Estimations for some functions of covariance matrix in high dimension under non-normality

(Last Modified: June 29, 2013*)

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Abstract
When we consider a statistical test in the high dimensional case, we often need estimators of the functions of the covariance matrix Σ. Especially, it is needed to estimate $a_2 = (1/p)\mathrm{tr}\Sigma^2$. The unbiased and consistent estimator of a_2 is proposed in preceding study when the population distribution is multivariate normal. But it is difficult to estimate in the non-normal case. So we propose the unbiased and consistent estimators for some functions of covariance matrix including a_2 under the non-normal case. Through the numerical simulation, we confirmed the accuracy of the approximation of the our proposed estimators.

1 Introduction

We consider the one-sample problem, that is, let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N$ be independent p-dimensional random vectors, each $\boldsymbol{x}_i$ can be expressed as

$$\boldsymbol{x}_i = \boldsymbol{\mu} + \Sigma^{1/2}\boldsymbol{z}_i, \tag{1}$$

and $\boldsymbol{z}_i$ has a distribution F with mean vector $\mathbf{0}$ and covariance matrix Σ, where Σ is a $p \times p$ positive definite. When we deal with statistical tests of the multivariate analysis for high-dimensional data, we often need the estimator of $\mathrm{tr}\Sigma^2$ (e.g. Schott [2], Srivastava and Fujikoshi [4], Chen and Qin [7], Fujikoshi et al. [8], and so on). Srivastava [3] proposed unbiased and consistent estimator of $\mathrm{tr}\Sigma^2/p (=: a_2)$ under some conditions and the assumption that F is normal. The estimator is given by

$$\tilde{a}_2 = \frac{n^2}{(n-1)(n+2)}\frac{1}{p}\left[\mathrm{tr}S^2 - \frac{1}{n}(\mathrm{tr}S)^2\right], \tag{2}$$

*The first version appeared on July 10, 2012. Some errors are corrected.

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where

$$\bar{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i, \qquad n = N-1,$$

$$S = \frac{1}{n}\sum_{i=1}^{N}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})'.$$

Subsequently, Srivastava [5] verified that this has consistency under the strong conditions even if F is not normal. Note that this estimator is not generally unbiased. Chen and Qin [7] and Chen et al [6] also proposed other consistent estimator of a_2. However these are not also unbiased and complicated form, respectively. Additionally, the consistency of these estimators is derived under strong moment conditions. So, we propose the unbiased consistent estimators of not only $\mathrm{tr}\Sigma^2$ but also $(\mathrm{tr}\Sigma)^2$ and κ_{11} in simple form, where

$$\kappa_{ij} := \mathrm{E}[\boldsymbol{z}'\Sigma^i\boldsymbol{z}\boldsymbol{z}'\Sigma^j\boldsymbol{z}] - 2\mathrm{tr}\Sigma^{i+j} - \mathrm{tr}\Sigma^i\mathrm{tr}\Sigma^j \tag{3}$$

and $\boldsymbol{z} \sim F$. For normal case, κ_{ij} is equal to zero (Magnus and Neudecker [1]). κ_{11} is one of the important parameters to indicate difference from the non-normal case.

In Section 2, we propose these estimators and prove the unbiasedness and the consistency. In Section 3, we verify the performance of the our proposed estimators by using Monte Carlo method. Technical details are provided in an Appendix.

2 Unbiasedness and consistency of estimators

Firstly, we set up the asymptotic framework A1 and assumption A2:

A1: $n/p \to c \in (0, \infty)$,
A2: $a_i := \mathrm{tr}\Sigma^i/p \to a_i^0 \in (0, \infty)$ (for $i = 1, \cdots, 4$).

When the estimation of $\mathrm{tr}\Sigma^2$ is considered, we often use $\mathrm{tr}S^2$ and $(\mathrm{tr}S)^2$. The expectations of these statistics are calculated as

$$\mathrm{E}[\mathrm{tr}S^2] = \frac{1}{N}\kappa_{11} + \frac{N}{N-1}\mathrm{tr}\Sigma^2 + \frac{1}{N-1}(\mathrm{tr}\Sigma)^2, \tag{4}$$

$$\mathrm{E}[(\mathrm{tr}S)^2] = \frac{1}{N}\kappa_{11} + \frac{2}{N-1}\mathrm{tr}\Sigma^2 + (\mathrm{tr}\Sigma)^2. \tag{5}$$

Since there are three unknown parameters in the expectations, it is difficult to estimate $\mathrm{tr}\Sigma^2$ by only $\mathrm{tr}S^2$ and $(\mathrm{tr}S)^2$. So we define another statistics as

$$Q := \frac{1}{N-1}\sum_{i=1}^{N}((\boldsymbol{x}_i - \bar{\boldsymbol{x}})'(\boldsymbol{x}_i - \bar{\boldsymbol{x}}))^2. \tag{6}$$

The expectation of Q is expressed as

$$\mathrm{E}[Q] = \frac{N^2-3N+3}{N^2}\kappa_{11} + \frac{2(N-1)}{N}\mathrm{tr}\Sigma^2 + \frac{N-1}{N}(\mathrm{tr}\Sigma)^2. \tag{7}$$

By solving simultaneous equations (4), (5), and (6), we obtain the following theorem.

Theorem 1 *For model (1), the unbiased estimators of* $\mathrm{tr}\Sigma^2$*,* $(\mathrm{tr}\Sigma)^2$*, and* κ_{11} *are obtained as*

$$\begin{aligned}
\widehat{\mathrm{tr}\Sigma^2} &= \frac{N-1}{N(N-2)(N-3)}\{(N-1)(N-2)\mathrm{tr}S^2 + (\mathrm{tr}S)^2 - NQ\},\\
(\widehat{\mathrm{tr}\Sigma})^2 &= \frac{N-1}{N(N-2)(N-3)}\{2\mathrm{tr}S^2 + (N^2-3N+1)(\mathrm{tr}S)^2 - NQ\},\\
\hat{\kappa}_{11} &= \frac{-1}{(N-2)(N-3)}\{2(N-1)^2\mathrm{tr}S^2 + (N-1)^2(\mathrm{tr}S)^2 - N(N+1)Q\}.
\end{aligned}$$

Next, we consider the consistency of $\hat{a}_2(:= \widehat{\mathrm{tr}\Sigma^2}/p)$ and $\widehat{a_1^2}(:= (\widehat{\mathrm{tr}\Sigma})^2/p^2)$. From the definitions of $\widehat{\mathrm{tr}\Sigma^2}$ and $(\widehat{\mathrm{tr}\Sigma})^2$, these statistics are expressed as

$$\begin{aligned}
\widehat{\mathrm{tr}\Sigma^2} &= \frac{1}{N(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2\\
&\quad -\frac{2}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\\
&\quad +\frac{1}{N(N-1)(N-2)(N-3)}\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l,\\
(\widehat{\mathrm{tr}\Sigma})^2 &= \frac{1}{N(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j\\
&\quad -\frac{2}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\\
&\quad +\frac{1}{N(N-1)(N-2)(N-3)}\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l.
\end{aligned}$$

Note that above form of $\widehat{\mathrm{tr}\Sigma^2}$ is the same as the one of Chen et al. [6]. But this form costs $O(N^4)$ for calculation about sample size. On the other hand, our proposed form costs only $O(N)$.

Using the calculations of Appendix B, we obtain the following lemma.

