}itc_1}{\sqrt{2c_2}} + \frac{(it)^2}{2}\right\} \\
&\quad \times\left\{1 + \frac{\sqrt{2}(it)^3 c_3}{3\sqrt{pc_2^3}} + \frac{(it)^4 c_4}{2pc_2^2} + \frac{(it)^6 c_3^2}{9pc_2^3}\right\} + O(p^{-\frac{3}{2}}), \\
\left|I_p + \frac{\sqrt{2}it}{n\sqrt{pc_2}}\Sigma\right|^{-\frac{n}{2}} &= \exp\left\{-\frac{\sqrt{p}itc_1}{\sqrt{2c_2}}\right\}\left\{1 + \frac{(it)^2}{2n}\right\} + O(p^{-\frac{3}{2}}).
\end{aligned}$$

The expectation of $g(S_h^{(\ell,m)}, S_e)$ can be also calculated as

$$\begin{aligned}
\mathrm{E}_{(\boldsymbol{z}_1^*, Z_2^*)}[g(S_h^{(\ell,m)}, S_e)] &= 1 - \frac{it}{2n\sqrt{2pc_2^3}}\mathrm{E}_{(\boldsymbol{z}_1^*, Z_2^*)}\left[W\left(\mathrm{tr}S_h^{(\ell,m)} - \frac{1}{n}\mathrm{tr}S_e\right)\right] + O(n^{-2}) \\
&= 1 + O(p^{-\frac{3}{2}}).
\end{aligned}$$

Summarizing these results, we have the following characteristic function:

$$C(t) = \exp\left\{\frac{(it)^2}{2}\right\}\left\{1 + \frac{1}{\sqrt{p}}d_3(it)^3 + \frac{1}{p}\{d_4(it)^4 + d_6(it)^6\} + \frac{1}{n}d_2(it)^2\right\} + O(p^{-\frac{3}{2}}),$$

where

$$d_2 = \frac{1}{2}, \quad d_3 = \frac{\sqrt{2}c_3}{3\sqrt{c_2^3}}, \quad d_4 = \frac{c_4}{2c_2^2}, \quad d_6 = \frac{c_3^2}{9c_2^3}.$$

B.3 Asymptotic distribution and Cornish-Fisher expansion

Inverting the characteristic function of $D^2_{\ell m}$, we have its density distribution,

$$f(z) = \phi(z)\left[1 + \frac{1}{\sqrt{p}} d_3 h_3(z) + \frac{1}{p}\{d_4 h_4(z) + d_6 h_6(z)\} + \frac{1}{n} d_2 h_2(z)\right] + O(p^{-\frac{3}{2}}).$$

Further, integrating the density function and using the consistent estimators, we obtain the distribution of $D^2_{\ell m}$,

$$\begin{aligned}\Pr\left\{D^2_{\ell m} \leq z\right\} &= \Phi(z) - \phi(z)\left\{\frac{1}{\sqrt{p}} \frac{\sqrt{2}\hat{c}_3}{3\sqrt{\hat{c}_2^3}} h_2(z) + \frac{1}{p}\left(\frac{\hat{c}_4}{2\hat{c}_2^2} h_3(z) + \frac{\hat{c}_3^2}{9\hat{c}_2^3} h_5(z)\right)\right. \\ &\quad \left. + \frac{1}{2n} h_1(z)\right\} + O(p^{-\frac{3}{2}}).\end{aligned}$$

Therefore, by Cornish-Fisher expansion, the upper 100α percentile of $D^2_{\ell m}$ can be expanded as

$$z_{\ell m}(\alpha) = z_\alpha + \frac{1}{\sqrt{p}} \frac{\sqrt{2}\hat{c}_3}{3\sqrt{\hat{c}_2^3}}(z_\alpha^2 - 1) + \frac{1}{p}\left\{\frac{\hat{c}_4}{2\hat{c}_2^2} z_\alpha(z_\alpha^2 - 3) - \frac{2\hat{c}_3^2}{9\hat{c}_2^3} z_\alpha(2z_\alpha^2 - 5)\right\} + \frac{1}{2n} z_\alpha.$$

Table 1 The simulated and approximate values

for pairwise comparison when $\Sigma = I_p$.

