On the distribution of test statistic using Song's kurtosis

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In this paper, we consider the multivariate normality test based on the sample measure of multivariate kurtosis defined by Song (2001). We derive expectation and variance of the multivariate sample kurtosis under normality, and propose test statistics using these expectation and variance. Moreover, we obtain an improved test statistic using the normalizing transformation. In order to evaluate accuracy of proposed test statistics, the numerical results by Monte Carlo simulation for some selected values of parameters are presented.

Key Words and Phrases: multivariate kurtosis; multivariate normality test; normalizing transformation.

1 Introduction

In statistical analysis, the test for normality is an important problem. This problem has been considered by many authors. Shapiro and Wilk (1965) derived test statistic using order statistic. This is called Shapiro-Wilk test as the univariate normality test. Multivariate extensions of the Shapiro-Wilk test were proposed by Malkovich and Afifi (1973), Royston (1983), Srivastava and Hui (1987) and so on. Mardia (1970), Srivastava (1984) and Song (2001) gave different definitions of the multivariate sample kurtosis. Mardia (1974) derived expectations and variances of multivariate sample skewness and kurtosis, and discussed their asymptotic distributions using

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expectation and variance of sample skewness or kurtosis. Okamoto and Seo (2010) derived the improved approximate χ^2 test statistic using the multivariate sample skewness of Srivastava (1984). Approximate accuracy of this test statistic is better than that of Srivastava (1984) especially for small sample size. Test statistics using the multivariate sample kurtosis of Srivastava (1984) were discussed by Seo and Ariga (2011). Sample measure of multivariate kurtosis by Song (2001) has not been much studied. Zografos (2008) derived an empirical estimator of Song's measure and its asymptotic distribution is investigated under the elliptic family of multivariate distributions.

In this paper, we derive test statistics using expectation and variance for multivariate sample kurtosis defined by Song (2001). In addition, we propose a test statistic which approximate accuracy for multivariate normal distribution is improved rather than them by normalizing transformation. Finally, we investigate accuracy of expectations, variances, upper percentiles, lower percentiles, skewness, kurtosis, type I error and power for these test statistics via a Monte Carlo simulation for selected values of parameters.

2 Test statistic using multivariate sample kurtosis

2.1 Song's measure of multivariate kurtosis

Let $\boldsymbol{x}$ be a p-dimensional random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. Then Song (2001) defined the population measure of multivariate kurtosis as follows:

$$\tau_{2,p} = \mathrm{Var}[\log(f(\boldsymbol{x}, \boldsymbol{\mu}, \Sigma))],$$

where $f(\boldsymbol{x}, \boldsymbol{\mu}, \Sigma)$ is a probability density function. We note that

$$f(\boldsymbol{x}, \boldsymbol{\mu}, \Sigma) = (2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) \right\}$$

and $\tau_{2,p}$ equals $p/2$ under a multivariate normal population.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be samples of size N from a multivariate population. Let $\overline{\boldsymbol{x}}$ and S be sample mean vector and sample covariance matrix as follows:

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_j,$$
$$S = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})',$$

respectively. Then we defined the sample measure of multivariate kurtosis as follows:

$$t_{2,p} = \frac{1}{N}\sum_{j=1}^{N}(y_j - \overline{y})^2,$$

where $y_j = \log(f(\boldsymbol{x}_j, \overline{\boldsymbol{x}}, S))$ and $\overline{y} = N^{-1}\sum_{j=1}^{N} y_j$. We note that

$$f(\boldsymbol{x}_j, \overline{\boldsymbol{x}}, S) = (2\pi)^{-\frac{p}{2}}|S|^{-\frac{1}{2}}\exp\left\{ -\frac{1}{2}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})'S^{-1}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})\right\}$$

under a multivariate normal population.