Lemma 1 *For model (1), the variances of* $\widehat{\mathrm{tr}\Sigma^2}$ *and* $(\widehat{\mathrm{tr}\Sigma})^2$ *defined by theorem 1 are derived as*

$$\begin{aligned}
\mathrm{Var}(\widehat{\mathrm{tr}\Sigma^2}) &= \frac{2}{N(N-1)}\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4] + \frac{4(N-3)}{N(N-2)}\kappa_{22} \\
&\quad - \frac{8}{N(N-1)(N-2)}\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2] \\
&\quad - \frac{2(N+1)(N-4)}{N(N-1)(N-2)(N-3)}(\mathrm{tr}\Sigma^2)^2 \\
&\quad + \frac{4(2N^2-13N+23)}{N(N-2)(N-3)}\mathrm{tr}\Sigma^4, \\
\mathrm{Var}((\widehat{\mathrm{tr}\Sigma})^2) &= \frac{2}{N(N-1)}\kappa_{11}^2 + \frac{4(2N-5)}{N(N-1)(N-2)}\kappa_{11}\mathrm{tr}\Sigma^2 + \frac{4}{N}\kappa_{11}(\mathrm{tr}\Sigma)^2 \\
&\quad - \frac{8}{N(N-1)(N-2)}\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2] \\
&\quad + \frac{8(N^2-6N+10)}{N(N-1)(N-2)(N-3)}(\mathrm{tr}\Sigma^2)^2 \\
&\quad + \frac{4(2N-3)}{N(N-1)}\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2 \\
&\quad + \frac{16}{N(N-1)(N-2)(N-3)}\mathrm{tr}\Sigma^4.
\end{aligned}$$

From this lemma, the following theorem is derived.

Theorem 2 *Assume model (1). Under the asymptotic framework A1 and assumptions A2 and C1,* $\hat{a}_2$ *is the consistent estimator of* a_2*. In addition,* $\widehat{a_1^2}$ *is the consistent estimator of* a_1^2 *under A1, A2 and C2. Here, conditions C1 and C2 are defined as follows:*

$$\begin{aligned}
&\text{C1: } \mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4] = o(p^4), \kappa_{22} = o(p^3) \qquad (\boldsymbol{z}_1, \boldsymbol{z}_2 \sim i.i.d.F), \\
&\text{C2: } \kappa_{11} = o(p^3).
\end{aligned}$$

We remark that $\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2] = o(p^{5/2})$ under the condition C1 and $\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2] = o(p^{7/2})$ under the condition C2. These results are derived using the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}
\mathrm{E}[|(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2|] &= \sqrt{\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4]\mathrm{E}[(\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2)^2]} \\
&= \sqrt{\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4]\mathrm{tr}\Sigma^4} \\
&= o(p^{5/2}), \\
\mathrm{E}[|\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2|] &= \sqrt{\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2(\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2)^2]\mathrm{E}[(\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2)^2]} \\
&= \sqrt{\{\kappa_{11} + 2\mathrm{tr}\Sigma^2 + (\mathrm{tr}\Sigma)^2\}^2\mathrm{tr}\Sigma^4} \\
&= o(p^{7/2}).
\end{aligned}$$

If F is normal distribution, then the variance of $\hat{a}_2$ can be expressed as

$$\begin{aligned}\mathrm{Var}(\hat{a}_2) &= \frac{2(4N^3-28N^2+67N-42)}{N(N-1)(N-2)(N-3)p}a_4 \\ &\quad +\frac{4(N^2-6N+11)}{N(N-1)(N-2)(N-3)}a_2^2 \\ &= \frac{8a_4}{Np}+\frac{4a_2^2}{N^2}+O(N^{-3}).\end{aligned}$$

By contrast, it can be expressed that

$$\begin{aligned}\mathrm{Var}(\tilde{a}_2) &= \frac{8(N+2)}{(N+1)(N-1)p}a_4 \\ &\quad +\frac{4}{(N-2)(N+1)}(a_2^2-a_4/p) \\ &= \frac{8a_4}{Np}+\frac{4a_2^2}{N^2}+O(N^{-3})\end{aligned}$$

under the assumption that F is multivariate normal (Srivastava [3]). Therefore, the variance of our proposed estimator is equal to the one of $\tilde{a}_2$ asymptotically.

Next we consider the estimation of κ_{11}. When F is normal distribution, $\kappa_{11}=0$. When $\boldsymbol{z}\sim F$ and the all elements of $\boldsymbol{z}=(z_1,\cdots,z_p)$ are i.i.d., we can calculate as $\kappa_{11}=(\gamma-3)\mathrm{tr}\Sigma\odot\Sigma=O(p)$, where $\gamma=E[z_i^4]$ and $\odot$ denotes the Hadamard product, that is, $(A\odot B)_{ij}=(A)_{ij}\cdot(B)_{ij}$. In the case that F is a specific multivariate distribution such as the multivariate t-distribution and the multivariate contaminated normal distribution, we obtain $\kappa_{11}=O(p^2)$. So, we consider the consistency of κ_{11} for the cases that all elements of $\boldsymbol{z}$ are i.i.d. or $\kappa_{11}=O(p^2)$.

When all elements of $\boldsymbol{z}$ are i.i.d., we assume that $E[z_i^8]<\infty$. Then the following relations holds:

$$\begin{aligned}E[(\boldsymbol{z}'\Sigma\boldsymbol{z}-\mathrm{tr}\Sigma)^4] &\leq C(\mathrm{tr}\Sigma^2)^2, \\ E[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4] &= 3(\mathrm{tr}\Sigma^2)^2+6\mathrm{tr}\Sigma^4+6(\gamma-3)\mathrm{tr}\Sigma^2\odot\Sigma^2 \\ &\quad +(\gamma-3)^2\mathrm{tr}(\Sigma\odot\Sigma)^2,\end{aligned}$$

where C is a constant (Appendix D). Additionally, we use the following inequations:

$$\begin{aligned}\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z})^2\boldsymbol{z}'\Sigma^2\boldsymbol{z}] &\leq \sqrt{\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z})^4]\mathrm{E}[(\boldsymbol{z}'\Sigma^2\boldsymbol{z})^2]}, \\ \mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2] &\leq \sqrt{\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2(\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2)^2]\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4]}, \\ \mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2] &\leq \sqrt{\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2(\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2)^2]\mathrm{E}[(\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2)^2]}, \\ \mathrm{E}[\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2] &\leq \sqrt{\mathrm{E}[(\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1)^2(\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2)^2]\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2]}, \\ \mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2] &\leq \sqrt{\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4]\mathrm{E}[(\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2)^2]}.\end{aligned}$$

We note that $E[(\boldsymbol{z}'\Sigma\boldsymbol{z})^4] = O(p^4)$. Since the above five relations hold without the assumption that all elements of $\boldsymbol{z}$ are i.i.d., we obtain the following theorem from the above relations and the result of Appendix C.

Theorem 3 *We assume the framework A1 and assumption A2. If all elements of $\boldsymbol{z} \sim F$ are i.i.d. and the eighth moment of the element is finite, then it holds that $\kappa_{11} = O(p)$ and $\hat{\kappa}_{11}/p$ is an unbiased and consistent estimator of κ_{11}/p. If we assume $\kappa_{11} = O(p^2)$, then $\hat{\kappa}_{11}/p^2$ is unbiased and consistent estimator of κ_{11}/p^2 under the assumptions that $E[(\boldsymbol{z}'\Sigma\boldsymbol{z})^4] = O(p^4)$, $\kappa_{13} = O(p^2)$, and $\kappa_{22} = O(p^2)$.*

3 Numerical simulations

In this section, we examine the performance of our proposed estimators by Monte Carlo simulation.

Since $\hat{a}_2$, $\widehat{a_1^2}$, and $\hat{\kappa}_{11}$ are invariant for $\boldsymbol{\mu}$, we assume $\boldsymbol{x}_i = \Sigma^{1/2}\boldsymbol{z}_i$ $(i = 1, \cdots, N)$, where $\boldsymbol{z}_i \sim$ i.i.d. as F. We give the covariance matrix as $\Sigma = (0.2^{|i-j|})$. We chose sample sizes and dimensions as $N = 40, 80, 120$ and $p = 40, 80, 120$. We consider the following four case of $\boldsymbol{z} = (z_1, \cdots, z_p)' \sim F$:

$D_1 : z_1, \cdots, z_p$ are i.i.d. as $N(0, 1)$,
$D_2 : z_1, \cdots, z_p$ are i.i.d. as t distribution with $p/4$ d.f.,
$D_3 : z_1, \cdots, z_p$ are i.i.d. as χ^2 distribution with 1 d.f.,
$D_4 : \boldsymbol{z}$ is distributed as p variate t-distribution with 10 d.f..