k	M	p	α	z^*_{p}	$z_{\alpha_{\mathrm{p}}}$	$P(z_{\alpha_{\mathrm{p}}})$	$P(z_{1\cdot\mathrm{p}})$
3	60	60	0.01	3.128	2.713	0.976	0.990
			0.05	2.320	2.128	0.929	0.954
			0.1	1.928	1.834	0.884	0.911
		90	0.01	3.080	2.713	0.978	0.989
			0.05	2.295	2.128	0.933	0.953
			0.1	1.902	1.834	0.888	0.911
		120	0.01	3.001	2.713	0.980	0.990
			0.05	2.268	2.128	0.934	0.953
			0.1	1.896	1.834	0.889	0.910
		150	0.01	2.968	2.713	0.982	0.990
			0.05	2.254	2.128	0.936	0.953
			0.1	1.883	1.834	0.891	0.910
		200	0.01	2.938	2.713	0.983	0.990
			0.05	2.236	2.128	0.938	0.953
			0.1	1.872	1.834	0.893	0.910
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	2.986	2.713	0.981	0.990
			0.05	2.256	2.128	0.936	0.953
			0.1	1.884	1.834	0.891	0.910
		150	0.01	2.965	2.713	0.982	0.990
			0.05	2.245	2.128	0.937	0.953
			0.1	1.875	1.834	0.892	0.910
		200	0.01	2.938	2.713	0.983	0.990
			0.05	2.229	2.128	0.939	0.953
			0.1	1.871	1.834	0.894	0.909

Table 2 The simulated and approximate values

for pairwise comparison when $\Sigma = I_p$.

k	M	p	α	z^*_{p}	$z_{\alpha_{\mathrm{p}}}$	$P(z_{\alpha_{\mathrm{p}}})$	$P(z_{1\cdot\mathrm{p}})$
6	60	60	0.01	3.820	3.209	0.961	0.990
			0.05	3.086	2.713	0.897	0.954
			0.1	2.729	2.475	0.845	0.914
		90	0.01	3.718	3.209	0.967	0.990
			0.05	3.027	2.713	0.906	0.953
			0.1	2.679	2.475	0.855	0.913
		120	0.01	3.629	3.209	0.971	0.990
			0.05	2.976	2.713	0.913	0.954
			0.1	2.645	2.475	0.863	0.915
		150	0.01	3.592	3.209	0.973	0.990
			0.05	2.945	2.713	0.917	0.955
			0.1	2.623	2.475	0.867	0.915
		200	0.01	3.538	3.209	0.975	0.990
			0.05	2.919	2.713	0.921	0.954
			0.1	2.604	2.475	0.871	0.914
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.640	3.209	0.972	0.990
			0.05	2.949	2.713	0.917	0.956
			0.1	2.624	2.475	0.866	0.916
		150	0.01	3.602	3.209	0.973	0.990
			0.05	2.937	2.713	0.919	0.954
			0.1	2.611	2.475	0.869	0.915
		200	0.01	3.531	3.209	0.976	0.990
			0.05	2.907	2.713	0.923	0.954
			0.1	2.590	2.475	0.874	0.914

Table 3 The simulated and approximate values

for pairwise comparison when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.2$.

k	M	p	α	z^*_{p}	$z_{\alpha_{\mathrm{p}}}$	$P(z_{\alpha_{\mathrm{p}}})$	$P(z_{1\cdot\mathrm{p}})$
3	60	60	0.01	3.198	2.713	0.974	0.990
			0.05	2.349	2.128	0.926	0.953
			0.1	1.944	1.834	0.882	0.910
		90	0.01	3.104	2.713	0.977	0.990
			0.05	2.317	2.128	0.930	0.953
			0.1	1.919	1.834	0.885	0.910
		120	0.01	3.043	2.713	0.980	0.990
			0.05	2.270	2.128	0.933	0.955
			0.1	1.905	1.834	0.887	0.910
		150	0.01	2.997	2.713	0.981	0.991
			0.05	2.262	2.128	0.935	0.954
			0.1	1.894	1.834	0.889	0.910
		200	0.01	2.962	2.713	0.982	0.991
			0.05	2.253	2.128	0.937	0.952
			0.1	1.887	1.834	0.891	0.909
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.017	2.713	0.980	0.990
			0.05	2.260	2.128	0.935	0.955
			0.1	1.889	1.834	0.890	0.911
		150	0.01	2.988	2.713	0.981	0.990
			0.05	2.248	2.128	0.936	0.954
			0.1	1.885	1.834	0.891	0.910
		200	0.01	2.978	2.713	0.982	0.990
			0.05	2.233	2.128	0.938	0.954
			0.1	1.874	1.834	0.892	0.910

Table 4 The simulated and approximate values

for pairwise comparison when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.2$.