Without loss of generality, we assume that $\Sigma = I$ and $\boldsymbol{\mu} = \mathbf{0}$ for calculating moments of the measure of multivariate kurtosis. Also, it is difficult to derive moments of $t_{2,p}$ when covariance matrix Σ is unknown. For this reason, we consider the case of known Σ. Expectation for the Song's sample measure of multivariate kurtosis $\mathrm{E}[t_{2,p}]$ is written by

$$\begin{aligned}\mathrm{E}[t_{2,p}] &= \frac{N-1}{N}(\mathrm{E}[y_j^2] - \mathrm{E}[y_j y_k]) \\ &= \frac{N-1}{N}\cdot\frac{1}{4}(\mathrm{E}[\chi_j^4] - \mathrm{E}[\chi_j^2\chi_k^2]),\end{aligned}$$

where $\chi_j^2 = (\boldsymbol{x}_j - \overline{\boldsymbol{x}})'(\boldsymbol{x}_j - \overline{\boldsymbol{x}})$ and $j \neq k$.

Since $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ are not independent, we put

$$\overline{\boldsymbol{x}}^{(\alpha)} = \frac{1}{N-1}\sum_{j=1, j\neq\alpha}^{N}\boldsymbol{x}_j,$$
$$\overline{\boldsymbol{x}} = \left(1 - \frac{1}{N}\right)\overline{\boldsymbol{x}}^{(\alpha)} + \frac{1}{N}\boldsymbol{x}_\alpha.$$

Then it can be shown that

$$\begin{aligned}\mathrm{E}[\chi_j^4] &= \mathrm{E}[\{(\boldsymbol{x}_j - \overline{\boldsymbol{x}})'(\boldsymbol{x}_j - \overline{\boldsymbol{x}})\}^2] \\ &= \left(\frac{N-1}{N}\right)\mathrm{E}[\{(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})'(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})\}^2].\end{aligned}$$

Note that

$$\overline{\boldsymbol{x}}^{(\alpha)} \sim N\left(\boldsymbol{0}, \frac{1}{N-1}I\right)$$

and

$$\frac{N-1}{N}(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)})'(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)}) \sim \chi_p^2.$$

Therefore we obtain

$$\mathrm{E}[\chi_j^4] = \left(\frac{N-1}{N}\right)p(p+2).$$

Similarly, we put

$$\begin{aligned}\overline{\boldsymbol{x}}^{(\alpha,\beta)} &= \frac{1}{N-2}\sum_{j=1, j\neq\alpha\neq\beta\neq j}^{N} \boldsymbol{x}_j \\ &= \frac{1}{\sqrt{N-2}}\boldsymbol{y},\end{aligned}$$

where $\boldsymbol{y} \sim N(\boldsymbol{0}, I)$. Then

$$\overline{\boldsymbol{x}}^{(\alpha)} = \left(1 - \frac{1}{N-1}\right)\overline{\boldsymbol{x}}^{(\alpha,\beta)} + \frac{1}{N-1}\boldsymbol{x}_\beta.$$

It is shown that

$$\begin{aligned}\mathrm{E}[\chi_\alpha^2\chi_\beta^2] = \frac{1}{N^4}\mathrm{E}\Bigg[\Bigg\{&(N-1)^2\left(-\frac{2\boldsymbol{x}_\alpha'\boldsymbol{y}}{\sqrt{N-2}} + \frac{\boldsymbol{y}'\boldsymbol{y}}{N-2} + \boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right) \\ &-2(N-1)\left(-\frac{\boldsymbol{x}_\alpha'\boldsymbol{y}}{\sqrt{N-2}} - \frac{\boldsymbol{x}_\beta'\boldsymbol{y}}{\sqrt{N-2}} + \frac{\boldsymbol{y}'\boldsymbol{y}}{N-2} + \boldsymbol{x}_\alpha'\boldsymbol{x}_\beta\right) \\ &-\frac{2\boldsymbol{x}_\beta'\boldsymbol{y}}{\sqrt{N-2}} + \frac{\boldsymbol{y}'\boldsymbol{y}}{N-2} + \boldsymbol{x}_\beta'\boldsymbol{x}_\beta\Bigg\} \\ &\times\Bigg\{(N-1)^2\left(-\frac{2\boldsymbol{x}_\beta'\boldsymbol{y}}{\sqrt{N-2}} + \frac{\boldsymbol{y}'\boldsymbol{y}}{N-2} + \boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right) \\ &-2(N-1)\left(-\frac{\boldsymbol{x}_\alpha'\boldsymbol{y}}{\sqrt{N-2}} - \frac{\boldsymbol{x}_\beta'\boldsymbol{y}}{\sqrt{N-2}} + \frac{\boldsymbol{y}'\boldsymbol{y}}{N-2} + \boldsymbol{x}_\alpha'\boldsymbol{x}_\beta\right) \\ &-\frac{2\boldsymbol{x}_\alpha'\boldsymbol{y}}{\sqrt{N-2}} + \frac{\boldsymbol{y}'\boldsymbol{y}}{N-2} + \boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\Bigg\}\Bigg] \\ = \frac{p\{pN(N-2)+p+2\}}{N^2}&\end{aligned}$$