We note that the distribution of F is standardized so that $\mathrm{Var}(\boldsymbol{z}) = I_p$. Under each situation, we calculate the estimates of $\hat{a}_2$, $\tilde{a}_2$, $\widehat{a_1^2}$, $\tilde{a}_1^2 (:= (\mathrm{tr}S/p)^2)$, and $\hat{\kappa}_{11}/p^i$, the unbiased sample variance of these estimates, and the true values of these parameters, where $i = 1$ in the case D_1, D_2, and D_3 and $i = 2$ in the case D_4. The number of replications is 10,000.

Table 1 shows the true value of a_2 and the estimates $\hat{a}_2$ and $\tilde{a}_2$. The values in parentheses show standard errors. The performance of $\tilde{a}_2$ is good for all distributions. Especially, even if the distribution is the normal distribution, the standard error of $\hat{a}_2$ is close to the one of $\tilde{a}_2$. By contrast, if the distribution is not near normal distribution, then the performance of $\tilde{a}_2$ is not so good. Table 2 shows the true value of a_1^2 and the estimates $\widehat{a_1^2}$, $\tilde{a}_1^2$. $\tilde{a}_1^2$ seems to overestimate. But we can hardly confirm the difference between $\widehat{a_1^2}$ and $\tilde{a}_1^2$ from this table. Table 3 shows the true value of κ_{11}/p^i and the estimates $\hat{\kappa}_{11}/p^i$, where $i = 1$ in the case D_1, D_2, and D_3 and $i = 2$ in the case D_4. When F is a symmetric distribution, the approximation seems to be good. However F is a chi square distribution, the approximation becomes bad. In such a case, we need large p and N.

4 Conclusions

We proposed the unbiased estimators of a_2, a_1^2, and κ_{11} as simple forms under non-normality and showed consistencies of them. Especially $\hat{a}_2$ is given as simple

Table 1: The true value of a_2 and the estimates $\hat{a}_2$ and $\tilde{a}_2$ of a_2 (and standard errors).

distribution	N	p	a_2	$\hat{a}_2$	$\tilde{a}_2$
D_1	40	40	1.081	1.081 (0.106)	1.081 (0.105)
		80	1.082	1.081 (0.085)	1.081 (0.084)
		120	1.083	1.082 (0.075)	1.082 (0.074)
	80	40	1.081	1.081 (0.068)	1.081 (0.068)
		80	1.082	1.082 (0.052)	1.082 (0.052)
		120	1.083	1.083 (0.046)	1.083 (0.045)
	120	40	1.081	1.081 (0.055)	1.081 (0.055)
		80	1.082	1.083 (0.040)	1.083 (0.040)
		120	1.083	1.083 (0.035)	1.083 (0.035)
D_2	40	40	1.081	1.080 (0.120)	1.104 (0.126)
		80	1.082	1.084 (0.088)	1.092 (0.088)
		120	1.083	1.083 (0.080)	1.088 (0.079)
	80	40	1.081	1.082 (0.079)	1.094 (0.081)
		80	1.082	1.082 (0.055)	1.087 (0.055)
		120	1.083	1.083 (0.045)	1.086 (0.047)
	120	40	1.081	1.082 (0.063)	1.090 (0.064)
		80	1.082	1.083 (0.043)	1.086 (0.043)
		120	1.083	1.082 (0.036)	1.084 (0.036)
D_3	40	40	1.081	1.081 (0.228)	1.368 (0.348)
		80	1.082	1.081 (0.165)	1.365 (0.255)
		120	1.083	1.084 (0.141)	1.369 (0.215)
	80	40	1.081	1.082 (0.151)	1.228 (0.191)
		80	1.082	1.084 (0.111)	1.231 (0.141)
		120	1.083	1.083 (0.092)	1.230 (0.118)
	120	40	1.081	1.081 (0.125)	1.180 (0.148)
		80	1.082	1.082 (0.088)	1.181 (0.103)
		120	1.083	1.082 (0.073)	1.180 (0.086)
D_4	40	40	1.081	1.082 (0.238)	1.421 (0.692)
		80	1.082	1.078 (0.224)	1.725 (0.741)
		120	1.083	1.085 (0.221)	2.059 (1.081)
	80	40	1.081	1.079 (0.162)	1.250 (0.250)
		80	1.082	1.082 (0.153)	1.415 (0.341)
		120	1.083	1.081 (0.150)	1.577 (0.467)
	120	40	1.081	1.081 (0.130)	1.196 (0.180)
		80	1.082	1.081 (0.124)	1.303 (0.217)
		120	1.083	1.081 (0.122)	1.416 (0.279)

Table 2: The true value of a_1^2 and the estimates $\widehat{a_1^2}$ and $\tilde{a}_1^2$ of a_1^2 (and standard errors).

distribution	N	p	a_1^2	$\widehat{a_1^2}$	$\tilde{a}_1^2$
D_1	40	40	1.000	1.000 (0.075)	1.001 (0.075)
		80	1.000	1.001 (0.053)	1.001 (0.053)
		120	1.000	0.999 (0.042)	1.000 (0.042)
	80	40	1.000	1.000 (0.052)	1.000 (0.052)
		80	1.000	1.000 (0.037)	1.001 (0.037)
		120	1.000	1.001 (0.030)	1.001 (0.030)
	120	40	1.000	1.000 (0.043)	1.000 (0.043)
		80	1.000	1.000 (0.030)	1.000 (0.030)
		120	1.000	1.000 (0.025)	1.000 (0.025)
D_2	40	40	1.000	0.999 (0.090)	1.001 (0.090)
		80	1.000	1.001 (0.057)	1.002 (0.057)
		120	1.000	1.000 (0.046)	1.001 (0.046)
	80	40	1.000	1.001 (0.064)	1.002 (0.064)
		80	1.000	1.000 (0.040)	1.002 (0.040)
		120	1.000	1.000 (0.032)	1.001 (0.032)
	120	40	1.000	1.000 (0.052)	1.002 (0.052)
		80	1.000	1.000 (0.032)	1.001 (0.032)
		120	1.000	1.000 (0.026)	1.000 (0.026)
D_3	40	40	1.000	1.002 (0.190)	1.011 (0.193)
		80	1.000	0.999 (0.133)	1.003 (0.134)
		120	1.000	1.001 (0.109)	1.004 (0.110)
	80	40	1.000	1.001 (0.131)	1.006 (0.132)
		80	1.000	1.002 (0.095)	1.004 (0.095)
		120	1.000	1.001 (0.077)	1.002 (0.077)
	120	40	1.000	1.000 (0.110)	1.003 (0.110)
		80	1.000	1.000 (0.077)	1.002 (0.077)
		120	1.000	0.999 (0.063)	1.001 (0.063)
D_4	40	40	1.000	1.000 (0.205)	1.010 (0.212)
		80	1.000	0.996 (0.191)	1.005 (0.195)
		120	1.000	1.002 (0.190)	1.011 (0.195)
	80	40	1.000	0.999 (0.143)	1.004 (0.145)
		80	1.000	0.999 (0.135)	1.004 (0.137)
		120	1.000	0.999 (0.134)	1.003 (0.136)
	120	40	1.000	1.000 (0.116)	1.003 (0.117)
		80	1.000	0.999 (0.111)	1.002 (0.112)
		120	1.000	0.999 (0.109)	1.002 (0.110)

Table 3: The true value of κ_{11}/p^i (i=1 or 2) and the estimates $\hat{\kappa}_{11}/p^i$ of κ_{11}/p^i (and standard errors).