k	M	p	α	z^*_{p}	$z_{\alpha_{\mathrm{p}}}$	$P(z_{\alpha_{\mathrm{p}}})$	$P(z_{1\cdot\mathrm{p}})$
6	60	60	0.01	3.880	3.209	0.958	0.990
			0.05	3.121	2.713	0.893	0.955
			0.1	2.751	2.475	0.840	0.917
		90	0.01	3.787	3.209	0.963	0.989
			0.05	3.056	2.713	0.901	0.954
			0.1	2.708	2.475	0.848	0.915
		120	0.01	3.698	3.209	0.967	0.990
			0.05	3.024	2.713	0.906	0.953
			0.1	2.678	2.475	0.855	0.913
		150	0.01	3.660	3.209	0.970	0.990
			0.05	2.978	2.713	0.913	0.954
			0.1	2.648	2.475	0.861	0.915
		200	0.01	3.585	3.209	0.974	0.990
			0.05	2.936	2.713	0.918	0.955
			0.1	2.613	2.475	0.869	0.915
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.659	3.209	0.969	0.991
			0.05	2.991	2.713	0.910	0.955
			0.1	2.657	2.475	0.859	0.914
		150	0.01	3.637	3.209	0.971	0.990
			0.05	2.953	2.713	0.916	0.955
			0.1	2.628	2.475	0.865	0.915
		200	0.01	3.576	3.209	0.974	0.990
			0.05	2.924	2.713	0.920	0.955
			0.1	2.606	2.475	0.870	0.914

Table 5 The simulated and approximate values

for pairwise comparison when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.5$.

k	M	p	α	z^*_{p}	$z_{\alpha_{\mathrm{p}}}$	$P(z_{\alpha_{\mathrm{p}}})$	$P(z_{1\cdot\mathrm{p}})$
3	60	60	0.01	3.439	2.713	0.965	0.990
			0.05	2.481	2.128	0.915	0.953
			0.1	2.017	1.834	0.871	0.908
		90	0.01	3.293	2.713	0.971	0.990
			0.05	2.402	2.128	0.922	0.953
			0.1	1.974	1.834	0.877	0.910
		120	0.01	3.215	2.713	0.974	0.990
			0.05	2.363	2.128	0.925	0.951
			0.1	1.946	1.834	0.881	0.908
		150	0.01	3.189	2.713	0.974	0.990
			0.05	2.351	2.128	0.926	0.954
			0.1	1.946	1.834	0.880	0.911
		200	0.01	3.120	2.713	0.977	0.990
			0.05	2.324	2.128	0.928	0.953
			0.1	1.929	1.834	0.883	0.909
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.198	2.713	0.974	0.990
			0.05	2.356	2.128	0.926	0.953
			0.1	1.945	1.834	0.881	0.909
		150	0.01	3.156	2.713	0.976	0.990
			0.05	2.320	2.128	0.929	0.954
			0.1	1.922	1.834	0.885	0.909
		200	0.01	3.088	2.713	0.978	0.991
			0.05	2.299	2.128	0.932	0.954
			0.1	1.909	1.834	0.887	0.909

Table 6 The simulated and approximate values

for pairwise comparison when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.5$.

k	M	p	α	z_{p}^{*}	$z_{\alpha_{\mathrm{p}}}$	$P(z_{\alpha_{\mathrm{p}}})$	$P(z_{1\cdot\mathrm{p}})$
6	60	60	0.01	4.273	3.209	0.935	0.990
			0.05	3.372	2.713	0.862	0.955
			0.1	2.936	2.475	0.807	0.916
		90	0.01	4.088	3.209	0.947	0.989
			0.05	3.240	2.713	0.878	0.954
			0.1	2.839	2.475	0.824	0.913
		120	0.01	3.940	3.209	0.954	0.990
			0.05	3.166	2.713	0.887	0.955
			0.1	2.784	2.475	0.834	0.915
		150	0.01	3.900	3.209	0.957	0.990
			0.05	3.134	2.713	0.892	0.953
			0.1	2.757	2.475	0.839	0.914
		200	0.01	3.786	3.209	0.963	0.989
			0.05	3.067	2.713	0.902	0.954
			0.1	2.705	2.475	0.850	0.914
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.954	3.209	0.955	0.990
			0.05	3.157	2.713	0.889	0.955
			0.1	2.771	2.475	0.835	0.914
		150	0.01	3.855	3.209	0.961	0.990
			0.05	3.091	2.713	0.898	0.955
			0.1	2.724	2.475	0.845	0.916
		200	0.01	3.767	3.209	0.965	0.990
			0.05	3.046	2.713	0.903	0.955
			0.1	2.698	2.475	0.852	0.914

Table 7 The simulated and approximate values

for comparison with a control when $\Sigma = I_p$.