and we can obtain

$$\mathrm{E}[t_{2,p}] = \frac{p(N-1)(N-2)}{2N^2}.$$

Similarly, we can obtain expectation of $t_{2,p}^2$ and variance of $t_{2,p}$ as

$$\begin{aligned}
\mathrm{E}[t_{2,p}^2] &= \frac{p(N-1)(N-2)\{p(N-2)(N+1)+4(3N-7)\}}{4N^4}, \\
\mathrm{Var}[t_{2,p}] &= \frac{p(N-1)(N-2)\{(p+6)N-2(p+7)\}}{2N^4}.
\end{aligned}$$

Also, expectation and variance of $t_{2,p}$ are expressed as $\mathrm{E}[t_{2,p}] = p/2 + O(N^{-1})$ and $\mathrm{Var}[t_{2,p}] = p(p+6)/(2N) + O(N^{-2})$. Therefore, we propose the following theorem.

THEOREM 2.1 *Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be random samples of size N drawn from $N_p(\boldsymbol{\mu}, \Sigma)$, where Σ is known. Then for large N,*

$$T = \frac{t_{2,p} - \dfrac{p}{2}}{\sqrt{\dfrac{p(p+6)}{2N}}} \to N(0,1)$$

and

$$T^* = \frac{t_{2,p} - \mathrm{E}[t_{2,p}]}{\sqrt{\mathrm{Var}[t_{2,p}]}} \to N(0,1).$$

2.2 The improved test statistic using normalizing transformation

In order to improve accuracy of test statistics T and T^*, we calculate $\mathrm{E}[t_{2,p}^3]$ and derive the normalizing transformation for the distribution of $\sqrt{N}(t_{2,p} - \tau_{2,p})$. The normalizing transformation of some statistic in multivariate analysis has been discussed by Konishi (1981, 1987) and so on.

By using the same way as mentioned above, we obtain expectation $\mathrm{E}[t_{2,p}^3]$ as follows:

$$\mathrm{E}[t_{2,p}^3] = \frac{p^3}{8} - \frac{3p^2(p-12)}{8N} - \frac{p(7p^2+124p-480)}{8N^2} + O(N^{-3}).$$

Let $Y = \sqrt{N}(t_{2,p} - \tau_{2,p})$, where $\tau_{2,p} = p/2$. Then the first cumulants of Y are expressed as the following form:

$$\begin{aligned}
\kappa_1(Y) &= \frac{1}{\sqrt{N}} a_1 + O\big(N^{-\frac{3}{2}}\big), \\
\kappa_2(Y) &= \sigma^2 + O(N^{-1}), \\
\kappa_3(Y) &= \frac{6}{\sqrt{N}} a_3 + O\big(N^{-\frac{3}{2}}\big),
\end{aligned}$$

where

$$a_1 = -\frac{3p}{2}, \quad \sigma = \sqrt{\frac{p(p+6)}{2}}, \quad a_3 = \frac{p(22p+61)}{6}.$$

Further the cumulative distribution function of Y/σ can be expanded for large N as

$$\Pr\left[\frac{\sqrt{N}(t_{2,p}-\tau_{2,p})}{\sigma} \le y\right] = \Phi(y) - \frac{1}{\sqrt{N}}\left\{\frac{a_1}{\sigma}\Phi^{(1)}(y) + \frac{a_3}{\sigma^3}\Phi^{(3)}(y)\right\} + O(N^{-1}),$$

where $\Phi(y)$ is the cumulative distribution function of $N(0,1)$ and $\Phi^{(j)}(y)$ is the jth derivation of $\Phi(y)$. Thus, in order to find an asymptotic expansion of Y, we evaluate only the first few cumulants of Y in expanded forms.