distribution	N	p	κ_{11}/p^i	$\hat{\kappa}_{11}/p^i$
D_1	40	40	0.000	-0.004 (0.548)
		80	0.000	0.005 (0.535)
		120	0.000	0.002 (0.540)
	80	40	0.000	-0.007 (0.369)
		80	0.000	0.000 (0.364)
		120	0.000	0.004 (0.362)
	120	40	0.000	0.002 (0.295)
		80	0.000	0.001 (0.295)
		120	0.000	0.004 (0.292)
D_2	40	40	1.000	0.994 (0.974)
		80	0.375	0.369 (0.643)
		120	0.231	0.236 (0.593)
	80	40	1.000	1.011 (0.708)
		80	0.375	0.370 (0.427)
		120	0.230	0.223 (0.401)
	120	40	1.000	0.996 (0.581)
		80	0.375	0.375 (0.350)
		120	0.231	0.228 (0.322)
D_3	40	40	12.000	12.069 (7.219)
		80	12.000	11.960 (5.623)
		120	12.000	11.975 (4.916)
	80	40	12.000	11.970 (4.984)
		80	12.000	12.047 (3.935)
		120	12.000	12.060 (3.552)
	120	40	12.000	12.020 (4.232)
		80	12.000	11.990 (3.125)
		120	12.000	11.957 (2.826)
D_4	40	40	0.351	0.356 (0.590)
		80	0.342	0.340 (0.314)
		120	0.339	0.341 (0.333)
	80	40	0.351	0.351 (0.247)
		80	0.342	0.342 (0.235)
		120	0.339	0.339 (0.255)
	120	40	0.351	0.350 (0.229)
		80	0.342	0.340 (0.185)
		120	0.339	0.341 (0.196)

from in comparison with Chen et al. [6]. We confirmed that the approximations of these estimators are good by numerical simulations. However we note that we need large p and N when we use $\hat{\kappa}_{11}$.

Acknowledgement

The second author was supported by grant from Japan Society for the Promotion of Science (JSPS), Grant-in-Aid for Scientific Research, Young Scientists (B), #23740085.

A Expectation of estimators

Let $\boldsymbol{y}_i = \boldsymbol{x}_i - \boldsymbol{\mu}$ $(i = 1, \cdots, N)$. Then the sample covariance matrix S can be expressed as

$$\begin{aligned}
S &= \frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{N-1}{N}\boldsymbol{y}_i - \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\boldsymbol{y}_j\right)\left(\frac{N-1}{N}\boldsymbol{y}_i - \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\boldsymbol{y}_j\right)' \\
&= \frac{N-1}{N^2}\sum_{i=1}^{N}\boldsymbol{y}_i\boldsymbol{y}_i' - \frac{1}{N^2}\sum_{i=1}^{N}\boldsymbol{y}_i\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\boldsymbol{y}_j'\right) - \frac{1}{N^2}\sum_{i=1}^{N}\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\boldsymbol{y}_j\right)\boldsymbol{y}_i' \\
&\quad + \frac{1}{N^2(N-1)}\sum_{i=1}^{N}\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\boldsymbol{y}_j\boldsymbol{y}_j' + \sum_{\substack{j,k=1\\ j\neq k, j\neq i, k\neq i}}^{N}\boldsymbol{y}_j\boldsymbol{y}_k'\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{y}_i\boldsymbol{y}_i' - \frac{1}{N^2}\sum_{i=1}^{N}\left(\boldsymbol{y}_i\sum_{\substack{j=1\\ j\neq i}}^{N}\boldsymbol{y}_j'\right) - \frac{1}{N^2}\sum_{i=1}^{N}\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\boldsymbol{y}_j\boldsymbol{y}_i'\right) \\
&\quad + \frac{N-2}{N^2(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{y}_i\boldsymbol{y}_j' \\
&= \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{y}_i\boldsymbol{y}_i' + \left(\frac{N-2}{N^2(N-1)} - \frac{2}{N^2}\right)\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{y}_i\boldsymbol{y}_j' \\
&= \frac{1}{N}\sum_{i=1}^{N}\Sigma^{1/2}\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma^{1/2} - \frac{1}{N(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\Sigma^{1/2}\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma^{1/2},
\end{aligned}$$

where $\boldsymbol{z}_i = \Sigma^{-1/2}\boldsymbol{y}_i$. From above expression and the definition of Q, we can express $\mathrm{tr}S^2$, $(\mathrm{tr}S)^2$, and Q as

$$
\begin{aligned}
\mathrm{tr}S^2 &= \frac{1}{N^2}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2 + \frac{1}{N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2 \\
&\quad + \frac{1}{N^2(N-1)^2}\sum_{\substack{i,j,k,l=1\\ i\neq l, j\neq k}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l \\
&\quad - \frac{2}{N^2(N-1)}\sum_{\substack{i,j,k=1\\ i\neq k}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k \\
&= \frac{1}{N^2}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2 + \frac{N^2-2N+2}{N^2(N-1)^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2 \\
&\quad + \frac{1}{N^2(N-1)^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j \\
&\quad - \frac{4}{N^2(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j \\
&\quad + \frac{2}{N^2(N-1)^2}\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k \\
&\quad - \frac{2N-4}{N^2(N-1)^2}\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k \\
&\quad + \frac{1}{N^2(N-1)^2}\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l, \\
(\mathrm{tr}S)^2 &= \frac{1}{N^2}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2 + \frac{1}{N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j \\
&\quad + \frac{1}{N^2(N-1)^2}\sum_{\substack{i,j,k,l=1\\ i\neq j, k\neq l}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{2}{N^2(N-1)} \sum_{\substack{i,j,k=1 \\ j\neq k}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k \\
= \quad & \frac{1}{N^2}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2 + \frac{2}{N^2(N-1)^2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2 \\
& +\frac{1}{N^2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j \\
& -\frac{4}{N^2(N-1)}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j \\
& -\frac{2}{N^2(N-1)}\sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k \\
& +\frac{4}{N^2(N-1)^2}\sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k \\
& +\frac{1}{N^2(N-1)^2}\sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l,
\end{aligned}
$$

$$
\begin{aligned}
Q \quad = \quad & \frac{1}{N-1}\sum_{i=1}^{N}\{(\boldsymbol{y}_i'\boldsymbol{y}_i)^2 + 4(\boldsymbol{y}_i'\bar{\boldsymbol{y}})^2 + (\bar{\boldsymbol{y}}'\bar{\boldsymbol{y}})^2 \\
& +2\boldsymbol{y}_i'\boldsymbol{y}_i\bar{\boldsymbol{y}}'\bar{\boldsymbol{y}} - 4\boldsymbol{y}_i'\boldsymbol{y}_i\boldsymbol{y}_i'\bar{\boldsymbol{y}} - 4\boldsymbol{y}_i'\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}'\bar{\boldsymbol{y}}\} \\
= \quad & \frac{1}{N-1}\sum_{i=1}^{N}(\boldsymbol{y}_i'\boldsymbol{y}_i)^2 + \frac{4}{N^2(N-1)}\sum_{i=1}^{N}\left(\boldsymbol{y}_i'\sum_{j=1}^{N}\boldsymbol{y}_j\right)^2 \\
& +\frac{2}{N^2(N-1)}\sum_{i=1}^{N}\boldsymbol{y}_i'\boldsymbol{y}_i\left(\sum_{j,k=1}^{N}\boldsymbol{y}_j'\boldsymbol{y}_k\right) \\
& -\frac{4}{N(N-1)}\sum_{i=1}^{N}\boldsymbol{y}_i'\boldsymbol{y}_i\boldsymbol{y}_i'\left(\sum_{j=1}^{N}\boldsymbol{y}_j\right) - \frac{3N}{N-1}(\bar{\boldsymbol{y}}'\bar{\boldsymbol{y}})^2 \\
= \quad & \frac{N^2-3N+3}{N^3}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2 + \frac{4N-6}{N^3(N-1)}\sum_{\substack{i,j=1 \\ i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2
\end{aligned}
$$

$$+\frac{2N-3}{N^3(N-1)}\sum_{\substack{i,j=1\\i\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j$$
$$-\frac{4(N^2-3N+3)}{N^3(N-1)}\sum_{\substack{i,j=1\\i\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j$$
$$+\frac{2N-6}{N^3(N-1)}\sum_{\substack{i,j,k=1\\i\neq j,j\neq k,k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k$$
$$+\frac{4N-12}{N^3(N-1)}\sum_{\substack{i,j,k=1\\i\neq j,j\neq k,k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k$$
$$-\frac{3}{N^3(N-1)}\sum_{\substack{i,j,k,l=1\\i\neq j\neq k\neq l,k\neq i\neq l\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l.$$

Therefore, we obtain the following lemma.