k	M	p	α	z_c^*	z_{α_c}	$P(z_{\alpha_c})$	$P(z_{1\cdot c})$
3	60	60	0.01	2.973	2.576	0.978	0.989
			0.05	2.126	1.960	0.933	0.952
			0.1	1.709	1.645	0.890	0.908
		90	0.01	2.880	2.576	0.980	0.990
			0.05	2.098	1.960	0.936	0.952
			0.1	1.700	1.645	0.890	0.907
		120	0.01	2.850	2.576	0.982	0.990
			0.05	2.081	1.960	0.938	0.952
			0.1	1.686	1.645	0.893	0.908
		150	0.01	2.789	2.576	0.983	0.991
			0.05	2.066	1.960	0.939	0.952
			0.1	1.683	1.645	0.893	0.907
		200	0.01	2.785	2.576	0.983	0.990
			0.05	2.058	1.960	0.940	0.951
			0.1	1.676	1.645	0.895	0.906
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	2.835	2.576	0.982	0.990
			0.05	2.072	1.960	0.939	0.952
			0.1	1.688	1.645	0.893	0.906
		150	0.01	2.813	2.576	0.983	0.990
			0.05	2.058	1.960	0.940	0.952
			0.1	1.680	1.645	0.894	0.906
		200	0.01	2.777	2.576	0.984	0.990
			0.05	2.045	1.960	0.941	0.952
			0.1	1.675	1.645	0.895	0.906

Table 8 The simulated and approximate values

for comparison with a control when $\Sigma = I_p$.

k	M	p	α	z_{c}^*	$z_{\alpha_{\mathrm{c}}}$	$P(z_{\alpha_{\mathrm{c}}})$	$P(z_{1\cdot\mathrm{c}})$
6	60	60	0.01	3.363	2.878	0.973	0.990
			0.05	2.571	2.326	0.922	0.955
			0.1	2.185	2.054	0.875	0.914
		90	0.01	3.258	2.878	0.976	0.990
			0.05	2.518	2.326	0.927	0.955
			0.1	2.152	2.054	0.882	0.914
		120	0.01	3.211	2.878	0.978	0.990
			0.05	2.508	2.326	0.929	0.953
			0.1	2.141	2.054	0.884	0.913
		150	0.01	3.181	2.878	0.980	0.990
			0.05	2.473	2.326	0.933	0.955
			0.1	2.123	2.054	0.887	0.914
		200	0.01	3.151	2.878	0.980	0.990
			0.05	2.462	2.326	0.933	0.953
			0.1	2.114	2.054	0.887	0.912
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.205	2.878	0.979	0.990
			0.05	2.485	2.326	0.930	0.954
			0.1	2.136	2.054	0.884	0.911
		150	0.01	3.173	2.878	0.980	0.990
			0.05	2.469	2.326	0.933	0.954
			0.1	2.119	2.054	0.888	0.912
		200	0.01	3.141	2.878	0.981	0.990
			0.05	2.446	2.326	0.935	0.954
			0.1	2.101	2.054	0.890	0.912

Table 9 The simulated and approximate values

for comparison with a control when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.2$.

k	M	p	α	z_{c}^*	$z_{\alpha_{\mathrm{c}}}$	$P(z_{\alpha_{\mathrm{c}}})$	$P(z_{1\cdot\mathrm{c}})$
3	60	60	0.01	2.996	2.576	0.977	0.990
			0.05	2.143	1.960	0.932	0.952
			0.1	1.721	1.645	0.888	0.907
		90	0.01	2.913	2.576	0.980	0.990
			0.05	2.109	1.960	0.935	0.953
			0.1	1.705	1.645	0.890	0.908
		120	0.01	2.902	2.576	0.980	0.990
			0.05	2.092	1.960	0.937	0.952
			0.1	1.695	1.645	0.891	0.907
		150	0.01	2.851	2.576	0.981	0.990
			0.05	2.086	1.960	0.937	0.952
			0.1	1.696	1.645	0.891	0.906
		200	0.01	2.824	2.576	0.982	0.990
			0.05	2.068	1.960	0.938	0.952
			0.1	1.688	1.645	0.892	0.906
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	2.865	2.576	0.982	0.990
			0.05	2.082	1.960	0.937	0.952
			0.1	1.693	1.645	0.891	0.906
		150	0.01	2.825	2.576	0.983	0.990
			0.05	2.067	1.960	0.938	0.952
			0.1	1.690	1.645	0.892	0.906
		200	0.01	2.798	2.576	0.983	0.990
			0.05	2.062	1.960	0.940	0.951
			0.1	1.678	1.645	0.895	0.906

Table 10 The simulated and approximate values

for comparison with a control when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.2$.