Next, we consider the normalizing transformation of Y/σ. Suppose that $g(t_{2,p})$ is a function of $t_{2,p}$ satisfying all the derivatives of g of orders 2 and less are continuous in a neighborhood of the population kurtosis $\tau_{2,p}$. Then the cumulative distribution function of $Y^* = \sqrt{N}\{g(t_{2,p}) - g(\tau_{2,p})\}/\{\sigma g'(\tau_{2,p})\}$ can be expanded for large N as

$$\Pr\left[\frac{\sqrt{N}\{g(t_{2,p})-g(\tau_{2,p})\}}{\sigma g'(\tau_{2,p})} \le y\right] = \Phi(y) - \frac{1}{\sqrt{N}}\{b_1\Phi^{(1)}(y) + b_3\Phi^{(3)}(y)\} + O(N^{-1}),$$

where

$$b_i = \frac{a_i}{\sigma^i} + \frac{\sigma g''(t_{2,p})}{2g'(t_{2,p})} \quad (i = 1, 3)$$

and $g'(t_{2,p}) \ne 0$. This result follows from Siotani, Hayakawa and Fujikoshi (1985). When we put

$$g(t_{2,p}) = -\frac{3p(p+6)^2}{4(22p+61)}\exp\left[-\frac{4(22p+61)}{3p(p+6)^2}t_{2,p}\right],$$

we can obtain

$$\Pr\left[\frac{\sqrt{N}\{g(t_{2,p})-g(\tau_{2,p})\}}{\sigma g''(\tau_{2,p})} \le y\right] = \Phi(y) - \frac{c}{\sqrt{N}}\Phi^{(1)}(y) + O(N^{-1}),$$

where

$$\begin{aligned} c &= \left(a_1 - \frac{a_3}{\sigma^2}\right)g'(\tau_{2,p}) \\ &= -\frac{9p^2+98p+122}{6(p+6)}\exp\left[-\frac{2(22p+61)}{3(p+6)^2}\right]. \end{aligned}$$

Therefore we have

$$\Pr\left[\frac{\sqrt{N}\left\{g(t_{2,p}) - g(\tau_{2,p}) - \frac{c}{N}\right\}}{\sigma g'(\tau_{2,p})} \le y\right] = \Phi(y) + O(N^{-1}).$$

Then we propose an improved test statistic T_{NT} as the following theorem.

THEOREM 2.2 *Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be random samples of size N drawn from $N_p(\boldsymbol{\mu}, \Sigma)$, where Σ is known. Then for large N, the normalizing transformation of $t_{2,p}$ is given by*

$$T_{NT} = \frac{1}{2}\sqrt{\frac{2N}{p(p+6)}}\left[\frac{1}{d}\big(\exp[d(2t_{2,p}-p)]-1\big) - \frac{p}{N}\{(p+6)d-3\}\right],$$

where

$$d = -\frac{2(22p+61)}{3p(p+6)^2}.$$

3 Simulation studies

Accuracy of asymptotic approximation for multivariate normality test statistics using the measure of multivariate sample kurtosis is evaluated via a Monte Carlo simulation study. Simulation parameters are as follows: $p = 3,\ 10,\ 20,\ 30$, $N = 20,\ 50,\ 100,\ 200,\ 400,\ 800$ and significance level $\alpha = 0.05$. As a numerical experiment, we carry out 1,000,000 replications for multivariate normal populations.

Table 1 gives simulated values for expectation and variance of test statistics T, T^* and T_{NT}. Simulated values of T^* have good accuracy regardless of N. Simulated values of expectation and variance for T and T_{NT} are good as N increases. Further, simulated values of T_{NT} rapidly converge in those of standard normal distribution rather than T. Table 2 presents the upper and lower 100α percentiles of T, T^* and T_{NT}. Although T^* has good accuracy for expectation and variance, the upper and lower percentiles are not good. The upper and lower percentiles of T_{NT} converge in those of standard normal distribution and are more stable rather than other test statistics. Table 3 presents the simulated values of skewness and kurtosis for T, T^* and T_{NT}. Particularly, values of T_{NT} are closer to those of standard normal distribution for all dimensions and sample sizes. Table 4 gives the simulated values for type I error of T, T^* and T_{NT}. This result shows that the values of T_{NT} are stable and closer to 0.05 for many parameters. Table 5 presents the simulated values of power for T, T^* and T_{NT} under contaminated normal distribution. We performed simulation with kurtosis parameter $\kappa = 1.78$ ($\varepsilon = 0.1$, $\sigma = 3$) and 3.24 ($\varepsilon = 0.1$, $\sigma = 4$). It is note that power of T, T^* and T_{NT} is almost the same. We confirmed almost same results under multivariate t distribution.