Lemma 2 *The expectations of trS^2, $(trS)^2$, and Q are calculated as*

$$\begin{aligned}
\mathrm{E}[\mathrm{tr}S^2] &= \frac{1}{N}\kappa_{11} + \frac{N}{N-1}\mathrm{tr}\Sigma^2 + \frac{1}{N-1}(\mathrm{tr}\Sigma)^2, \\
\mathrm{E}[(\mathrm{tr}S)^2] &= \frac{1}{N}\kappa_{11} + \frac{2}{N-1}\mathrm{tr}\Sigma^2 + (\mathrm{tr}\Sigma)^2, \\
\mathrm{E}[Q] &= \frac{N^2-3N+3}{N^2}\kappa_{11} + \frac{2(N-1)}{N}\mathrm{tr}\Sigma^2 + \frac{N-1}{N}(\mathrm{tr}\Sigma)^2,
\end{aligned}$$

where $\kappa_{11} = \mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z})^2] - 2\mathrm{tr}\Sigma^2 - (\mathrm{tr}\Sigma)^2$ and $\boldsymbol{z} \sim F$.

B Some variances and covariances

For the calculations of variances of $\widehat{\mathrm{tr}\Sigma^2}$, $\widehat{(\mathrm{tr}\Sigma)^2}$, and $\hat{\kappa}_{11}$, we need some variances and covariances for functions of $\boldsymbol{z}_i \sim F$. We express the results of the calculations as under.

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2\right) &= N\{\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^4] - (\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2])^2\}, \\
\mathrm{Var}\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2\right) &= 2\sum_{\substack{i,j=1\\i\neq j}}^{N} E[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^4]
\end{aligned}$$

$$
\begin{aligned}
& +4 \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} E[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k)^2] \\
& + \sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} E[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2(\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l)^2] \\
& -N^2(N-1)^2(\mathrm{tr}\Sigma^2)^2 \\
= \;& 2N(N-1)\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4] \\
& +4N(N-1)(N-2)\mathrm{E}[(\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1)^2] \\
& +N(N-1)(N-2)(N-3)(\mathrm{tr}\Sigma^2)^2 \\
& -N^2(N-1)^2(\mathrm{tr}\Sigma^2)^2 \\
= \;& 2N(N-1)\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4] \\
& +4N(N-1)(N-2)\kappa_{22} \\
& +8N(N-1)(N-2)\mathrm{tr}\Sigma^4 \\
& -2N(N-1)(\mathrm{tr}\Sigma^2)^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j\right) = \;& 2\sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2(\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j)^2] \\
& +4 \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_k] \\
& + \sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_k\boldsymbol{z}_l'\Sigma\boldsymbol{z}_l] \\
& -N^2(N-1)^2(\mathrm{tr}\Sigma)^4 \\
= \;& 2N(N-1)(\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2])^2 \\
& +4N(N-1)(N-2)(\mathrm{tr}\Sigma)^2\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2] \\
& +N(N-1)(N-2)(N-3)(\mathrm{tr}\Sigma)^4 \\
& -N^2(N-1)^2(\mathrm{tr}\Sigma)^4 \\
= \;& 2N(N-1)\kappa_{11}^2 + 8N(N-1)\kappa_{11}\mathrm{tr}\Sigma^2 \\
& +4N(N-1)^2\kappa_{11}(\mathrm{tr}\Sigma)^2 \\
& +8N(N-1)(\mathrm{tr}\Sigma^2)^2 \\
& +8N(N-1)^2\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j \right) &= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2] \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2] \\
&\quad + \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_k\boldsymbol{z}_k'\Sigma\boldsymbol{z}_j] \\
&= N(N-1)\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1] \\
&\quad +N(N-1)\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2] \\
&\quad +N(N-1)(N-2)\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2],
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Var}\left(\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k \right) \\
&\quad = 2\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2(\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k)^2] \\
&\qquad +4\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\boldsymbol{z}_k'\Sigma\boldsymbol{z}_j\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j] \\
&\qquad +2\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j(\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l)^2] \\
&\quad = 2N(N-1)(N-2)\mathrm{tr}\Sigma^2 E[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2] \\
&\qquad +4N(N-1)(N-2)\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2] \\
&\qquad +2N(N-1)(N-2)(N-3)\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2 \\
&\quad = 4N(N-1)(N-2)\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2] \\
&\qquad +2N(N-1)(N-2)\mathrm{tr}\Sigma^2\{\kappa_{11}+2\mathrm{tr}\Sigma^2\} \\
&\qquad +2N(N-1)(N-2)^2\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2,
\end{aligned}
$$

$$
\mathrm{Var}\left(\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k \right)
$$

$$
\begin{aligned}
&= 2 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k)^2] \\
&\quad +4 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\boldsymbol{z}_j'\Sigma\boldsymbol{z}_i] \\
&\quad +2 \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\boldsymbol{z}_l'\Sigma\boldsymbol{z}_j\boldsymbol{z}_l'\Sigma\boldsymbol{z}_k] \\
&= 2N(N-1)(N-2)\mathrm{E}[(\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1)^2] \\
&\quad +4N(N-1)(N-2)\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2] \\
&\quad +2N(N-1)(N-2)(N-3)\mathrm{tr}\Sigma^4 \\
&= 4N(N-1)(N-2)\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2] \\
&\quad +2N(N-1)(N-2)\{\kappa_{22} + (\mathrm{tr}\Sigma^2)^2\} \\
&\quad +2N(N-1)^2(N-2)\mathrm{tr}\Sigma^4,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Var}\left(\sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l \right) \\
&= 16 \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\boldsymbol{z}_j'\Sigma\boldsymbol{z}_l] \\
&\quad +8 \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l)^2] \\
&= 8N(N-1)(N-2)(N-3)\{2\mathrm{tr}\Sigma^4 + (\mathrm{tr}\Sigma^2)^2\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Cov}\left(\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2, \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j \right) \\
&= 2 \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^3\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j] \\
&\quad + \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_k]
\end{aligned}
$$

$$
\begin{aligned}
& -N^2(N-1)(\mathrm{tr}\Sigma)^2\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2] \\
= & \ 2N(N-1)\mathrm{tr}\Sigma\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^3] \\
& -2N(N-1)(\mathrm{tr}\Sigma)^2\{\kappa_{11}+2\mathrm{tr}\Sigma^2+(\mathrm{tr}\Sigma)^2\},
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{Cov}\left(\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k,\ \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\right) \\
= & \ 2\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k] \\
& +4\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\boldsymbol{z}_j'\Sigma\boldsymbol{z}_i] \\
& +2\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{N} \mathrm{E}[\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\boldsymbol{z}_l'\Sigma\boldsymbol{z}_j\boldsymbol{z}_l'\Sigma\boldsymbol{z}_k] \\
= & \ 2N(N-1)(N-2)\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^3\boldsymbol{z}_1] \\
& +4N(N-1)(N-2)\mathrm{E}[\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2] \\
& +2N(N-1)(N-2)(N-3)\mathrm{tr}\Sigma\mathrm{tr}\Sigma^3 \\
= & \ 2N(N-1)(N-2)(\kappa_{13}+2\mathrm{tr}\Sigma^4) \\
& +4N(N-1)(N-2)\mathrm{E}[\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2] \\
& +2N(N-1)(N-2)^2\mathrm{tr}\Sigma\mathrm{tr}\Sigma^3.
\end{aligned}
$$