k	M	p	α	z_{c}^*	$z_{\alpha_{\mathrm{c}}}$	$P(z_{\alpha_{\mathrm{c}}})$	$P(z_{1\cdot\mathrm{c}})$
6	60	60	0.01	3.432	2.878	0.970	0.990
			0.05	2.607	2.326	0.918	0.954
			0.1	2.207	2.054	0.871	0.913
		90	0.01	3.346	2.878	0.973	0.989
			0.05	2.554	2.326	0.923	0.954
			0.1	2.171	2.054	0.878	0.913
		120	0.01	3.270	2.878	0.976	0.990
			0.05	2.527	2.326	0.927	0.954
			0.1	2.149	2.054	0.882	0.914
		150	0.01	3.223	2.878	0.977	0.990
			0.05	2.509	2.326	0.929	0.953
			0.1	2.143	2.054	0.883	0.912
		200	0.01	3.187	2.878	0.979	0.990
			0.05	2.476	2.326	0.932	0.954
			0.1	2.124	2.054	0.886	0.913
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.247	2.878	0.977	0.990
			0.05	2.511	2.326	0.927	0.954
			0.1	2.148	2.054	0.882	0.911
		150	0.01	3.220	2.878	0.978	0.990
			0.05	2.493	2.326	0.931	0.954
			0.1	2.128	2.054	0.887	0.912
		200	0.01	3.172	2.878	0.980	0.990
			0.05	2.466	2.326	0.933	0.954
			0.1	2.116	2.054	0.888	0.912

Table 11 The simulated and approximate values

for comparison with a control when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.5$.

k	M	p	α	z_c^*	z_{α_c}	$P(z_{\alpha_c})$	$P(z_{1 \cdot c})$
3	60	60	0.01	3.211	2.576	0.970	0.990
			0.05	2.248	1.960	0.923	0.951
			0.1	1.780	1.645	0.879	0.906
		90	0.01	3.065	2.576	0.975	0.991
			0.05	2.183	1.960	0.929	0.952
			0.1	1.736	1.645	0.885	0.908
		120	0.01	3.059	2.576	0.975	0.989
			0.05	2.172	1.960	0.929	0.951
			0.1	1.743	1.645	0.884	0.905
		150	0.01	2.972	2.576	0.978	0.990
			0.05	2.142	1.960	0.932	0.952
			0.1	1.725	1.645	0.886	0.907
		200	0.01	2.933	2.576	0.979	0.990
			0.05	2.121	1.960	0.933	0.952
			0.1	1.713	1.645	0.888	0.906
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.023	2.576	0.977	0.990
			0.05	2.145	1.960	0.933	0.952
			0.1	1.715	1.645	0.888	0.908
		150	0.01	2.961	2.576	0.978	0.990
			0.05	2.129	1.960	0.934	0.952
			0.1	1.706	1.645	0.890	0.908
		200	0.01	2.921	2.576	0.980	0.990
			0.05	2.108	1.960	0.935	0.952
			0.1	1.700	1.645	0.890	0.907

Table 12 The simulated and approximate values

for comparison with a control when $\Sigma = (\rho^{|i-j|})$ with $\rho = 0.5$.

k	M	p	α	z_c^*	z_{α_c}	$P(z_{\alpha_c})$	$P(z_{1 \cdot c})$
6	60	60	0.01	3.712	2.878	0.959	0.989
			0.05	2.758	2.326	0.903	0.954
			0.1	2.306	2.054	0.856	0.913
		90	0.01	3.567	2.878	0.964	0.990
			0.05	2.680	2.326	0.911	0.953
			0.1	2.253	2.054	0.864	0.913
		120	0.01	3.459	2.878	0.968	0.990
			0.05	2.632	2.326	0.916	0.954
			0.1	2.217	2.054	0.869	0.914
		150	0.01	3.419	2.878	0.971	0.990
			0.05	2.595	2.326	0.919	0.954
			0.1	2.191	2.054	0.874	0.914
		200	0.01	3.353	2.878	0.973	0.989
			0.05	2.567	2.326	0.922	0.953
			0.1	2.184	2.054	0.876	0.912
	120	60	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		90	0.01	–	–	–	–
			0.05	–	–	–	–
			0.1	–	–	–	–
		120	0.01	3.444	2.878	0.969	0.990
			0.05	2.613	2.326	0.917	0.955
			0.1	2.212	2.054	0.871	0.913
		150	0.01	3.381	2.878	0.971	0.990
			0.05	2.598	2.326	0.918	0.953
			0.1	2.207	2.054	0.873	0.910
		200	0.01	3.301	2.878	0.975	0.990
			0.05	2.542	2.326	0.925	0.955
			0.1	2.164	2.054	0.879	0.913