In conclusion, the normalizing transformation statistic T_{NT} proposed in this paper is considerably good normal approximation even for small sample size and is useful for multivariate normality test. When Σ is unknown for multivariate normal populations, it will be our future problem.

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Table 1: Expectation and variance.

		Expectation				Variance			
p	N	T	T^*	T_{NT}	$N(0,1)$	T	T^*	T_{NT}	$N(0,1)$
3	20	-0.265	0.000	0.065		0.763	1.004	0.719	
	50	-0.171	0.000	0.019		0.902	1.003	0.855	
	100	-0.121	0.001	0.009		0.950	1.002	0.917	
	200	-0.085	0.001	0.004		0.975	1.001	0.955	
	400	-0.061	0.000	0.001		0.986	0.999	0.975	
	800	-0.043	0.000	0.001		0.993	0.999	0.987	
10	20	-0.363	0.000	0.024		0.764	1.000	0.792	
	50	-0.234	0.000	0.007		0.902	1.001	0.908	
	100	-0.165	0.001	0.004		0.949	1.000	0.951	
	200	-0.119	0.000	0.000		0.974	0.999	0.975	
	400	-0.083	0.001	0.001		0.988	1.001	0.988	
	800	-0.059	0.001	0.001		0.994	1.001	0.994	
20	50	-0.259	0.001	0.005		0.903	1.001	0.914	
	100	-0.185	0.000	0.002		0.948	0.998	0.954	
	200	-0.131	0.000	0.001		0.978	1.003	0.980	
	400	-0.093	0.000	0.000		0.986	0.998	0.987	
	800	-0.065	0.001	0.001		0.992	0.998	0.992	
30	50	-0.271	-0.001	0.004		0.900	0.998	0.910	
	100	-0.193	-0.001	0.001		0.949	0.999	0.954	
	200	-0.137	0.000	0.000		0.975	1.000	0.977	
	400	-0.097	0.000	0.000		0.984	0.997	0.986	
	800	-0.067	0.001	0.001		0.991	0.997	0.992	
	∞				0				1

Table 2: Upper and lower 5 percentiles.

		Upper 5 percentile				Lower 5 percentile			
p	N	T	T^*	T_{NT}	$N(0,1)$	T	T^*	T_{NT}	$N(0,1)$
3	20	1.379	1.886	1.514		-1.259	-1.141	-1.285	
	50	1.588	1.855	1.562		-1.413	-1.310	-1.490	
	100	1.650	1.819	1.597		-1.489	-1.404	-1.562	
	200	1.673	1.783	1.621		-1.541	-1.474	-1.600	
	400	1.669	1.742	1.628		-1.573	-1.522	-1.618	
	800	1.668	1.717	1.637		-1.601	-1.562	-1.634	
10	20	1.267	1.865	1.580		-1.493	-1.293	-1.351	
	50	1.485	1.811	1.628		-1.581	-1.419	-1.507	
	100	1.560	1.772	1.644		-1.613	-1.484	-1.562	
	200	1.602	1.743	1.654		-1.631	-1.532	-1.595	
	400	1.622	1.716	1.655		-1.641	-1.567	-1.616	
	800	1.634	1.698	1.656		-1.643	-1.589	-1.624	
20	50	1.440	1.790	1.649		-1.647	-1.461	-1.491	
	100	1.524	1.753	1.661		-1.661	-1.515	-1.551	
	200	1.575	1.728	1.666		-1.668	-1.556	-1.589	
	400	1.596	1.699	1.659		-1.663	-1.580	-1.606	
	800	1.616	1.687	1.660		-1.658	-1.598	-1.618	
30	50	1.415	1.774	1.651		-1.674	-1.478	-1.479	
	100	1.506	1.742	1.665		-1.682	-1.528	-1.543	
	200	1.561	1.719	1.670		-1.680	-1.564	-1.582	
	400	1.583	1.690	1.659		-1.675	-1.588	-1.605	
	800	1.607	1.681	1.660		-1.665	-1.602	-1.615	
	∞				1.645				-1.645

Table 3: Skewness and kurtosis.