C Evaluation of the variance of $\hat{\kappa}_{11}$

From the definition of $\hat{\kappa}_{11}$, we can express $\hat{\kappa}_{11}$ as

$$
\begin{aligned}
\hat{\kappa}_{11} = & \ \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2 - \frac{2}{N(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2 \\
& -\frac{1}{N(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j \\
& -\frac{4}{N(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j
\end{aligned}
$$

$$+\frac{4}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\i\neq j,j\neq k,k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k$$

$$+\frac{8}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\i\neq j,j\neq k,k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k$$

$$-\frac{6}{N(N-1)(N-2)(N-3)}\sum_{\substack{i,j,k,l=1\\i\neq j\neq k\neq l,k\neq i\neq l\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l.$$

From above expression and Appendix B, we obtain the following result:

$$\begin{aligned}
\mathrm{Var}(\hat{\kappa}_{11}) \leq\ & 4\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2-\frac{1}{N(N-1)}\sum_{\substack{i,j=1\\i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j\right)\\
& +2\mathrm{Var}\left(-\frac{2}{N(N-1)}\sum_{\substack{i,j=1\\i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2\right.\\
& \left.-\frac{4}{N(N-1)}\sum_{\substack{i,j=1\\i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\right)\\
& +2\mathrm{Var}\left(\frac{4}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\i\neq j,j\neq k,k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\right.\\
& +\frac{8}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\i\neq j,j\neq k,k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\\
& \left.-\frac{6}{N(N-1)(N-2)(N-3)}\sum_{\substack{i,j,k,l=1\\i\neq j\neq k\neq l,k\neq i\neq l\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l\right)\\
\leq\ & 4\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i)^2-\frac{1}{N(N-1)}\sum_{\substack{i,j=1\\i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_j\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{32}{N^2(N-1)^2}\mathrm{Var}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{N}(\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j)^2\right)\\
&+\frac{128}{N^2(N-1)^2}\mathrm{Var}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\right)\\
&+2\mathrm{Var}\left(\frac{4}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\ i\neq j,j\neq k,k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_i\boldsymbol{z}_j'\Sigma\boldsymbol{z}_k\right.\\
&+\frac{8}{N(N-1)(N-2)}\sum_{\substack{i,j,k=1\\ i\neq j,j\neq k,k\neq i}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_i'\Sigma\boldsymbol{z}_k\\
&\left.-\frac{6}{N(N-1)(N-2)(N-3)}\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l,k\neq i\neq l\neq j}}^{N}\boldsymbol{z}_i'\Sigma\boldsymbol{z}_j\boldsymbol{z}_k'\Sigma\boldsymbol{z}_l\right)\\
=\ & 4\left\{\frac{1}{N}\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1-\mathrm{tr}\Sigma)^4]\right.\\
&\left.-\frac{(N-3)}{N(N-1)}\kappa_{11}^2-\frac{4(N-3)}{N(N-1)}\kappa_{11}\mathrm{tr}\Sigma^2-\frac{4(N-3)}{N(N-1)}(\mathrm{tr}\Sigma^2)^2\right\}\\
&+\left\{\frac{64}{N(N-1)}\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^4]+\frac{128(N-2)}{N(N-1)}\kappa_{22}\right.\\
&\left.+\frac{256(N-2)}{N(N-1)}\mathrm{tr}\Sigma^4-\frac{64}{N(N-1)}(\mathrm{tr}\Sigma^2)^2\right\}\\
&+\left\{\frac{128}{N(N-1)}\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1]\right.\\
&+\frac{128}{N(N-1)}\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2]\\
&\left.+\frac{128(N-2)}{N(N-1)}\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2]\right\}\\
&+\left\{\frac{128}{N(N-1)(N-2)}\mathrm{E}[\boldsymbol{z}_1'\Sigma\boldsymbol{z}_1\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2]\right.\\
&+\frac{512}{N(N-1)(N-2)}\mathrm{E}[\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_1\boldsymbol{z}_2'\Sigma\boldsymbol{z}_2\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2]\\
&+\frac{512}{N(N-1)(N-2)}\mathrm{E}[(\boldsymbol{z}_1'\Sigma\boldsymbol{z}_2)^2\boldsymbol{z}_1'\Sigma^2\boldsymbol{z}_2]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{64}{N(N-1)(N-2)}\kappa_{11}\mathrm{tr}\Sigma^2\\
&+\frac{256}{N(N-1)(N-2)}\kappa_{22}+\frac{256}{N(N-1)(N-2)}\kappa_{13}\\
&+\frac{192(2N-3)}{N(N-1)(N-2)(N-3)}(\mathrm{tr}\Sigma^2)^2\\
&+\frac{64}{N(N-1)}\mathrm{tr}\Sigma^2(\mathrm{tr}\Sigma)^2+\frac{256}{N(N-1)}\mathrm{tr}\Sigma\mathrm{tr}\Sigma^3\\
&+\left.\frac{128(2N^2-4N+3)}{N(N-1)(N-2)(N-3)}\mathrm{tr}\Sigma^4\right\}.
\end{aligned}
$$

D Some moments of $\boldsymbol{z} \sim F$

In this section, we assume that all elements of $\boldsymbol{z} = (z_1, \cdots, z_p) \sim F$ are i.i.d. and $\mathrm{E}[z_i^8] < \infty$. Let $\boldsymbol{z}(= (z_1, \cdots, z_p))$ and $\boldsymbol{y}(= (y_1, \cdots, y_p))$ be $\boldsymbol{x}, \boldsymbol{y} \sim F$. We note that $\gamma_i = E[z_j^i]$. Then we can calculate some moments as follows:

$$
\begin{aligned}
\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{y})^4] &= \mathrm{E}\left[\left(\sum_{i,j=1}^{p}\sigma_{ij}x_iy_j\right)^4\right]\\
&= \mathrm{E}\left[\left(\sum_{i,j=1}^{p}\sigma_{ij}^2x_i^2y_j^2+\sum_{\substack{i,j,k=1\\ j\neq k}}^{p}\sigma_{ij}\sigma_{ik}x_i^2y_jy_k\right.\right.\\
&\qquad\left.\left.+\sum_{\substack{i,j,k=1\\ i\neq j}}^{p}\sigma_{ik}\sigma_{jk}x_ix_jy_k^2+\sum_{\substack{i,j,k,l=1\\ i\neq j,k\neq l}}^{p}\sigma_{ik}\sigma_{jl}x_ix_jy_ky_l\right)^2\right]\\
&= \sum_{i,j=1}^{p}\sigma_{ij}^4\mathrm{E}[x_i^4y_j^4]+2\sum_{\substack{i,j,k=1\\ j\neq k}}^{p}\sigma_{ij}^2\sigma_{ik}^2E[x_i^4y_j^2y_k^2]\\
&\qquad+\sum_{\substack{i,j,k,l=1\\ i\neq j,k\neq l}}^{p}\sigma_{ik}^2\sigma_{jl}^2\mathrm{E}[x_i^2x_j^2y_k^2y_l^2]+4\sum_{\substack{i,j,k=1\\ j\neq k}}^{p}\sigma_{ij}^2\sigma_{ik}^2\mathrm{E}[x_i^4y_j^2y_k^2]\\
&\qquad+4\sum_{\substack{i,j,k=1\\ i\neq j,k\neq l}}^{p}\sigma_{ik}\sigma_{il}\sigma_{jk}\sigma_{jl}\mathrm{E}[x_i^2x_j^2y_k^2y_l^2]\\
&\qquad+2\sum_{\substack{i,j,k,l=1\\ i\neq j,k\neq l}}^{p}\sigma_{ik}^2\sigma_{jl}^2\mathrm{E}[x_i^2x_j^2y_k^2y_l^2]
\end{aligned}
$$

$$
\begin{aligned}
& +2 \sum_{\substack{i,j,k,l=1\\ i\neq j, k\neq l}}^{p} \sigma_{ik}\sigma_{jl}\sigma_{il}\sigma_{jk}\mathrm{E}[x_i^2 x_j^2 y_k^2 y_l^2] \\
= \; & \gamma_4^2 \sum_{i,j=1}^{p} \sigma_{ij}^4 + 6\gamma_4 \sum_{\substack{i,j,k=1\\ j\neq k}}^{p} \sigma_{ij}^2\sigma_{ik}^2 \\
& +3 \sum_{\substack{i,j,k,l=1\\ i\neq j, k\neq l}}^{p} \sigma_{ik}^2\sigma_{jl}^2 + 6 \sum_{\substack{i,j,k,l=1\\ i\neq j, k\neq l}}^{p} \sigma_{ik}\sigma_{jl}\sigma_{il}\sigma_{jk} \\
= \; & (\gamma_4-3)^2\mathrm{tr}(\Sigma\odot\Sigma)^2 + 6(\gamma_4-3)\mathrm{tr}(\Sigma^2\odot\Sigma^2) \\
& +3(\mathrm{tr}\Sigma^2)^2 + 6\mathrm{tr}\Sigma^4,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z})^4] = \; & \mathrm{E}\left[\left(\sum_{i=1}^{p}\sigma_{ii}z_i^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{p}\sigma_{ij}z_iz_j\right)^4\right] \\
= \; & \sum_{i=1}^{p}\sigma_{ii}^4\mathrm{E}[z_i^8] + 4\sum_{\substack{i,j=1\\ i\neq j}}^{p}\sigma_{ii}^3\sigma_{jj}\mathrm{E}[z_i^6] + 3\sum_{\substack{i,j=1\\ i\neq j}}^{p}\sigma_{ii}^2\sigma_{jj}^2\mathrm{E}[z_i^4z_j^4] \\
& +6 \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}^2\sigma_{jj}\sigma_{kk}\mathrm{E}[z_i^4] \\
& + \sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{p} \sigma_{ii}\sigma_{jj}\sigma_{kk}\sigma_{ll} \\
& +24 \sum_{\substack{i,j=1\\ i\neq j}}^{p} \sigma_{ii}^2\sigma_{jj}\sigma_{ij}\mathrm{E}[z_i^5z_j^3] \\
& +24 \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}\sigma_{jj}\sigma_{kk}\sigma_{ij}\mathrm{E}[z_i^3z_j^3] \\
& +24 \sum_{\substack{i,j=1\\ i\neq j}}^{p} \sigma_{ii}^2\sigma_{ij}^2\mathrm{E}[z_i^6] + 12 \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}^2\sigma_{jk}^2\mathrm{E}[z_i^4] \\
& +24 \sum_{\substack{i,j=1\\ i\neq j}}^{p} \sigma_{ii}\sigma_{jj}\sigma_{ij}^2\mathrm{E}[z_i^4z_j^4] + 48 \sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}\sigma_{jj}\sigma_{ik}^2\mathrm{E}[z_i^4]
\end{aligned}
$$

$$
\begin{aligned}
& +12 \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{p} \sigma_{ii}\sigma_{jj}\sigma_{kl}^2 \\
& +48 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}\sigma_{jj}\sigma_{ki}\sigma_{kj}\mathrm{E}[z_i^3 z_k^3] \\
& +32 \sum_{\substack{i,j=1 \\ i\neq j}}^{p} \sigma_{ii}\sigma_{ij}^3\mathrm{E}[z_i^5 z_j^3] + 16 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}\sigma_{jk}^3\mathrm{E}[z_j^3 z_k^3] \\
& +96 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}\sigma_{ij}\sigma_{jk}^2\mathrm{E}[z_i^3 z_j^3] \\
& +96 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ii}\sigma_{ij}\sigma_{ik}\sigma_{jk}\mathrm{E}[z_i^4] \\
& +32 \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{p} \sigma_{ii}\sigma_{jk}\sigma_{jl}\sigma_{kl} \\
& +8 \sum_{\substack{i,j=1 \\ i\neq j}}^{p} \sigma_{ij}^4\mathrm{E}[z_i^4 z_j^4] + 48 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ij}^2\sigma_{ik}^2\mathrm{E}[z_i^4] \\
& +96 \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{p} \sigma_{ij}^2\sigma_{ik}\sigma_{jk}\mathrm{E}[z_i^3 z_k^3] \\
& +12 \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{p} \sigma_{ij}^2\sigma_{kl}^2 + 48 \sum_{\substack{i,j,k,l=1 \\ i\neq j\neq k\neq l, k\neq i\neq l\neq j}}^{p} \sigma_{ij}\sigma_{kl}\sigma_{ik}\sigma_{jl} \\
= \; & (\gamma_8 - 28\gamma_6 - 56\gamma_5\gamma_3 - 35\gamma_4^2 + 420\gamma_4 + 560\gamma_3^2 - 630) \\
& \times \mathrm{tr}(\Sigma \odot \Sigma \odot \Sigma \odot \Sigma) \\
& +4(\gamma_6 - 15\gamma_4 - 10\gamma_3^2 + 30)\mathrm{tr}\Sigma\mathrm{tr}(\Sigma \odot \Sigma \odot \Sigma) \\
& +3(\gamma_4 - 3)^2\{\mathrm{tr}(\Sigma \odot \Sigma)\}^2 + 6(\gamma_4 - 3)(\mathrm{tr}\Sigma)^2\mathrm{tr}(\Sigma \odot \Sigma) \\
& +(\mathrm{tr}\Sigma)^4 + 24\gamma_3(\gamma_5 - 10\gamma_3)\boldsymbol{\delta}(\Sigma \odot \Sigma)'\Sigma\boldsymbol{\delta}(\Sigma) \\
& +24\gamma_3^2\mathrm{tr}\Sigma\boldsymbol{\delta}(\Sigma)'\Sigma\boldsymbol{\delta}(\Sigma) \\
& +24(\gamma_6 - 15\gamma_4 - 10\gamma_3^2 + 30)\mathrm{tr}\{(\Sigma \odot \Sigma)\Sigma^2\} \\
& +12(\gamma_4 - 3)\mathrm{tr}\Sigma^2\mathrm{tr}(\Sigma \odot \Sigma) \\
& +24(\gamma_4 - 3)^2\boldsymbol{\delta}(\Sigma)'(\Sigma \odot \Sigma)\boldsymbol{\delta}(\Sigma) + 48(\gamma_4 - 3)\mathrm{tr}\Sigma\mathrm{tr}(\Sigma \odot \Sigma^2) \\
& +12(\mathrm{tr}\Sigma)^2\mathrm{tr}\Sigma^2 + 48\gamma_3^2\boldsymbol{\delta}(\Sigma)'\Sigma^2\boldsymbol{\delta}(\Sigma)
\end{aligned}
$$