		Skewness				Kurtosis			
p	N	T	T^*	T_{NT}	$N(0,1)$	T	T^*	T_{NT}	$N(0,1)$
3	20	1.854	1.854	0.117		10.057	10.057	2.436	
	50	1.170	1.170	0.033		5.763	5.763	2.690	
	100	0.817	0.817	0.020		4.336	4.336	2.829	
	200	0.574	0.574	0.017		3.660	3.660	2.916	
	400	0.406	0.406	0.016		3.334	3.334	2.962	
	800	0.287	0.287	0.014		3.161	3.161	2.975	
10	20	1.191	1.191	0.298		5.717	5.717	2.844	
	50	0.759	0.759	0.183		4.131	4.131	2.942	
	100	0.529	0.529	0.130		3.534	3.534	2.964	
	200	0.373	0.373	0.092		3.265	3.265	2.987	
	400	0.266	0.266	0.068		3.140	3.140	2.997	
	800	0.192	0.192	0.052		3.073	3.073	2.999	
20	50	0.610	0.610	0.266		3.690	3.690	3.053	
	100	0.431	0.431	0.189		3.347	3.347	3.032	
	200	0.304	0.304	0.134		3.165	3.165	3.010	
	400	0.215	0.215	0.095		3.086	3.086	3.009	
	800	0.152	0.152	0.068		3.036	3.036	2.998	
30	50	0.551	0.551	0.303		3.563	3.563	3.113	
	100	0.389	0.389	0.215		3.277	3.277	3.056	
	200	0.272	0.272	0.150		3.139	3.139	3.032	
	400	0.186	0.186	0.100		3.065	3.065	3.015	
	800	0.136	0.136	0.075		3.034	3.034	3.008	
	∞				0				3

Table 4: Type I error.

p	N	T	T^*	T_{NT}	$N(0,1)$
3	20	0.036	0.065	0.034	
	50	0.046	0.065	0.041	
	100	0.050	0.064	0.045	
	200	0.052	0.062	0.047	
	400	0.052	0.059	0.048	
	800	0.052	0.057	0.049	
10	20	0.029	0.066	0.044	
	50	0.039	0.063	0.048	
	100	0.044	0.061	0.050	
	200	0.046	0.059	0.051	
	400	0.048	0.057	0.051	
	800	0.049	0.055	0.051	
20	50	0.036	0.062	0.050	
	100	0.040	0.060	0.051	
	200	0.044	0.058	0.052	
	400	0.046	0.055	0.051	
	800	0.047	0.054	0.051	
30	50	0.034	0.061	0.051	
	100	0.039	0.059	0.052	
	200	0.043	0.057	0.052	
	400	0.044	0.055	0.051	
	800	0.047	0.054	0.052	
	∞				0.050

Table 5: Power under contaminated normal distribution.

κ		1.78			3.24		
p	N	T	T^*	T_{NT}	T	T^*	T_{NT}
3	20	0.739	0.764	0.736	0.461	0.518	0.456
	50	0.961	0.965	0.959	0.766	0.791	0.756
	100	0.998	0.998	0.998	0.939	0.946	0.935
	200	1.000	1.000	1.000	0.996	0.997	0.996
	400	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000
10	20	0.873	0.880	0.876	0.761	0.795	0.777
	50	0.994	0.994	0.994	0.969	0.974	0.971
	100	1.000	1.000	1.000	0.999	0.999	0.999
	200	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000
20	50	0.995	0.995	0.995	0.991	0.992	0.991
	100	1.000	1.000	1.000	1.000	1.000	1.000
	200	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000
30	50	0.995	0.995	0.995	0.994	0.994	0.994
	100	1.000	1.000	1.000	1.000	1.000	1.000
	200	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000