$$
\begin{aligned}
&+32\gamma_3(\gamma_5-10\gamma_3)\mathrm{tr}[\Sigma\odot\{\Sigma(\Sigma\odot\Sigma)\}]+16\gamma_3^2\mathrm{tr}\Sigma\mathrm{tr}\{\Sigma(\Sigma\odot\Sigma)\}\\
&+96\gamma_3^2\boldsymbol{\delta}(\Sigma)'\Sigma\boldsymbol{\delta}(\Sigma^2)+96(\gamma_4-3)\mathrm{tr}(\Sigma\odot\Sigma^3)+32\mathrm{tr}\Sigma\mathrm{tr}\Sigma^3\\
&+8(\gamma_4-3)^2\mathrm{tr}(\Sigma\odot\Sigma)^2+48(\gamma_4-3)\mathrm{tr}(\Sigma^2\odot\Sigma^2)\\
&+96\gamma_3^2\mathrm{tr}\{\Sigma^2(\Sigma\odot\Sigma)\}+12(\mathrm{tr}\Sigma^2)^2+48\mathrm{tr}\Sigma^4
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z})^3] &= \mathrm{E}\left[\left(\sum_{i=1}^p\sigma_{ii}z_i^2+\sum_{\substack{i,j=1\\ i\neq j}}^p\sigma_{ij}z_iz_j\right)^3\right]\\
&= \sum_{i=1}^p\sigma_{ii}^3\mathrm{E}[z_i^6]+3\sum_{\substack{i,j=1\\ i\neq j}}^p\sigma_{ii}^2\sigma_{jj}\mathrm{E}[z_i^4]+\sum_{\substack{i,j,k=1\\ i\neq j,j\neq k,k\neq i}}^p\sigma_{ii}\sigma_{jj}\sigma_{kk}\\
&\quad+6\sum_{\substack{i,j=1\\ i\neq j}}^p\sigma_{ii}\sigma_{jj}\sigma_{ij}\mathrm{E}[z_i^3z_j^3]+12\sum_{\substack{i,j=1\\ i\neq j}}^p\sigma_{ii}\sigma_{ij}^2\mathrm{E}[z_i^4]\\
&\quad+6\sum_{\substack{i,j,k=1\\ i\neq j,j\neq k,k\neq i}}^p\sigma_{ii}\sigma_{jk}^2+4\sum_{\substack{i,j=1\\ i\neq j}}^p\sigma_{ij}^3\mathrm{E}[z_i^3z_j^3]\\
&\quad+8\sum_{\substack{i,j,k=1\\ i\neq j,j\neq k,k\neq i}}^p\sigma_{ij}\sigma_{ik}\sigma_{jk}\\
&= (\gamma_6-15\gamma_4-10\gamma_3^2+30)\mathrm{tr}(\Sigma\odot\Sigma\odot\Sigma)\\
&\quad+3(\gamma_4-3)\mathrm{tr}\Sigma\mathrm{tr}(\Sigma\odot\Sigma)+12(\gamma_4-3)\mathrm{tr}(\Sigma\odot\Sigma^2)\\
&\quad+4\gamma_3^2\mathrm{tr}\{\Sigma(\Sigma\odot\Sigma)\}+6\gamma_3^2\boldsymbol{\delta}(\Sigma)'\Sigma\boldsymbol{\delta}(\Sigma)\\
&\quad+(\mathrm{tr}\Sigma)^3+6\mathrm{tr}\Sigma\mathrm{tr}\Sigma^2+8\mathrm{tr}\Sigma^3,
\end{aligned}
$$

where $\Sigma=(\sigma_{ij})$, $\boldsymbol{\delta}(A)=(a_{11},\cdots,a_{pp})'$ $(A=(a_{ij}):p\times p)$. From the expansions of $\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z})^4]$ and $\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z})^3]$, we can derive

$$
\begin{aligned}
\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z}-\mathrm{tr}\Sigma)^4] &= (\gamma_8-28\gamma_6-56\gamma_5\gamma_3-35\gamma_4^2+420\gamma_4+560\gamma_3^2-630)\\
&\quad\times\mathrm{tr}(\Sigma\odot\Sigma\odot\Sigma\odot\Sigma)\\
&\quad+3(\gamma_4-3)^2\{\mathrm{tr}(\Sigma\odot\Sigma)\}^2\\
&\quad+24\gamma_3(\gamma_5-10\gamma_3)\boldsymbol{\delta}(\Sigma\odot\Sigma)'\Sigma\boldsymbol{\delta}(\Sigma)\\
&\quad+24(\gamma_6-15\gamma_4-10\gamma_3^2+30)\mathrm{tr}\{(\Sigma\odot\Sigma)\Sigma^2\}\\
&\quad+12(\gamma_4-3)\mathrm{tr}\Sigma^2\mathrm{tr}(\Sigma\odot\Sigma)\\
&\quad+24(\gamma_4-3)^2\boldsymbol{\delta}(\Sigma)'(\Sigma\odot\Sigma)\boldsymbol{\delta}(\Sigma)\\
&\quad+48\gamma_3^2\boldsymbol{\delta}(\Sigma)'\Sigma^2\boldsymbol{\delta}(\Sigma)\\
&\quad+32\gamma_3(\gamma_5-10\gamma_3)\mathrm{tr}[\Sigma\odot\{\Sigma(\Sigma\odot\Sigma)\}]\\
&\quad+96\gamma_3^2\boldsymbol{\delta}(\Sigma)'\Sigma\boldsymbol{\delta}(\Sigma^2)+96(\gamma_4-3)\mathrm{tr}(\Sigma\odot\Sigma^3)\\
&\quad+8(\gamma_4-3)^2\mathrm{tr}(\Sigma\odot\Sigma)^2+48(\gamma_4-3)\mathrm{tr}(\Sigma^2\odot\Sigma^2)
\end{aligned}
$$

$$+96\gamma_3^2\mathrm{tr}\{\Sigma^2(\Sigma\odot\Sigma)\}+12(\mathrm{tr}\Sigma^2)^2+48\mathrm{tr}\Sigma^4.$$

Using the inequality $2ab \le a^2+b^2$ for any real numbers a and b, above all terms are bounded by $C(\mathrm{tr}\Sigma^2)^2$, where C is a positive constant. For example,

$$\begin{aligned}\mathrm{tr}(\Sigma\odot\Sigma^3) &= \sum_{i,j,k=1}^{p}\sigma_{ii}\sigma_{ij}\sigma_{jk}\sigma_{ki}\\ &\le \sum_{i,j,k=1}^{p}(\sigma_{ii}^2+\sigma_{ij}^2)(\sigma_{jk}^2+\sigma_{ki}^2)\\ &\le 4\left(\sum_{i,j=1}^{p}\sigma_{ij}^2\right)^2 = 4(\mathrm{tr}\Sigma^2)^2.\end{aligned}$$

Since the other terms can be evaluated by similar derivation, we obtain the relation $\mathrm{E}[(\boldsymbol{z}'\Sigma\boldsymbol{z}-\mathrm{tr}\Sigma)^4] \le C(\mathrm{tr}\Sigma^2)^2$.

References

[1] J. R. Magnus, H. Neudecker, The commutation matrix: some properties and applications, Ann. Statist. 7 (1979) 381-894.

[2] J. R. Schott, Some high-dimensional tests for a one-way MANOVA, J. Multivariate Anal. 98 (2007) 1825-1839.

[3] M. S. Srivastava, Some tests concerning the covariance matrix in high dimensional data, J. Japan Statist. Soc. 35 (2005) 251-272.

[4] M. S. Srivastava, Y. Fujikoshi, Multivariate analysis of variance with fewer observations than the dimension J. Multivariate Anal. 97 (2006) 1927-1940.

[5] M. S. Srivastava, T. Kollo, D. V. Rosen, Some tests for the covariance matrix with fewer observations than the dimension under non-normality, J. Multivariate Anal. 102 (2011) 1090-1103.

[6] S. X. Chen, L.-X. Zhang, P.-S. Zhong, Tests for high-dimensional covariance matrices, J. Am. Statist. Assoc. 105 (2010) 810-819.

[7] S. X. Chen, Y.-L. Qin, A two-sample test for high-dimensional data with applications to gene-set testing, Ann. Statist. 38 (2010) 808-835.

[8] Y. Fujikoshi, T. Himeno, H. Wakaki, Asymptotic results of a high dimensional MANOVA test and power comparison when the dimension is large compared to the sample size, J. Japan Statist. Soc. 34 (2004) 19-26.